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Vorticity Alignment Dynamics in Fluids & MHD

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The Large Eddy Simulation (LES) turbulence model of Misra and Pullin [1] is based on alignment dynamics, in which each fluid parcel carries an unit vector that responds to the mean flow as a Lundgren spiral vortex. We analyze this model in the framework of its constrained variational principle and Ertel theorem (equality of cross derivatives with respect to Lagrangian coordinate and time.)

Outline:
(1) Define Ertel's theorem, Ohkitani's relation, vorticity frame dynamics and alignment dynamics for Euler's equations.
(2) Use Ertel's theorem to derive Lagrangian dynamics of the Frenet-Serret curvature and torsion of vortex lines
(3) Represent Euler vorticity alignments with strain rate S & pressure Hessian P as quaternions. These yield the Cayley-Klein parameters of $S\hat{\boldsymbol{\omega}}$ & $P\hat{\boldsymbol{\omega}}$
(4) Recover evolution of S-alignment $\boldsymbol{\zeta} = [\alpha, \boldsymbol{\chi}]$ driven by P-alignment $\boldsymbol{\zeta}_p = [\alpha_p, \boldsymbol{\chi}_p]$ as dynamics of quaternions
(5) Apply this structure to LES models (Misra-Pullin [1], LANS$-\alpha$ [2]) and ideal MHD.

Theorem: (Ertel 1942) *If $\boldsymbol{\omega}$ satisfies the $3D$ incompressible Euler equations then any differentiable function μ satisfies*

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right).$$

Proof: In characteristic (Lie-derivative) form, the vorticity equation is,

$$\frac{D}{Dt}\Big(\boldsymbol{\omega}\cdot\frac{\partial}{\partial\boldsymbol{x}}\Big) = \Big(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\Big)\cdot\frac{\partial}{\partial\boldsymbol{x}} = 0 \quad \text{along} \quad \frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x},t)$$

So $\boldsymbol{\omega}\cdot\frac{\partial}{\partial\boldsymbol{x}}(t) = \boldsymbol{\omega}\cdot\frac{\partial}{\partial\boldsymbol{x}}(0) = d/ds$ is a Lagrangian invariant (Cauchy 1859) That is, the derivatives commute

$$\left[\frac{D}{Dt}\,,\,\frac{d}{ds}\right] = 0 \quad \text{with} \quad \frac{d}{ds}\equiv\boldsymbol{\omega}\cdot\nabla$$

and Ertel's theorem follows. Hence, a vortex line with arclength derivative $\boldsymbol{\omega}\cdot\nabla = d/ds$ is frozen into the flow of the fluid velocity $\boldsymbol{u}(\boldsymbol{x},t)$, and Ertel's theorem follows by equality of cross derivatives in t and s.

Corollary: $D\mu/Dt = 0$ implies $D(\boldsymbol{\omega}\cdot\nabla\mu)/Dt = 0$ (e.g. PV in GFD).

Theorem: (Ohkitani 1993) *The vortex stretching vector $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$ obeys*

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right) = -P\,\boldsymbol{\omega}$$

where P the Hessian matrix of the pressure $P = \{p_{,ij}\} = \left\{\frac{\partial^2 p}{\partial x_i\partial x_j}\right\}$ Thus,

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega}\cdot P\boldsymbol{\omega})}{Dt} = -\,P\boldsymbol{\omega} \quad \text{(Ohkitani's relation)}$$

Remark: Ohkitani's relation shows how P-alignments drive the dynamics of S-alignments.

Proof: Take $\mu = \boldsymbol{u}$ in Ertel's theorem.

