Vorticity alignment dynamics in fluids & MHD

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Collaborators and main reference for this talk

JD Gibbon, DDH, RM Kerr & I Roulstone, 2005:
To appear in Nonlinearity
Notation: The 3D incompressible Euler fluid

The equations for Eulerian fluid velocity $u$ in 3D are

$$\frac{Du}{Dt} = -\nabla p, \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla \quad \text{and} \quad \text{div} \, u = 0$$

Taking the curl yields the vorticity equation $(\omega = \text{curl} \, u)$

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u = S\omega$$

The vortex stretching vector is $S\omega$ with $S = \frac{1}{2}(\nabla u + \nabla u^T)$, or

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and preservation of $\text{div} \, u = 0$ determines the pressure $p$ as

$$-\Delta p = u_{i,j}u_{j,i} = |\nabla u|^2 = \text{Tr} \, S^2 - \frac{1}{2} \omega^2.$$
Outline for the talk

1. Define **Ertel’s theorem, Ohkitani’s relation, vorticity frame dynamics and alignment dynamics** for Euler’s equations.

2. Use Ertel’s theorem to derive **Lagrangian dynamics** of the Frenet-Serret curvature and torsion of vortex lines.

3. Represent Euler vorticity alignments with $S$ & $P$ as **quaternions**. These yield the **Cayley-Klein parameters** of
   \[ S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega} \quad \text{and} \quad P\hat{\omega} = \alpha_p \hat{\omega} + \chi_p \times \hat{\omega} \]

4. Derive **dynamics of quaternions** for $S$-alignment $\zeta = [\alpha, \chi]$ driven by $P$-alignment $\zeta_p = [\alpha_p, \chi_p]$

5. Apply this structure to LES models (LAE−$\alpha$, MP97mod2’) and MHD
Define vorticity growth rate \((\alpha)\) and swing rate \((\chi)\)

The material rates of change of \(|\omega|\) and \(\hat{\omega}\) are given by

\[
\frac{D\omega}{Dt} = S\omega \quad \text{with} \quad S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega}
\]

- The scalar \(\alpha = \hat{\omega} \cdot S\hat{\omega}\) is the vorticity growth rate
  \[
  \frac{D|\omega|}{Dt} = \alpha |\omega| \quad \alpha > 0 \quad \text{stretching} \quad \alpha < 0 \quad \text{shrinking}
  \]

- The 3-vector \(\chi = \hat{\omega} \times S\hat{\omega}\) is the vorticity swing rate
  \[
  \frac{D\hat{\omega}}{Dt} = \chi \times \hat{\omega}, \quad \hat{\omega} \times \frac{D\hat{\omega}}{Dt} = \chi \quad \text{(frequency)}
  \]

**Remark:** If \(\omega\) aligns with an eigenvector \(S\hat{\omega} = \lambda \hat{\omega}\), then \(\chi = 0\).

For such alignment, the vorticity direction is **frozen** into the flow.
3D vortex stretching and alignment

The curl of Euler’s equation yields vorticity dynamics

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u = S\omega$$

whose the strain-rate matrix $S$ has components $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

For $S$-alignment: $S\omega = \lambda \omega$, the vorticity stretches (shrinks) depending on whether the corresponding eigenvalue $\lambda$ is positive (negative).

- How long will the vorticity grow, before getting misaligned?
  - This depends on the Lagrangian rates of change of $\alpha = \hat{\omega} \cdot S\hat{\omega}$ and of the vorticity swing rate $\chi = \hat{\omega} \times S\hat{\omega}$. For this we need $\frac{D^2\omega}{Dt^2}$!

- Seek alignment-parameter dynamics ($\frac{D\alpha}{Dt}$ and $\frac{D\chi}{Dt}$)

- Is vorticity alignment dynamics cause, effect, or both?

