Vorticity alignment dynamics in fluids & MHD

DD Holm

Mathematics Department

Imperial College London, London SW7 2AZ

d.holm@imperial.ac.uk

Computer, Computational and Statistical Sciences

Los Alamos National Security (LANS)

dholm@lanl.gov

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Collaborators and main reference for this talk

JD Gibbon, DDH, RM Kerr & I Roulstone, 2005:

http://arxiv.org/abs/nlin.CD/0512034

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Notation: The $3D$ incompressible Euler fluid

The equations for **Eulerian fluid velocity** $\boldsymbol{u}$ in 3D are

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p\,, \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0$$

Taking the curl yields the **vorticity equation** $(\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u})$

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$$

The **vortex stretching vector** is $S\boldsymbol{\omega}$ with $S = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, or

$$S_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

and preservation of $\operatorname{div}\boldsymbol{u} = 0$ determines the **pressure** p as

$$-\Delta p = u_{i,j}u_{j,i} =: |\nabla\boldsymbol{u}|^2 = \mathrm{Tr}\,S^2 - \tfrac{1}{2}\,\omega^2\,.$$

Outline for the talk

1. Define **Ertel's theorem, Ohkitani's relation, vorticity frame dynamics and alignment dynamics** for Euler's equations.

2. Use Ertel's theorem to derive **Lagrangian dynamics** of the Frenet-Serret curvature and torsion of vortex lines

3. Represent Euler vorticity alignments with S & P as **quaternions**. These yield the **Cayley-Klein parameters** of

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} \quad \text{and} \quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$$

4. Derive **dynamics of quaternions** for S-alignment $\boldsymbol{\zeta} = [\alpha,\,\boldsymbol{\chi}]$ driven by P-alignment $\boldsymbol{\zeta}_p = [\alpha_p,\,\boldsymbol{\chi}_p]$

5. Apply this structure to LES models (LAE$-\alpha$, MP97mod2') and MHD

Define vorticity growth rate (α) and swing rate (χ)

The material rates of change of $|\boldsymbol{\omega}|$ and $\hat{\boldsymbol{\omega}}$ are given by

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \text{with} \quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}$$

- *The scalar $\alpha = \hat{\boldsymbol{\omega}} \cdot S\hat{\boldsymbol{\omega}}$ is the vorticity growth rate*

$$\frac{D|\boldsymbol{\omega}|}{Dt} = \alpha\,|\boldsymbol{\omega}| \qquad \begin{array}{ll} \alpha > 0 & \text{stretching} \\ \alpha < 0 & \text{shrinking} \end{array}$$

- *The 3-vector $\boldsymbol{\chi} = \hat{\boldsymbol{\omega}} \times S\hat{\boldsymbol{\omega}}$ is the vorticity swing rate*

$$\frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}\,, \qquad \hat{\boldsymbol{\omega}} \times \frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi} \quad \text{(frequency)}$$

Remark: If $\boldsymbol{\omega}$ aligns with an eigenvector $S\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}$, then $\boldsymbol{\chi} = 0$.
For such alignment, the vorticity direction is **frozen** *into the flow.*

$3D$ vortex stretching and alignment

The curl of Euler's equation yields vorticity dynamics

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}$$

whose the strain-rate matrix S has components $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

For S-alignment: $S\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$, the vorticity stretches (shrinks) depending on whether the corresponding eigenvalue λ is positive (negative).

- How long will the vorticity grow, before getting misaligned?
  - This depends on the Lagrangian rates of change of $\alpha = \hat{\boldsymbol{\omega}} \cdot S\hat{\boldsymbol{\omega}}$ and of the vorticity swing rate $\boldsymbol{\chi} = \hat{\boldsymbol{\omega}} \times S\hat{\boldsymbol{\omega}}$. For this we need $\frac{D^2\boldsymbol{\omega}}{Dt^2}$!
- Seek alignment-parameter dynamics $\left(\frac{D\alpha}{Dt} \text{ and } \frac{D\boldsymbol{\chi}}{Dt}\right)$
- Is vorticity alignment dynamics cause, effect, or both?
- A key to the answer will be Ertel's Theorem (1942)

Ertel's Theorem (1942)

Theorem: (Ertel 1942) *If $\boldsymbol{\omega}$ satisfies the $3D$ incompressible Euler equations then an arbitrary differentiable function μ satisfies*

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu)=\boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right)\,.$$

Proof: In characteristic (Lie-derivative) form, the vorticity equation is,

$$\frac{D}{Dt}\Big(\boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}\Big)=\Big(\frac{D\boldsymbol{\omega}}{Dt}-\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\Big)\cdot\frac{\partial}{\partial\mathbf{x}}=0\quad\text{along}\quad\frac{d\mathbf{x}}{dt}=\boldsymbol{u}(\mathbf{x},t)$$

So $\boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}(t)=\boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}(0)$ (Cauchy 1859) and the derivatives commute

$$\left[\frac{D}{Dt}\,,\,\boldsymbol{\omega}\cdot\nabla\right]=0$$

Hence, Ertel's theorem follows.

Corollary: $D\mu/Dt=0$ implies $D(\boldsymbol{\omega}\cdot\nabla\mu)/Dt=0$ (e.g. PV in GFD).

Some Ertel references

- Ertel; *Ein Neuer Hydrodynamischer Wirbelsatz*, Met. Z. **59**, 271-281, (1942).

- Hoskins, McIntyre, & Robertson; *On the use & significance of isentropic potential vorticity maps*, Quart. J. Roy. Met. Soc., **111**, 877-946, (1985).

- Ohkitani; *Eigenvalue problems in 3D Euler flows*, Phys. Fluids, **A5**, 2570, (1993).

- Viudez; *On the relation between Beltrami's material vorticity and Rossby-Ertel's Potential*, J. Atmos. Sci. (2001).

Define Ohkitani's relation & the pressure Hessian

Ohkitani took $\mu = \boldsymbol{u}$ in Ertel's theorem (Phys. Fluids, **A5**, 2570, 1993).

Result: The vortex stretching vector $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}$ obeys

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u})}{Dt} = \boldsymbol{\omega} \cdot \nabla \left(\frac{D\boldsymbol{u}}{Dt} \right) = -P\,\boldsymbol{\omega}$$

where P the Hessian matrix of the pressure

$$P = \{p_{,ij}\} = \left\{ \frac{\partial^2 p}{\partial x_i \partial x_j} \right\}$$

Thus,

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} \quad (\text{Ohkitani's relation})$$

So, P-alignments drive dynamics of S-alignments!

