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A High Order WENO Finite Difference Scheme for Incompressible Fluids and Magnetohydrodynamics

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We present a high-order accurate weighted-essentially-non-oscillatory (WENO) finite difference scheme for solving the motion of incompressible fluids, both for non-magnetic and magnetohydrodynamic (MHD) systems. WENO schemes were originally developed for the compressible Euler equations. They are based on essentially non-oscillatory (ENO) schemes. The key idea in ENO schemes is to approximate the fluxes at the cell boundaries with high order accuracy by using the smoothest stencil among several candidates, and at the same time to avoid spurious oscillations near shocks and discontinuities. The WENO schemes go one step further by taking a weighted average of all candidates. The weights are adjusted by the local smoothness of the solution so that essentially zero weights are given to non-smooth stencils while optimal weights are prescribed in smooth regions. Near discontinuities, WENO schemes and ENO schemes behave in much the same way but, in the smooth regions of the solution, WENO schemes act more like an upstream centered scheme. In principle, any r-th order accurate ENO scheme can be converted into a (2r-1)-th order accurate WENO scheme.

WENO schemes were first developed in a finite volume formulation by Liu, Osher and Chan [1994] for one-dimensional conservation laws. A finite difference version for multidimensional conservation laws was created by Jiang and Shu [1996]. The finite difference WENO scheme was applied to compressible MHD by Jiang and Wu [1999]. Their code forms the basis of this incompressible code.

As in many modern shock capturing methods, WENO schemes are based on local characteristic decomposition of waves and on upwind methods. These two features are equally important for incompressible systems. The wave decomposition of this incompressible MHD code is accomplished by using the characteristic, Elsasser variables. In the code, the WENO method is used in the spatial discretization. High-order Runge-Kutta methods are employed for time integration and the fractional-step method of Kim and Moin [1985] is used to enforce the incompressibility condition.

Numerical results from our new 5th-order accurate code demonstrate that the scheme perform well for one-dimensional Riemann problems, a two-dimensional double-shear flow problem, and the two-dimensional Orszag-Tang MHD vortex system. They establish that the WENO code is numerical stable even when there are no explicit dissipation terms. The code competes on equal terms with pseudo-spectral and spectral methods in regions where the solution is smooth. It can also treat discontinuous data; the method has an advantage over spectral methods in regions where gradients are large.

References

[1] G.-S. Jiang and C.-W. Shu 1996. *Efficient implementation of weighted ENO schemes*, J. Comput. Phys. **126**, 202 – 228.

[2] G.-S. Jiang and C.-C. Wu 1999. *A high-order WENO finite difference scheme for the equations of ideal magnetohydrodynamics*, J. Comput. Phys. **150**, 561 – 594.

[3] J. Kim and P. Moin 1985. *Application of a fractional-step method to incompressible Navier-Stokes equations*, J. Comput. Phys. **59**, 308 – 323.

[4] X-D. Liu, S. Osher and T. Chan 1994. *Weighted essentially non-oscillatory schemes*, J. Comput. Phys. **115**, 200 – 212.