- A key to the answer will be Ertel’s Theorem (1942)
Ertel’s Theorem (1942)

**Theorem:** (Ertel 1942) If $\omega$ satisfies the 3D incompressible Euler equations then an arbitrary differentiable function $\mu$ satisfies

$$\frac{D}{Dt}(\omega \cdot \nabla \mu) = \omega \cdot \nabla \left( \frac{D\mu}{Dt} \right).$$

**Proof:** In characteristic (Lie-derivative) form, the vorticity equation is,

$$\frac{D}{Dt} \left( \omega \cdot \frac{\partial}{\partial x} \right) = \left( \frac{D\omega}{Dt} - \omega \cdot \nabla u \right) \cdot \frac{\partial}{\partial x} = 0 \quad \text{along} \quad \frac{dx}{dt} = u(x, t)$$

So $\omega \cdot \frac{\partial}{\partial x}(t) = \omega \cdot \frac{\partial}{\partial x}(0)$ (Cauchy 1859) and the derivatives commute

$$\left[ \frac{D}{Dt}, \omega \cdot \nabla \right] = 0$$

Hence, Ertel’s theorem follows.

**Corollary:** $D\mu/Dt = 0$ implies $D(\omega \cdot \nabla \mu)/Dt = 0$ (e.g. PV in GFD).
Some Ertel references


Define Ohkitani’s relation & the pressure Hessian

Ohkitani took $\mu = u$ in Ertel’s theorem (Phys. Fluids, A5, 2570, 1993).

**Result:** The vortex stretching vector $\omega \cdot \nabla u = S\omega$ obeys

$$\frac{D^2 \omega}{Dt^2} = \frac{D(\omega \cdot \nabla u)}{Dt} = \omega \cdot \nabla \left( \frac{Du}{Dt} \right) = -P \omega$$

where $P$ the Hessian matrix of the pressure

$$P = \{p_{ij}\} = \left\{ \frac{\partial^2 p}{\partial x_i \partial x_j} \right\}$$

Thus,

$$\frac{D^2 \omega}{Dt^2} = \frac{DS\omega}{Dt} = -P \omega \quad \text{(Ohkitani’s relation)}$$

So, $P$-alignments drive dynamics of $S$-alignments!
Vorticity accelerations $- \alpha_p \& - \chi_p$ of $|\omega| \& \hat{\omega}$

The material accelerations of $|\omega|$ and $\hat{\omega}$ are given by Ohkitani as

$$\frac{D^2 \omega}{Dt^2} = - P \omega \quad \text{with} \quad P \hat{\omega} = \alpha_p \hat{\omega} + \chi_p \times \hat{\omega}$$

- **Scalar** $\alpha_p = \hat{\omega} \cdot P \hat{\omega}$ gives acceleration of vorticity magnitude

  $$\frac{D^2 |\omega|}{Dt^2} = - \alpha_p |\omega| \quad \alpha_p > 0 \quad \text{decelerating} \quad \alpha_p < 0 \quad \text{accelerating}$$

- **3-vector** $\chi_p = \hat{\omega} \times P \hat{\omega}$ gives acceleration of vorticity direction

  $$\frac{D^2 \hat{\omega}}{Dt^2} = - \chi_p \times \hat{\omega}$$

**Remark:** If $\omega$ aligns with an eigenvector $P \hat{\omega} = \lambda \hat{\omega}$, then $\chi_p = 0$. For such alignment, $P \perp S \hat{\omega} = \chi \times \hat{\omega}$ is frozen into the flow.
Vorticity and alignment dynamics

• Vorticity is driven by $S$

$$\frac{D\omega}{Dt} = S\omega$$

• Alignment is driven by $P$

$$\frac{DS\omega}{Dt} = -P\omega$$

with

$$u = \text{curl}^{-1}\omega, \quad \text{tr} \ P = -|\nabla u|^2$$

• The latter involves the pressure Hessian $P$.
• This pressure dependence produces nonlocal effects.
Lagrangian frame dynamics: tracking the orientation of vorticity following a fluid particle