Vorticity accelerations $-\alpha_p$ & $-\chi_p$ of $|\omega|$ & $\hat{\omega}$

The material accelerations of $|\boldsymbol{\omega}|$ and $\hat{\boldsymbol{\omega}}$ are given by Ohkitani as

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = -\,P\boldsymbol{\omega} \quad \text{with} \quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}}$$

- *Scalar $\alpha_p = \hat{\boldsymbol{\omega}} \cdot P\hat{\boldsymbol{\omega}}$ gives acceleration of vorticity magnitude*

$$\frac{D^2|\boldsymbol{\omega}|}{Dt^2} = -\,\alpha_p\,|\boldsymbol{\omega}| \qquad \begin{array}{ll} \alpha_p > 0 & \text{decelerating} \\ \alpha_p < 0 & \text{accelerating} \end{array}$$

- *3-vector $\boldsymbol{\chi}_p = \hat{\boldsymbol{\omega}} \times P\hat{\boldsymbol{\omega}}$ gives acceleration of vorticity direction*

$$\frac{D^2\hat{\boldsymbol{\omega}}}{Dt^2} = -\,\boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}}$$

Remark: If $\boldsymbol{\omega}$ aligns with an eigenvector $P\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}$, then $\boldsymbol{\chi}_p = 0$.
For such alignment, $\mathrm{P}_\perp S\hat{\boldsymbol{\omega}} = \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}$ is frozen into the flow.

Vorticity and alignment dynamics

- **Vorticity is driven by S**

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega}$$

- **Alignment is driven by P**

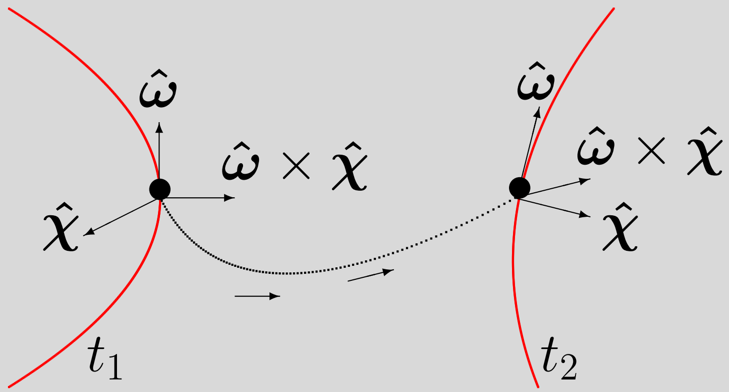
$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega}$$

with

$$\boldsymbol{u} = \mathrm{curl}^{-1}\boldsymbol{\omega}, \quad \mathrm{tr}\,P = -\,|\nabla\boldsymbol{u}|^2$$

- **The latter involves the pressure Hessian P.**
- **This pressure dependence produces nonlocal effects.**

Lagrangian frame dynamics: tracking the orientation of vorticity following a fluid particle



The figure shows a vortex line at two times t_1 & t_2, the Lagrangian trajectory of one of its vortex line elements, and the orientations of the orthonormal frame $\{\hat{\boldsymbol{\omega}},\, \hat{\boldsymbol{\chi}},\, (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})\}$ attached to it at the two times.

Alignment variables $\alpha(\boldsymbol{x},t)$, $\boldsymbol{\chi}(\boldsymbol{x},t)$ and $\alpha_p(\boldsymbol{x},t)$, $\boldsymbol{\chi}_p(\boldsymbol{x},t)$

$S\hat{\boldsymbol{\omega}}$ lies in the $(\hat{\boldsymbol{\omega}},\, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})$ plane and $P\hat{\boldsymbol{\omega}}$ in the $(\hat{\boldsymbol{\omega}},\, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}}_p)$ plane

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}\,, \qquad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}}$$

where $(\alpha,\, \boldsymbol{\chi})$ & $(\alpha_p,\, \boldsymbol{\chi}_p)$ define $S\hat{\boldsymbol{\omega}}$ & $P\hat{\boldsymbol{\omega}}$ as stretched & rotated $\hat{\boldsymbol{\omega}}$

$$\alpha = \hat{\boldsymbol{\omega}} \cdot S\hat{\boldsymbol{\omega}}\,, \quad \boldsymbol{\chi} = \hat{\boldsymbol{\omega}} \times S\hat{\boldsymbol{\omega}}\,,$$

$$\alpha_p = \hat{\boldsymbol{\omega}} \cdot P\hat{\boldsymbol{\omega}}\,, \quad \boldsymbol{\chi}_p = \hat{\boldsymbol{\omega}} \times P\hat{\boldsymbol{\omega}} =: -c_1\hat{\boldsymbol{\chi}} \times \hat{\boldsymbol{\omega}} - c_2\hat{\boldsymbol{\chi}}$$

Evolution of vorticity alignment parameters

We have

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \& \quad \frac{D^2\boldsymbol{\omega}}{Dt^2} = -\,P\boldsymbol{\omega}$$

where $S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}$ and $P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$

As we know, P-alignment drives S-alignment. That is,

$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} \quad \text{or} \quad \frac{D}{Dt}(\alpha\,\boldsymbol{\omega} + \boldsymbol{\chi}\times\boldsymbol{\omega}) = -\,(\alpha_p\,\boldsymbol{\omega} + \boldsymbol{\chi}_p\times\boldsymbol{\omega})$$

A direct calculation shows that P-parameters $[\alpha_p,\,\boldsymbol{\chi}_p]$ drive S-parameters $[\alpha,\,\boldsymbol{\chi}]$ in the following alignment-parameter dynamics

$$\boxed{\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 = -\,\alpha_p \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} = -\,\boldsymbol{\chi}_p}$$

We'll first derive and analyze evolution equations for comoving frame $\{\hat{\boldsymbol{\omega}},\,\hat{\boldsymbol{\chi}},\,\hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}}\}$, then we'll interpret the alignment-parameter dynamics.

