GTP Workshop on
Modeling MHD Turbulence; Applications to Planetary and Stellar dynamos
at NCAR, 27-30 June, 2006, Boulder, CO, USA

Euler-Lagrangian means in rotating, magnetohydrodynamic flows

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1. Background

In large Reynolds number turbulence, motion occurs on a wide range of length scales varying from the large size L of the of the system down to the very short length viscous length scale l_ν ($\ll L$). Only on that latter length l_ν is viscous dissipation important. For buoyancy driven MHD systems the problem is complicated by the fact that there are in addition other dissipation lengths such as the thermal and magnetic diffusion length scales l_κ and l_η, which may be of very disparate values depending on the Prandtl numbers l_ν/l_κ and l_ν/l_η. Since the length scale range between L and $l_{\max} \equiv \max(l_\nu,\, l_\kappa,\, l_\eta)$ is so large, it remains problematic, how to deal with the short lengths $l\,(\ll L)$, even when they remain large compared to the diffusion lengths $l \gg l_{\max}$ This is exactly the range that has motivates our enquiry and to which we restrict attention.

In rotating MHD systems, it is well known that the Lagrangian (rather than the Eulerian) representation can often be used very effectively, when $l \gg l_{\max}$. The idea is most readily appreciated in the context of the advection without diffusion of a passive scalar quantity such as temperature, for which its material derivative vanishes. Then the temperature remains constant following fluid particles. Likewise in the case of magnetic field in a perfectly conducting fluid, magnetic flux is conserved on material surfaces. Then the magnetic field at a point moving with the fluid is readily derived in the Lagrangian framework simply by properties of the coordinate transformation relating the current position of fluid particles to their original positions.

The properties mentioned are kinematic in nature and ultimately provide a useful description of the advected quantities. To actually determine their temporal evolution, we need to take advantage of the frozen field results when considering the equation of motion. The simplest application of the idea is through the investigation of the stability of a static state. Since the pressure gradient in the equation of motion does not transform nicely from a Lagrangian point of view, it is better to consider the equation of motion in its Eulerian form. The Eulerian values of the perturbation values of frozen quantities like the magnetic field, which appear in the equation of motion, are determined from their Lagrangian description in terms of the small fluid particle displacement. In this way, equations like the temperature and magnetic induction equations are bypassed leaving only equations for the fluid particle displacement. Even when the background state is moving the essence of this procedure may still be used, albeit a hybrid Eulerian–Lagrangian approach must be adopted instead, as explained in §2, and expressions for the perturbation velocity become more complicated (see, e.g., Frieman and Rotenberg [1]). Though we have outlined the linear ideas in terms of stability calculations, the idea is also useful in the description of wave motions.

Once the fluid particle displacements are no longer small, two distinctive situations need to be distinguished. On the one hand, the displacements may increase indefinitely, as is common in turbulence. For such problems involving (say) the transport of a passive scalar, the Lagrangian procedure has been adopted and used to obtain Eulerian values at quadratic order in the displacement. Then averaging may be used to determine the evolution of the Eulerian mean quantity. On the other, when the particle path displacements $\boldsymbol{\xi}$ (introduced in §2 below and employed in [1]) though finite remain of moderate size, as exemplified by wave turbulence, then the hybrid Eulerian–Lagrangian approach of §2, which builds on the early work of Eckart [2], provides a good way of addressing the evolution of the mean fields correct to $\mathcal{O}(|\boldsymbol{\xi}|^2)$. It was developed by Soward [3] in the dynamo context and Andrews and McIntyre [4] in the atmospheric science context.

