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Scaling laws for dynamos in rotating spherical shells and application to planetary magnetic fields

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We study numerically an extensive set of dynamo models in rotating spherical shells with the geometry of Earth's core, covering a wide range of control parameters. The Ekman numner E varies between 10^{-6} and 3×10^{-4}, the magnetic Prandtl number Pm between 0.06 and 10, the Prandtl number Pr between 0.1 and 10, and the Rayleigh number Ra is up to 50 times critical. Convection is driven by a fixed temperature contrast between rigid boundaries. There are two distinct classes of solutions with strong and weak dipole contributions to the magnetic field, respectively. The transition from dipolar to non-dipolar dynamos is found when the scale-dependent Rossby number, $Ro_\ell = U/(\Omega\ell)$, exceeds a value of ≈ 0.12 independent of the values of E, Pr and Pm (U is the rms-velocity, Ω rotation rate, and ℓ a characteristic length scale of the flow). Since Ro_ℓ measures the importance of inertial forces to the Coriolis force, dipolar dynamos break down when inertia starts to play a significant role in the force balance.

We find that in the dipolar regime the minimum magnetic Reynolds number Rm for self-sustained dynamos is independent of the magnetic Prandtl number Pm in the range 40 - 50. However, dynamos at low Pm exist only at sufficiently low Ekman number E. The lowest magnetic Prandtl number at which we find a self-sustained dipolar dynamo varies as $Pm \sim E^{3/4}$. At low Pm the hydrodynamic Reynolds number must be large to exceed the critical value of Rm. The associated inertial effects have an adverse influence on the dynamo and a low Ekman number is required to balance them by strong rotational forces.

For dynamos in the dipolar regime we attempt to establish scaling laws that fit our numerical results. Assuming that diffusive effects do not play a primary role, we introduce non-dimensional parameters that are independent of any diffusivity. As the primary control parameter, we define a modified Rayleigh number based on the advected heat (or buoyancy) flux Q, $Ra^*_Q \sim Q/(\Omega^3 D^4)$, where D is the shell thickness. Ra^*_Q is equivalent to the non-dimensional power generated by buoyancy forces. Characteristic properties of the solution are described by the Rossby number $Ro = U/(\Omega D)$ for the flow velocity, the Lorentz number $Lo = B/([\mu\rho]^1/2\Omega D)$ for the magnetic field strength B, and a modified Nusselt number $Nu^* \sim Q/(\Omega \Delta T D^3)$ for the heat transport efficiency. To first approximation, all our dynamo results can be collapsed into simple power-law dependencies on the modified Rayleigh number, with approximate exponents of 2/5, 1/2 and 1/3 for the Rossby number, modified Nusselt number and Lorentz number, respectively. Residual dependencies on the parameters related to diffusion are weak. The Ekman number and hydrodynamic Prandtl number seem to have no effect, but an influence on the magnetic Prandtl number, with a power law exponent of order 1/10, may exist. A similar weak dependency on Pm has been found before in a scaling law for the ohmic dissipation time in numerical dynamo models, but has been rejected because it did not agree well with the ohmic dissipation observed in the Karlsruhe dynamo experiment, where Pm is much smaller than in the models.

The Elsasser number Λ, which is the conventional measure for the ratio of Lorentz force to Coriolis force, is found to vary widely. Our scaling laws are in agreement with the assumption that the magnetic field strength is controlled by the available power and not necessarily by a force balance. In fact, the scaling law for the Lorentz number requires for a good fit the introduction of a correction factor which accounts for the fraction of energy dissipated by viscous rather than by ohmic dissipation. We try to assess the relative importance of the various forces by studying sources and sinks of enstrophy (vorticity squared). In general Coriolis and buoyancy forces are of the same order, inertia and viscous forces make smaller and variable contributions, and the Lorentz force is highly variable. We can give only a partial theoretical basis for our scaling law. The missing piece is an explanation for the empirical 2/5-exponent in the law for the Rossby number.

We use our scaling law for the Rossby number to deduce the Rayleigh number of the Earth's core. Using core flow velocity estimates obtained from geomagnetic secular variation, we obtain Ra_Q^* to be about 3×10^{-13} and an associated buoyancy flux of 3×10^4 kg/sec. When we assume that this represents predominantly the compositional flux of light element which is rejected when the inner core solidifies, we predict a small growth rate of the inner core of order 0.1 mm/yr and an inner core age of the order 3.5 Gyr.

When we take a power law exponent of 1/3 in the scaling law for the Lorentz number and ignore the possible weak dependence on Pm, a surprising implication is the independence of magnetic field strength B on both the conductivity and the rotation rate. B is basically controlled by the buoyancy flux. For our estimate of the buoyancy flux we obtain a magnetic field strength of order 1 mT inside the core. This is slightly low compared to previous estimates, but is still reasonable and in agreement with a core field estimate from the possible observation of torsional oscillations.

Applying our scaling laws to other planetary dynamos, we find that the observed excess luminosity of Jupiter implies an internal field of 8 mT, in agreement with Jupiter's external field being ten times stronger than that of the Earth. For Saturn the predicted magnetic field seems too strong and Mercury's very weak field cannot be explained by a very low buoyancy flux in the core, because this would correspond to a subcritical magnetic Reynolds number. Possibly Earth and Jupiter fall into the same class of dynamos as are realized in our simulations, whereas different conditions (differential rotation, strongly different inner core size) lead to different dynamos in the other two planets.

Challenges for the future are (1) to establish a more complete theoretical basis for the scaling laws, (2) further explore there range of validity, and (3) to clarify the role of the magnetic Prandtl number. For the latter two points the comparison with future laboratory dynamo experiments will be very helpful.