Lagrangian frame dynamics

One computes $\dfrac{D\hat{\boldsymbol{\chi}}}{Dt} = -c_1\chi^{-1}(\hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}}) \quad \& \quad \dfrac{D(\hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}})}{Dt} = \chi\,\hat{\boldsymbol{\omega}} + c_1\chi^{-1}\hat{\boldsymbol{\chi}}\,.$

The various Lagrangian time derivatives may be assembled into

$$\begin{aligned}
\frac{D\hat{\boldsymbol{\omega}}}{Dt} &= \boldsymbol{\mathcal{D}}\times\hat{\boldsymbol{\omega}}\\
\frac{D(\hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}})}{Dt} &= \boldsymbol{\mathcal{D}}\times(\hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}})\\
\frac{D\hat{\boldsymbol{\chi}}}{Dt} &= \boldsymbol{\mathcal{D}}\times\hat{\boldsymbol{\chi}}
\end{aligned}$$

The "Darboux vector" $\boldsymbol{\mathcal{D}}$ is defined as

$$\boldsymbol{\mathcal{D}} = \boldsymbol{\chi} - \frac{c_1}{\chi}\hat{\boldsymbol{\omega}} \quad \text{with} \quad |\boldsymbol{\mathcal{D}}|^2 = \chi^2 + \frac{c_1^2}{\chi^2}$$

and one sees that $c_1 = \hat{\boldsymbol{\omega}}\cdot(\hat{\boldsymbol{\chi}}\times\boldsymbol{\chi}_p)$ depends on the pressure Hessian.

What can we deduce from vorticity frame dynamics? Where are we going next?

1. Note similarity of vorticity frame dynamics to Frenet-Serret equations for space curves in three dimensions.

2. Use Ertel's theorem to derive Lagrangian dynamics of the Frenet-Serret curvature and torsion

3. Represent vorticity alignments with S & P as quaternions. These yield the Cayley-Klein parameters of $S\hat{\boldsymbol{\omega}}$ & $P\hat{\boldsymbol{\omega}}$

4. Recover dynamics of S-alignment $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$ driven by P-alignment $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p]$ in quaternionic form

Frame dynamics for F

Use $\hat{\boldsymbol{\omega}}$ etc as row-vectors to define the 3×3 orthogonal frame-matrix

$$F(t,s) = \begin{pmatrix} \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\chi}} \\ \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}} \end{pmatrix}, \qquad F^T = F^{-1}$$

The matrix $F(t,s)$ specifies the evolution in time t of an orthonormal frame attached to any given Lagrangian label s along the vortex line.

The previous frame dynamics may now be re-written using F as,

$$\frac{DF}{Dt} = BF(t,s) \qquad \text{where} \qquad B = \begin{pmatrix} 0 & 0 & -\chi \\ 0 & 0 & -c_1/\chi \\ \chi & c_1/\chi & 0 \end{pmatrix}$$

with $B_{ij} = \epsilon_{ijk}\mathcal{D}_k$ for Darboux components $\mathcal{D}_k$ and $\boldsymbol{\chi}_p \cdot \hat{\boldsymbol{\chi}} \times \hat{\boldsymbol{\omega}} = -c_1$.

Frenet-Serret equations

The unit tangent $\hat{\boldsymbol{\omega}}$, normal $\hat{\boldsymbol{n}}$ & binormal $\hat{\boldsymbol{b}}$ of a vortex line define another 3×3 orthogonal frame-matrix $N(t, s)$ whose orientation varies with its arclength s according to the Frenet-Serret equations,

$$N = \begin{pmatrix} \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{b}} \end{pmatrix}, \quad \frac{\partial N}{\partial s} = AN(t, s) \quad \text{where} \quad A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

Here κ and τ are the curvature and torsion of the vortex line.

The matrix $N(t, s)$ is also orthogonal: $N^T = N^{-1}$.

The solution of Frenet-Serret for $N(t, s)$ determines an orthonormal frame at each point s along the vortex line at a given time t.

Frame dynamics for the Frenet-Serret matrix N

The frames N & F are related by a rotation $R(\phi)$ around the unit tangent vector $\hat{\boldsymbol{\omega}}$ by an angle $\phi(t, s)$

$$N = R(\phi)F$$

where

$$\frac{DF}{Dt} = BF(t, s)$$

Consequently,the Frenet-Serret matrix N satisfies

$$\frac{\partial N}{\partial s} = AN(t, s) \quad \text{and} \quad \frac{DN}{Dt} = BN(t, s) + \text{ linear correx}$$

where the arclength derivative along a vortex line is defined as,

$$\frac{\partial}{\partial s} = \boldsymbol{\omega} \cdot \nabla \,.$$

Evolving the curvature and torsion of a vortex line

Ertel's Theorem tells us that the derivatives in t and s commute

$$\left[\frac{D}{Dt}, \frac{\partial}{\partial s}\right] = 0\,.$$

This commutation relation implies equality of cross derivatives of N. That is, $N_{ts} = N_{st}$. Hence,

$$\frac{DA}{Dt} = \frac{\partial B}{\partial s} - [\,A\,,\;B\,]\,,$$

with $A = (\partial N/\partial s)N^{-1}$, $B = (DN/Dt)N^{-1}$ and $[\,A\,,\;B\,] = AB - BA$.

Hence,

$$\boxed{-\,\boldsymbol{\kappa}\,\frac{\boldsymbol{D\tau^{-1}}}{\boldsymbol{Dt}} = \boldsymbol{\chi} + \text{Correx linear in } \boldsymbol{\chi}}$$

Look at the case of straight vortices $\kappa = 0$.

Alignment: cause and effect!

- Thus, swing rate $\chi \neq 0$ implies time-dependence of vortex torsion τ. And $\kappa = 0$ implies $\chi = 0$, so **straight vortices don't swing!**

- We now seek alignment-parameter dynamics of growth rate (α) and swing rate $(\boldsymbol{\chi})$ for a **combined** scalar and vector quantity denoted $\boldsymbol{\zeta} = [\alpha, \, \boldsymbol{\chi}]$.

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}$$

We rewrite this as **quaternionic multiplication**:

$$[0, \, S\hat{\boldsymbol{\omega}}] = [\alpha, \, \boldsymbol{\chi}] \circledast [0, \, \hat{\boldsymbol{\omega}}]$$

which expresses parallel & perpendicular decomposition of $S\hat{\boldsymbol{\omega}}$.