Definitions: Vorticity growth rate (α) and swing rate ($\boldsymbol{\chi}$)
The material rates of change of magnitude $|\boldsymbol{\omega}|$ and direction $\hat{\boldsymbol{\omega}}$ are given by

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \text{with} \quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} = (S\hat{\boldsymbol{\omega}})_{\|} + (S\hat{\boldsymbol{\omega}})_{\perp}$$

- *The scalar $\alpha = \hat{\boldsymbol{\omega}}\cdot S\hat{\boldsymbol{\omega}}$ is the vorticity growth rate*

$$\frac{D|\boldsymbol{\omega}|}{Dt} = \alpha\,|\boldsymbol{\omega}| \qquad \begin{array}{ll} \alpha > 0 & \text{stretching} \\ \alpha < 0 & \text{shrinking} \end{array}$$

- *The 3-vector $\boldsymbol{\chi} = \hat{\boldsymbol{\omega}}\times S\hat{\boldsymbol{\omega}}$ is the vorticity swing rate*

$$\frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}\,, \qquad \hat{\boldsymbol{\omega}}\times\frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi} \quad \text{(frequency)}$$

Remark: If $\boldsymbol{\omega}$ aligns with an eigenvector $S\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}$, then $\boldsymbol{\chi} = 0$.
*For such alignment, the vorticity **direction** is frozen into the flow.*

We decompose the statement "P-alignment drives S-alignment" into parallel and perpendicular components:

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \& \quad \frac{D^2\boldsymbol{\omega}}{Dt^2} = -\,P\boldsymbol{\omega} \quad \text{with} \quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} \quad \& \quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$$

We then find Lagrangian dynamics for the orthonormal co-moving frame $\{\hat{\boldsymbol{\omega}},\, \hat{\boldsymbol{\chi}},\, \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}}\}$

$$\frac{D}{Dt}\begin{pmatrix} \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\chi}} \\ \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}} \end{pmatrix} = \boldsymbol{\mathcal{D}}\times\begin{pmatrix} \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\chi}} \\ \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}} \end{pmatrix}$$

The "Darboux vector" $\boldsymbol{\mathcal{D}}$ is defined as $\boldsymbol{\mathcal{D}} = \boldsymbol{\chi} - c_1\hat{\boldsymbol{\omega}}/\chi$ and $c_1 = \hat{\boldsymbol{\omega}}\cdot(\hat{\boldsymbol{\chi}}\times\boldsymbol{\chi}_p)$ depends on the pressure Hessian. Furthermore, the P-parameters $[\alpha_p,\, \boldsymbol{\chi}_p]$ drive S-parameters $[\alpha,\, \boldsymbol{\chi}]$ in the Lagrangian alignment-parameter dynamics

$$\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 = -\,\alpha_p \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} = -\,\boldsymbol{\chi}_p$$

Vorticity frame dynamics for the three-dimensional incompressible Euler equations also has an elegant representation in terms of quaternions. In this representation, alignment-parameter dynamics is unified into a single equation and Ertel's theorem relates the Lagrangian evolution in time of the orthonormal frame $\{\hat{\boldsymbol{\omega}},\, \hat{\boldsymbol{\chi}},\, \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}}\}$ to the Frenet-Serret equations for the curvature and torsion of a vortex line. The equations for ideal MHD also fall naturally into this framework.

References

[1] A. Misra and D.I. Pullin 1997. *A vortex-based subgrid stress model for large-eddy simulation*, Phys. Fluids **9**, 2443 – 2454.

[2] C. Foias, D. D. Holm and E. S. Titi 2001. *The Navier-Stokes-alpha model of fluid turbulence*, Physica D **152**, 505 – 519.

[3] H. Ertel 1942 *Ein Neuer Hydrodynamischer Wirbelsatz*, Met. Z. **59**, 271 – 281.

[4] K. Ohkitani 1993. Eigenvalue problems in three-dimensional Euler flows. *Phys. Fluids A*, **5**, 2570–2572.

[5] J.D. Gibbon, D.D. Holm, R.M. Kerr and I. Roulstone 2005. *Quaternions and particle dynamics in the Euler fluid equations.* nlin.cd/0512034