2. Methodology

We relate the actual position $\mathbf{x}^*$ of a fluid element at time t^* to a reference position $\mathbf{x}$ at time t by a mapping $\mathbf{x}^* = \mathbf{x}^*(\mathbf{x}, t)$, $t^* = t$. It is simply a time dependent co-ordinate transformation which is complicated by the the motion $\mathbf{w}^*(\mathbf{x}, t) = \partial\mathbf{x}^*/\partial t$. The underlying idea is to construct the governing equations relative to the reference frame. To that end we transform our field variables such as the flow velocity $\mathbf{v}^*(\mathbf{x}^*, t^*)$ to form, in the language of the General Tensor Calculus, contravariant and covariant vectors $\mathbf{v}$ and $\mathbf{V}$ defined by

$$v_i^* \;=\; v_j \partial x_i^*/\partial x_j \;=\; V_j \partial x_j/\partial x_i^* \qquad \text{together with} \qquad w_i^* \;:=\; \partial x_i^*/\partial t \;=\; w_j \partial x_i^*/\partial x_j \;=\; W_j \partial x_j/\partial x_i^* \,.$$

In view of the pressure gradient in the equation of motion, it is convenient to write it in covariant form with momentum proportional to $\mathbf{V}$, which in turn relates to circulation $\oint \mathbf{v}^*{\cdot}\mathrm{d}\mathbf{x}^* = \oint \mathbf{V}{\cdot}\mathrm{d}\mathbf{x}$. On the other hand, the rate of working of a body force $\mathbf{F}^*$ is $\mathbf{v}^*{\cdot}\mathbf{F}^* \;=\; \mathbf{v}{\cdot}\mathbf{F}$, where $\mathbf{F}$ is the resulting covariant body force in the transformed equation of motion. The material derivative needed in the advection of momentum (or any other advected quantity for that matter) takes the form

$$\mathrm{D}/\mathrm{D}t^* \;:=\; \partial/\partial t^* \,+\, \mathbf{v}^*{\cdot}\boldsymbol{\nabla}^* \;=\; \partial/\partial t \,+ \mathbf{u}{\cdot}\boldsymbol{\nabla} \;=:\; \mathrm{D}/\mathrm{D}t \,,$$

in which $\mathbf{u} = \mathbf{v} - \mathbf{w}$ is the contravariant form of the advective velocity $\mathbf{v}^* - \mathbf{w}^*$ in the moving $\mathbf{x}^*(\mathbf{x}, t)$–frame.

The three velocities $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{V}$, which we have identified, have an important role to play in the hybrid Eulerian–Lagrangian approach. In practise to use them we consider small displacements $\mathbf{x}^* - \mathbf{x}$ and write

$$\mathbf{v}^*(\mathbf{x}^*, t^*) \;=\; \mathrm{D}\mathbf{x}^*/\mathrm{D}t^* \;=\; \mathbf{u} \,+\, \mathrm{D}\boldsymbol{\xi}/\mathrm{D}t \,, \qquad \text{where} \qquad \mathbf{x}^*(\mathbf{x}, t) \;=\; \mathbf{x} \,+\, \boldsymbol{\xi}(\mathbf{x}, t) \qquad (L \gg |\boldsymbol{\xi}| \gg l_{\max}) \,.$$

We take statistical averages $\overline{\cdots}$ and demand that $\overline{\mathbf{u}} = \mathbf{u}$ and $\overline{\boldsymbol{\xi}} = \mathbf{0}$. By this device $\mathbf{u}$ is the the Lagrangian average of $\mathbf{v}^*(\mathbf{x}^*, t^*)$ (i.e. at fixed $\mathbf{x}$ following the motion of the fluctuating displacement $\boldsymbol{\xi}$).

3. Results

We outline the equations for the contravariant and covariant vector fields that emerge from the governing equations of the full rotating MHD system (as reviewed in [5]). Holm [6] calls them the Generalised Lagrangian Mean (GLM) equations. We consider their expansions up to $\mathcal{O}(|\boldsymbol{\xi}|^2)$ extending on the earlier developments of [3], [4] and [6]. Holm has also proposed an Eulerian counterpart which he calls the generalised lagrangian mean (glm) equations derived from Hamilton's principle applied to an averaged Lagrangian. The new equations are motivated by the wish to have an Eulerian formulation which contains the merits of the GLM system (such as the conservation of mean circulation for Euler's equations). We consider the relation between the GLM and glm systems as formulated in terms of Eulerian variables. The absence of certain $\mathcal{O}(|\boldsymbol{\xi}|^2)$ terms in the glm system suggests that they have been filtered out on averaging the Lagrangian.

References

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