What about using quaternions? (Hamilton 1843)

Quaternions combine scalar q & 3-vector $\boldsymbol{q}$ into a tetrad $\mathfrak{q} = [q,\, \boldsymbol{q}]$ as

$$\mathfrak{q} = [q,\, \boldsymbol{q}] = qI - \boldsymbol{q}\cdot\boldsymbol{\sigma}\,, \quad \text{with} \quad \boldsymbol{q}\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} q_i \sigma_i$$

The Pauli spin matrices $\boldsymbol{\sigma}$ obey the relations $\sigma_i\sigma_j = -\delta_{ij} - \epsilon_{ijk}\sigma_k$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

By this definition tetrads obey the multiplication rule denoted $\circledast$

$$\mathfrak{p} \circledast \mathfrak{q} = [pq - \boldsymbol{p}\cdot\boldsymbol{q}\,,\, p\boldsymbol{q} + q\boldsymbol{p} + \boldsymbol{p}\times\boldsymbol{q}]$$

Vorticity dynamics suggests alignment tetrads $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$, $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p]$

$$\text{which satisfy} \quad [0, S\hat{\boldsymbol{\omega}}] = \boldsymbol{\zeta} \circledast [0, \hat{\boldsymbol{\omega}}] \quad \text{and} \quad [0, P\hat{\boldsymbol{\omega}}] = \boldsymbol{\zeta}_p \circledast [0, \hat{\boldsymbol{\omega}}]$$

as parallel and perpendicular decompositions.

Are quaternions a good idea?

Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way. – Lord Kelvin (William Thompson)

O'Connor, J. J. & Robertson, E. F. 1998 *Sir William Rowan Hamilton*,

http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Hamilton.html

Hamilton was vindicated – quaternions are now used in the robotics and avionics industries to track objects undergoing a sequence of tumbling rotations and are also heavily used in graphics.

- *Quaternions & rotation Sequences: a Primer with Applications to Orbits, Aerospace & Virtual Reality*, J. B. Kuipers, Princeton University Press, 1999.
- *Visualizing quaternions,* by Andrew J. Hanson, MK-Elsevier, 2006.

Quaternions & Cayley-Klein parameters I

The **dot product** of two quaternions $\mathfrak{p} := [p\,,\,\boldsymbol{p}]$ and $\mathfrak{q} := [q\,,\,\boldsymbol{q}]$ is defined as

$$\mathfrak{p}\cdot\mathfrak{q} := pq + \boldsymbol{p}\cdot\boldsymbol{q}$$

The **magnitude** of quaternion $\mathfrak{q}$ is

$$|\mathfrak{q}| := (\mathfrak{q}\cdot\mathfrak{q})^{1/2} = (q^2 + \boldsymbol{q}\cdot\boldsymbol{q})^{1/2}$$

One defines the **conjugate** of $\mathfrak{q} := [q\,,\,\boldsymbol{q}]$ as $\mathfrak{q}^* = [q\,,\,-\boldsymbol{q}]$
So, product $\mathfrak{q}\circledast\mathfrak{q}^* = (\mathfrak{q}\cdot\mathfrak{q})\mathfrak{e}$, where $\mathfrak{e} = [1, 0]$ is the **identity**.

Hence

$$\mathfrak{q}^{-1} := \mathfrak{q}^*/(\mathfrak{q}\cdot\mathfrak{q}) \qquad \text{is the **inverse** of quaternion } \mathfrak{q}$$

under $\circledast$ product. (Recall that vectors don't have inverses.)

Quaternions & Cayley-Klein parameters II

Consider the map under the quaternionic product (which is associative)

$$\mathfrak{r} \to \mathfrak{r}' = \hat{\mathfrak{p}} \circledast \mathfrak{r} \circledast \hat{\mathfrak{p}}^*$$

where $\hat{\mathfrak{p}}$ is a **unit quaternion**, $\hat{\mathfrak{p}} \cdot \hat{\mathfrak{p}} = 1$, so $\hat{\mathfrak{p}} \circledast \hat{\mathfrak{p}}^* = \mathfrak{e} = [1, 0]$

The inverse map is

$$\mathfrak{r} = \hat{\mathfrak{p}}^* \circledast \mathfrak{r}' \circledast \hat{\mathfrak{p}}$$

If $\mathfrak{r} = [0, \boldsymbol{r}]$ then $\mathfrak{r}' = [0, \boldsymbol{r}'] = [0, \boldsymbol{r} + 2p(\boldsymbol{p} \times \boldsymbol{r}) + 2\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{r})]$

For $\hat{\mathfrak{p}} := \pm[\cos\frac{\theta}{2}\,,\, \sin\frac{\theta}{2}\,\hat{\boldsymbol{n}}]$, this is a **rotation** of $\boldsymbol{r}$ by angle θ about $\hat{\boldsymbol{n}}$.

In $\hat{\mathfrak{p}} = [p\,,\, \boldsymbol{p}]$, p & $\boldsymbol{p}$ are the **Cayley-Klein parameters** of the rotation.

$\therefore$ Composition of rotations $\simeq$ Multiplication of $(\pm)$ unit quaternions

Alignment $S\hat{\omega}$ vs $\hat{\omega}$ & Cayley-Klein parameters

Consider the unit quaternion relation with $\hat{\mathfrak{p}} := \pm[\cos\frac{\theta}{2}\,,\,\sin\frac{\theta}{2}\,\hat{\boldsymbol{\chi}}]$,

$$
\begin{aligned}
|S\hat{\boldsymbol{\omega}}|^{-1}[0, S\hat{\boldsymbol{\omega}}] &= \hat{\mathfrak{p}} \circledast [0, \hat{\boldsymbol{\omega}}] \circledast \hat{\mathfrak{p}}^* = [0, \cos\theta\,\hat{\boldsymbol{\omega}} + \sin\theta\,\hat{\boldsymbol{\chi}} \times \hat{\boldsymbol{\omega}}] \\
&= |S\hat{\boldsymbol{\omega}}|^{-1}\boldsymbol{\zeta} \circledast [0, \hat{\boldsymbol{\omega}}] = (\alpha^2 + \chi^2)^{-1/2}\,[0, \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}]
\end{aligned}
$$

where $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$. Thus, the unit vector $|S\hat{\boldsymbol{\omega}}|^{-1}S\hat{\boldsymbol{\omega}}$ is a rotation of $\hat{\boldsymbol{\omega}}$ by angle θ around $\hat{\boldsymbol{\chi}}$ with

$$
\cos\theta = \frac{\alpha}{(\alpha^2 + \chi^2)^{1/2}} \quad \text{and} \quad \sin\theta = \frac{\chi}{(\alpha^2 + \chi^2)^{1/2}}
$$

Hence,

Alignment parameters α and $\boldsymbol{\chi}$ define $S\hat{\boldsymbol{\omega}}$ as a stretching of $\hat{\boldsymbol{\omega}}$ by $(\alpha^2 + \chi^2)^{1/2}$ & rotation of $\hat{\boldsymbol{\omega}}$ by $\theta = \tan^{-1}\chi/\alpha$ about $\hat{\boldsymbol{\chi}}$.

Likewise for $P\hat{\boldsymbol{\omega}}$ and its alignment parameters α_p and $\boldsymbol{\chi}_p$ relative to $\hat{\boldsymbol{\omega}}$.

The angle θ is the **misalignment** between $S\hat{\boldsymbol{\omega}}$ & $\hat{\boldsymbol{\omega}}$.

The Euler equations in quaternionic form

Define velocity & pressure tetrads $\mathcal{U}$ & $\boldsymbol{\Pi}$ and the 4-derivative $\boldsymbol{\nabla}$ as

$$\mathcal{U} = [0, \boldsymbol{u}] \qquad \boldsymbol{\Pi} = [p, 0] \qquad \boldsymbol{\nabla} = [0, \nabla]$$

Then Euler's fluid equation is written in quaternionic form as

$$\frac{D\mathcal{U}}{Dt} = -\boldsymbol{\nabla} \circledast \boldsymbol{\Pi}$$

The vorticity tetrad $\boldsymbol{\Omega}$ is formed from

$$\boldsymbol{\nabla} \circledast \mathcal{U} = [-\mathrm{div}\,\boldsymbol{u}\,,\ \mathrm{curl}\,\boldsymbol{u}] = [0, \boldsymbol{\omega}] =: \boldsymbol{\Omega}$$

Operating with $\boldsymbol{\nabla}\circledast$ on Euler's equation above produces

$$[\Delta p,\, 0] = \left[-\left|\nabla \boldsymbol{u}\right|^2,\, \frac{D\boldsymbol{\omega}}{Dt} - S\boldsymbol{\omega} \right]$$

Identifying terms yields $\Delta p = -\left|\nabla \boldsymbol{u}\right|^2$ and Euler's vorticity equation.

Theorem: *The vorticity tetrad* $\boldsymbol{\Omega}(\boldsymbol{x}, t) = [0, \boldsymbol{\omega}]$ *satisfies*

$$\frac{D\boldsymbol{\Omega}}{Dt} = \boldsymbol{\zeta} \circledast \boldsymbol{\Omega} \qquad \text{(Frozen-in tetrad field)}$$

$$\frac{D^2\boldsymbol{\Omega}}{Dt^2} + \boldsymbol{\zeta}_p \circledast \boldsymbol{\Omega} = 0 \qquad \text{(Ohkitani's relation)}$$

where $\boldsymbol{\zeta} = [\alpha, \boldsymbol{\chi}]$ and $\boldsymbol{\zeta}_p = [\alpha_p, \boldsymbol{\chi}_p]$.

Consequently, the growth & swing rate tetrad $\boldsymbol{\zeta}(\boldsymbol{x}, t) = [\alpha, \boldsymbol{\chi}]$ *satisfies*

$$\boxed{\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0}$$

Remark: The $\boldsymbol{\zeta}$-equation is a Riccati equation driven by $\boldsymbol{\zeta}_p$ which, in turn, depends on the other variables through the pressure Hessian P.

The growth/swing rate tetrad $\boldsymbol{\zeta}(\boldsymbol{x}, t) = [\alpha, \boldsymbol{\chi}]$ evolves by quadratic nonlinearity and is driven by the $P-$alignment tetrad $\boldsymbol{\zeta}_p = [\alpha_p, \boldsymbol{\chi}_p]$.

Proof:

$$\frac{D\boldsymbol{\Omega}}{Dt} = [0,\ \underbrace{\alpha\,\boldsymbol{\omega} + \boldsymbol{\chi} \times \boldsymbol{\omega}}_{S\boldsymbol{\omega}}\,] = [\alpha\,,\boldsymbol{\chi}] \circledast [0,\ \boldsymbol{\omega}] = \boldsymbol{\zeta} \circledast \boldsymbol{\Omega}\,.$$

$$P\boldsymbol{\omega} = \alpha_p\,\boldsymbol{\omega} + \boldsymbol{\chi}_p \times \boldsymbol{\omega} \qquad \Rightarrow \qquad [0,\ P\boldsymbol{\omega}] = \boldsymbol{\zeta}_p \circledast \boldsymbol{\Omega}$$

Use Ertel's Theorem to express Ohkitani's relation as

$$\frac{D^2\boldsymbol{\Omega}}{Dt^2} = \frac{D}{Dt}[0,\ S\boldsymbol{\omega}] = -[0,\ P\boldsymbol{\omega}] = -\,\boldsymbol{\zeta}_p \circledast \boldsymbol{\Omega}$$

Compare this relation with $D^2\boldsymbol{\Omega}/Dt^2 = D/Dt(\boldsymbol{\zeta} \circledast \boldsymbol{\Omega})$ to find

$$0 = \frac{D\boldsymbol{\zeta}}{Dt} \circledast \boldsymbol{\Omega} + \boldsymbol{\zeta} \circledast (\boldsymbol{\zeta} \circledast \boldsymbol{\Omega}) + \boldsymbol{\zeta}_p \circledast \boldsymbol{\Omega}$$

The equation for ζ follows, because $\circledast$ is associative. ■

Quaternion alignment dynamics in components

The alignment equation for tetrads $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$ with $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p\,]$ is

$$\boxed{\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0}$$

Recall the components of the tetrad multiplication rule

$$\mathfrak{p} \circledast \mathfrak{q} = [pq - \boldsymbol{p} \cdot \boldsymbol{q}\,,\, p\boldsymbol{q} + q\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}]$$

So $\boldsymbol{\zeta} \circledast \boldsymbol{\zeta} = [\alpha^2 - \chi^2, 2\alpha\boldsymbol{\chi}]$ in components & the alignment variables α, $\boldsymbol{\chi}$ are driven by α_p, $\boldsymbol{\chi}_p$ according to

$$\boxed{\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 + \alpha_p = 0 \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} + \boldsymbol{\chi}_p = 0}$$

where $\quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}} \quad$ and $\quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}}$

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \& \quad \boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}\,, \qquad \frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} \quad \& \quad \mathrm{tr}\,P = -\,|\nabla \boldsymbol{u}|^2$$

Alignment dynamics in polar coordinates

In polar coordinates given by the stretching rate along $\hat{\boldsymbol{\omega}}$ as the radius $r = (\alpha^2+\chi^2)^{1/2} = |S\hat{\boldsymbol{\omega}}|$ and the angle $\theta = \tan^{-1}\chi/\alpha$ of rotation about the comoving $\hat{\boldsymbol{\chi}}$ axis from $\hat{\boldsymbol{\omega}}$ to $S\hat{\boldsymbol{\omega}}$, the alignment dynamics derived from

$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega}$$

becomes, upon using

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \chi\,\hat{\boldsymbol{\chi}}\times\hat{\boldsymbol{\omega}} = r(\cos\theta\,\hat{\boldsymbol{\omega}} + \sin\theta\,\hat{\boldsymbol{\chi}}\times\hat{\boldsymbol{\omega}})\,,$$

the 2×2 system in polar coordinates,

$$\frac{D}{Dt}\frac{\sin\theta}{r} + \cos 2\theta \;=\; \frac{\alpha_p}{r^2}$$

$$\frac{D}{Dt}\frac{\cos\theta}{r} - \sin 2\theta \;=\; \frac{\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\chi}_p}{r^2}$$

where one recalls that $\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\chi}_p = -c_2$ and $\theta = 0$ **is perfect alignment**.

A simple solution: the Burgers vortex

The most elementary Burgers vortex solution is (with $\gamma_0 = const$)

$$\boldsymbol{u} = (-\tfrac{1}{2}\gamma_0 x + \psi_y,\ -\tfrac{1}{2}\gamma_0 y - \psi_x,\ z\gamma_0) \qquad \Rightarrow \qquad \boldsymbol{\omega} = (0,\ 0,\ \omega_3)$$

$$\omega_3(r,t) = e^{\gamma_0 t}\omega_0\left(r\, e^{\frac{1}{2}\gamma_0 t}\right)$$ (note exponential growth)

Thus, for the Burgers vortex one computes

$$\alpha = \gamma_0\,, \qquad \boldsymbol{\chi} = 0\,, \qquad \alpha_p = -\,\gamma_0^2$$

$$\boldsymbol{\zeta} = [\gamma_0,\ 0] \qquad\qquad \boldsymbol{\zeta}_p = -[\gamma_0^2,\ 0]$$

Conclusions: Burgers tubes/sheets are scalar objects: they don't swing.

(In fact, they are steady solutions of the ζ-equation.)

When tubes & sheets bend then $\boldsymbol{\chi} \neq 0$ and $\boldsymbol{\zeta}$ becomes a full tetrad driven by $\boldsymbol{\zeta}_p$ which is coupled back through the pressure Hessian P.

When do $[\alpha,\,\chi]$ tetrad equations arise in fluids?

- First, we need a Frozen-in Vector Field, $\boldsymbol{\varpi}\cdot\nabla$, so that

$$\frac{D\boldsymbol{u}}{Dt}=\boldsymbol{\mathcal{F}}\quad\text{implies}\quad\frac{D\boldsymbol{\varpi}}{Dt}=\boldsymbol{\varpi}\cdot\nabla\boldsymbol{u}\quad\text{for}\quad\boldsymbol{\varpi}=Q_{op}\boldsymbol{u}$$

- This will produce an Ertel Theorem and Ohkitani relation

$$\Big[\frac{D}{Dt}\,,\,\boldsymbol{\varpi}\cdot\nabla\Big]=0\,,\quad\text{so}\quad\frac{D^2\boldsymbol{\varpi}}{Dt^2}=\frac{D}{Dt}(\boldsymbol{\varpi}\cdot\nabla\boldsymbol{u})=\boldsymbol{\varpi}\cdot\nabla\boldsymbol{\mathcal{F}}$$

- In turn these will produce orthonormal Frame Dynamics for $\widehat{\boldsymbol{\varpi}}$, whose alignment parameters will satisfy Quaternion equations.

- Other examples:
  (1) Lagrangian Averaged Euler-alpha (LAE$-\alpha$) equations
  (2) MP97mod2' as Euler-Poincaré equations
  (3) Ideal MHD and LAMHD-alpha.

Lagrangian Averaged Euler-alpha (LAE$-\alpha$) model

Lagrangian averaging preserves Kelvin's circulation theorem, which leads to a frozen-in vector field and thereby produces Ertel's theorem.

The LAE$-\alpha$ motion equation is

$$\frac{D\boldsymbol{w}}{Dt} + \nabla \boldsymbol{u}^T \cdot \boldsymbol{w} = -\nabla p \quad \text{for} \quad \boldsymbol{w} = \boldsymbol{u} - \alpha^2 \Delta \boldsymbol{u} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0$$

or, in Kelvin circulation form,

$$\frac{D}{Dt}(\boldsymbol{w} \cdot d\boldsymbol{x}) = -dp \quad \text{along} \quad \frac{D\boldsymbol{x}}{Dt} = \boldsymbol{u}$$

Stokes-ing (or taking d) and $\nabla \cdot \boldsymbol{u} = 0$ yield a Frozen-in Vector Field

$$\frac{D\boldsymbol{\varpi}}{Dt} = \boldsymbol{\varpi} \cdot \nabla \boldsymbol{u} \quad \text{for} \quad \boldsymbol{\varpi} = \nabla \times \boldsymbol{w}$$

Ertel Theorem & Ohkitani relation for LAE$-\alpha$

The LAE$-\alpha$ motion equation may also be written using $\boldsymbol{u} = G * \boldsymbol{w}$ as

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\mathcal{F}} = -G * (\nabla p + 4\alpha^2 \nabla \cdot \Omega S)$$

where $2\Omega = \nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T$ and $G* = (1 - \alpha^2 \Delta)^{-1}$ denotes convolution with the Greens function for the Helmholtz operator.

The Ertel Theorem and Ohkitani relation for LAE$-\alpha$ are then

$$\left[\frac{D}{Dt}, \boldsymbol{\varpi} \cdot \nabla\right] = 0 \,, \quad \text{and} \quad \frac{D}{Dt}(\boldsymbol{\varpi} \cdot \nabla \boldsymbol{u}) = \frac{D^2 \boldsymbol{\varpi}}{Dt^2} = \boldsymbol{\varpi} \cdot \nabla \boldsymbol{\mathcal{F}}$$

where $\boldsymbol{\varpi} = \nabla \times \boldsymbol{w}$ and $\boldsymbol{w} = (1 - \alpha^2 \Delta) \boldsymbol{u}$

The rest (Dynamics of Vorticity Frames and Quaternionic Alignment Parameters) follows the pattern of Euler fluids.

Ertel Theorem & Ohkitani relation for MP97mod2'

The MP97mod2' motion equation may be written as

$$\frac{\tilde{D}\tilde{\boldsymbol{U}}}{\tilde{D}t} = -\nabla(p+2q) + \nabla\cdot 2q\,\hat{\boldsymbol{\omega}}\otimes\hat{\boldsymbol{\omega}} =: \tilde{\boldsymbol{\mathcal{F}}}\,,$$

where $\nabla\cdot\tilde{\boldsymbol{U}}=0$ and $\boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{U}}\cdot\boldsymbol{l}=0$ determine p & q, and

$$\frac{\partial\boldsymbol{l}}{\partial t} = \mathrm{curl}\,(\tilde{\boldsymbol{U}}\times\boldsymbol{l})\,,\quad \hat{\boldsymbol{\omega}} = \boldsymbol{l}/|\boldsymbol{l}|\,,\quad |\boldsymbol{l}|^2=1 \quad \text{with} \quad \nabla\cdot\boldsymbol{l}=0$$

The Ertel Theorem and Ohkitani relation for MP97mod2' are then

$$\left[\frac{D}{Dt}\,,\,\boldsymbol{l}\cdot\nabla\right]=0\,,\ \text{so}\ \frac{D\boldsymbol{l}}{Dt}=\boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{U}}\ \text{and}\ \frac{D}{Dt}(\boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{U}}) = \frac{D^2\boldsymbol{l}}{Dt^2} = \boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{\mathcal{F}}}$$

Together, Ertel and Ohkitani conveniently deliver

$$\frac{D}{Dt}(\boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{U}}\cdot\boldsymbol{l}) = \boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{\mathcal{F}}}\cdot\boldsymbol{l} + |\boldsymbol{l}\cdot\nabla\tilde{\boldsymbol{U}}|^2$$

The equation system for Lagrange multipliers p & q

Preservation of $\nabla \cdot \tilde{\boldsymbol{U}} = 0$ and $\boldsymbol{l} \cdot \nabla \tilde{\boldsymbol{U}} \cdot \boldsymbol{l} = 0$ determines Lagrange multipliers p & q from the system

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\nabla \cdot \tilde{\boldsymbol{U}}) = -\,|\nabla \tilde{\boldsymbol{U}}|^2 + \nabla \cdot \tilde{\boldsymbol{\mathcal{F}}} \\ 0 &= \frac{\partial}{\partial t}(\boldsymbol{l} \cdot \nabla \tilde{\boldsymbol{U}} \cdot \boldsymbol{l}) = -\tilde{\boldsymbol{U}} \cdot \nabla(\boldsymbol{l} \cdot \nabla \tilde{\boldsymbol{U}} \cdot \boldsymbol{l}) + |\boldsymbol{l} \cdot \nabla \tilde{\boldsymbol{U}}|^2 + \boldsymbol{l} \cdot \nabla \tilde{\boldsymbol{\mathcal{F}}} \cdot \boldsymbol{l} \end{aligned}$$

where the MP97mod2' force $\tilde{\boldsymbol{\mathcal{F}}}$ depends linearly on p & q as

$$\frac{\tilde{D}\tilde{\boldsymbol{U}}}{\tilde{D}t} = -\nabla(p + 2q) + \nabla \cdot (2q\, \boldsymbol{l} \otimes \boldsymbol{l}) =: \tilde{\boldsymbol{\mathcal{F}}}$$

The rest (Dynamics of $\hat{\boldsymbol{\omega}}$ Frames and Quaternionic Alignment Parameters) follows for MP97mod2' as for Euler fluids, provided the p, q system may be solved at each time step.

Ertel Theorem & Ohkitani relation for Ideal MHD

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla p\,, \qquad \frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}\,,$$

and div $\boldsymbol{u} = 0 =$ div $\boldsymbol{B}$. Notice that $\boldsymbol{B}$ is a Frozen-in Vector Field. The Elsasser variables & $(\pm)$ material derivatives are defined as

$$\boldsymbol{w}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}\,; \qquad \frac{D^{\pm}}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{w}^{\pm}\cdot\nabla$$

The magnetic field $\boldsymbol{B}$ and $\boldsymbol{w}^{\pm}$ with div $\boldsymbol{w}^{\pm} = 0$ satisfy (note $\pm$ vs $\mp$)

$$\frac{D^{\pm}\boldsymbol{w}^{\mp}}{Dt} = -\nabla p \quad \text{and} \quad \frac{D^{\pm}\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{w}^{\pm}$$

Ertel's Theorem and Ohkitani's relation for ideal MHD are then

$$\left[\frac{D^{\pm}}{Dt}\,,\,\boldsymbol{B}\cdot\nabla\right] = 0\,, \quad \text{and} \quad \frac{D^{\pm}}{Dt}(\boldsymbol{B}\cdot\nabla\boldsymbol{w}^{\mp}) = \frac{D^{\pm}}{Dt}\frac{D^{\mp}}{Dt}\boldsymbol{B} = -P\boldsymbol{B}$$

Definition of $\alpha^{\pm}$ and $\chi^{\pm}$ in Elsasser variables

The stretching rates $\alpha^{\pm}$ & swing rates $\boldsymbol{\chi}^{\pm}$ for evolving magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{B}}$ along the $\pm$ characteristics are given by

$$\frac{D^{\pm}B}{Dt} = \alpha^{\pm}B\,, \qquad \frac{D^{\pm}\hat{\boldsymbol{B}}}{Dt} = \boldsymbol{\chi}^{\pm}\times\hat{\boldsymbol{B}}$$

where

$$\alpha^{\pm} = \hat{\boldsymbol{B}}\cdot(\hat{\boldsymbol{B}}\cdot\nabla\boldsymbol{w}^{\pm}) \qquad \boldsymbol{\chi}^{\pm} = \hat{\boldsymbol{B}}\times(\hat{\boldsymbol{B}}\cdot\nabla\boldsymbol{w}^{\pm})$$

As Moffatt (1985) suggested, $\boldsymbol{B}$ in ideal MHD is analogous to vorticity $\boldsymbol{\omega}$ in Euler fluids – except MHD has two $\pm$ characteristic velocities!

Per Moffatt's suggestion, we introduce the MHD analogs of α_p & $\boldsymbol{\chi}_p$,

$$\alpha_{pb} = \hat{\boldsymbol{B}}\cdot P\hat{\boldsymbol{B}} \qquad \boldsymbol{\chi}_{pb} = \hat{\boldsymbol{B}}\times P\hat{\boldsymbol{B}}$$

Lagrangian frame dynamics for ideal MHD

The 2 sets of orthonormal vectors $\{\hat{\boldsymbol{B}},\ (\hat{\boldsymbol{B}}\times\hat{\boldsymbol{\chi}}^{\pm}),\ \hat{\boldsymbol{\chi}}^{\pm}\}$ acted on by their opposite Lagrangian time derivatives are found to obey

$$
\begin{aligned}
\frac{D^{\mp}\hat{\boldsymbol{B}}}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp}\times\hat{\boldsymbol{B}}\,,\\
\frac{D^{\mp}(\hat{\boldsymbol{B}}\times\hat{\boldsymbol{\chi}}^{\pm})}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp}\times(\hat{\boldsymbol{B}}\times\hat{\boldsymbol{\chi}}^{\pm})\,,\\
\frac{D^{\mp}\hat{\boldsymbol{\chi}}^{\pm}}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp}\times\hat{\boldsymbol{\chi}}^{\pm}
\end{aligned}
$$

where the $(\mp)$ pair of Darboux vectors $\boldsymbol{\mathcal{D}}^{\mp}$ are defined as

$$
\boldsymbol{\mathcal{D}}^{\mp}=\boldsymbol{\chi}^{\mp}-\frac{c_1^{\mp}}{\chi^{\mp}}\hat{\boldsymbol{B}}\,,\qquad c_1^{\mp}=\hat{\boldsymbol{B}}\cdot\left[\hat{\boldsymbol{\chi}}^{\pm}\times(\boldsymbol{\chi}_{pb}+\alpha^{\pm}\boldsymbol{\chi}^{\mp})\right].
$$

The corresponding Frenet-Serret frames and their Lagrangian parameter evolution may again be found, by using Ertel's theorem for ideal MHD.

Quaternionic alignment dynamics for ideal MHD

Tetrads $\boldsymbol{\Omega_B} = [0,\ \boldsymbol{B}]$, $\boldsymbol{\zeta}^{\pm} = [\alpha^{\pm},\ \boldsymbol{\chi}^{\pm}]$ and $\boldsymbol{\zeta}_{pb} = [\alpha_{pb},\ \boldsymbol{\chi}_{pb}]$ are used to express the following

Theorem: *The magnetic field tetrad $\boldsymbol{\Omega_B}$ satisfies the two relations*

$$\frac{D^{\pm}\boldsymbol{\Omega_B}}{Dt} = \boldsymbol{\zeta}^{\pm} \circledast \boldsymbol{\Omega_B}\,, \qquad \text{(Frozen-in tetrads)}$$

$$\frac{D^{\mp}}{Dt}\left(\frac{D^{\pm}\boldsymbol{\Omega_B}}{Dt}\right) + \boldsymbol{\zeta}_{pb} \circledast \boldsymbol{\Omega_B} = 0\,, \qquad \text{(Ohkitani relations)}$$

Consequently, the tetrads $\boldsymbol{\zeta}^{\pm}$ satisfy the coupled Riccati equations

$$\boxed{\frac{D^{\mp}\boldsymbol{\zeta}^{\pm}}{Dt} + \boldsymbol{\zeta}^{\pm} \circledast \boldsymbol{\zeta}^{\mp} + \boldsymbol{\zeta}_{pb} = 0}$$

Quaternionic MHD alignment eqns in components

Alignment dynamics of tetrads $\boldsymbol{\zeta}^{\pm} = [\alpha^{\pm},\, \boldsymbol{\chi}^{\pm}]$ with $\boldsymbol{\zeta}_{pb} = [\alpha_{pb},\, \boldsymbol{\chi}_{pb}]$ is

$$\boxed{\frac{D^{\mp}\boldsymbol{\zeta}^{\pm}}{Dt} + \boldsymbol{\zeta}^{\pm} \circledast \boldsymbol{\zeta}^{\mp} + \boldsymbol{\zeta}_{pb} = 0}$$

Recall the components of the tetrad multiplication rule

$$\mathfrak{p} \circledast \mathfrak{q} = [pq - \boldsymbol{p}\cdot\boldsymbol{q}\,,\, p\boldsymbol{q} + q\boldsymbol{p} + \boldsymbol{p}\times\boldsymbol{q}]$$

So $\boldsymbol{\zeta}^{\pm} \circledast \boldsymbol{\zeta}^{\mp} = [\alpha^{\pm}\alpha^{\mp} - \boldsymbol{\chi}^{\pm}\cdot\boldsymbol{\chi}^{\mp}, \alpha^{\pm}\boldsymbol{\chi}^{\mp} + \alpha^{\mp}\boldsymbol{\chi}^{\pm} + \boldsymbol{\chi}^{\pm}\times\boldsymbol{\chi}^{\mp}]$ in components & the alignment variables $[\alpha^{\pm},\, \boldsymbol{\chi}^{\pm}]$ with $\boldsymbol{\zeta}_{pb} = [\alpha_{pb},\, \boldsymbol{\chi}_{pb}]$ evolve by

$$\frac{D^{\mp}\alpha^{\pm}}{Dt} + \alpha^{\pm}\alpha^{\mp} - \boldsymbol{\chi}^{\pm}\cdot\boldsymbol{\chi}^{\mp} = -\,\alpha_{pb}$$

$$\text{and}\quad \frac{D^{\mp}\boldsymbol{\chi}^{\pm}}{Dt} + \alpha^{\pm}\boldsymbol{\chi}^{\mp} + \alpha^{\mp}\boldsymbol{\chi}^{\pm} + \boldsymbol{\chi}^{\pm}\times\boldsymbol{\chi}^{\mp} = -\,\boldsymbol{\chi}_{pb}$$

This is quaternionic alignment dynamics for ideal MHD.

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