The figure shows a vortex line at two times $t_1$ & $t_2$, the Lagrangian trajectory of one of its vortex line elements, and the orientations of the orthonormal frame $\{\hat{\omega}, \hat{\chi}, (\hat{\omega} \times \hat{\chi})\}$ attached to it at the two times.
Alignment variables $\alpha(x, t)$, $\chi(x, t)$ and $\alpha_p(x, t)$, $\chi_p(x, t)$

$S\hat{\omega}$ lies in the $(\hat{\omega}, \hat{\omega} \times \hat{\chi})$ plane and $P\hat{\omega}$ in the $(\hat{\omega}, \hat{\omega} \times \hat{\chi}_p)$ plane

$$S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega}, \quad P\hat{\omega} = \alpha_p \hat{\omega} + \chi_p \times \hat{\omega}$$

where $(\alpha, \chi)$ & $(\alpha_p, \chi_p)$ define $S\hat{\omega}$ & $P\hat{\omega}$ as stretched & rotated $\hat{\omega}$

$$\alpha = \hat{\omega} \cdot S\hat{\omega}, \quad \chi = \hat{\omega} \times S\hat{\omega},$$

$$\alpha_p = \hat{\omega} \cdot P\hat{\omega}, \quad \chi_p = \hat{\omega} \times P\hat{\omega} =: -c_1 \hat{\chi} \times \hat{\omega} - c_2 \hat{\chi}$$
Evolution of vorticity alignment parameters

We have
\[
\frac{D\omega}{Dt} = S\omega \quad \& \quad \frac{D^2\omega}{Dt^2} = -P\omega
\]
where \( S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega} \) and \( P\hat{\omega} = \alpha_p \hat{\omega} + \chi_p \times \hat{\omega} \)

As we know, \( P \)-alignment drives \( S \)-alignment. That is,
\[
\frac{DS\omega}{Dt} = -P\omega \quad \text{or} \quad \frac{D}{Dt}(\alpha \omega + \chi \times \omega) = -(\alpha_p \omega + \chi_p \times \omega)
\]

A direct calculation shows that \( P \)-parameters \([\alpha_p, \chi_p]\) drive \( S \)-parameters \([\alpha, \chi]\) in the following alignment-parameter dynamics

\[
\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 = -\alpha_p \quad \text{and} \quad \frac{D\chi}{Dt} + 2\alpha \chi = -\chi_p
\]

We’ll first derive and analyze evolution equations for comoving frame \{\(\hat{\omega}, \hat{\chi}, \hat{\omega} \times \hat{\chi}\)\}, then we’ll interpret the alignment-parameter dynamics.
Lagrangian frame dynamics

One computes

\[ \frac{D \hat{\chi}}{Dt} = -c_1 \chi^{-1}(\hat{\omega} \times \hat{\chi}) \quad \& \quad \frac{D(\hat{\omega} \times \hat{\chi})}{Dt} = \chi \hat{\omega} + c_1 \chi^{-1} \hat{\chi}. \]

The various Lagrangian time derivatives may be assembled into

\[ \frac{D\hat{\omega}}{Dt} = \mathcal{D} \times \hat{\omega} \]
\[ \frac{D(\hat{\omega} \times \hat{\chi})}{Dt} = \mathcal{D} \times (\hat{\omega} \times \hat{\chi}) \]
\[ \frac{D\hat{\chi}}{Dt} = \mathcal{D} \times \hat{\chi} \]

The “Darboux vector” \( \mathcal{D} \) is defined as

\[ \mathcal{D} = \chi - \frac{c_1}{\chi} \hat{\omega} \quad \text{with} \quad |\mathcal{D}|^2 = \chi^2 + \frac{c_1^2}{\chi^2} \]

and one sees that \( c_1 = \hat{\omega} \cdot (\hat{\chi} \times \chi_p) \) depends on the pressure Hessian.
What can we deduce from vorticity frame dynamics? Where are we going next?

1. Note similarity of vorticity frame dynamics to Frenet-Serret equations for space curves in three dimensions.

2. Use Ertel’s theorem to derive Lagrangian dynamics of the Frenet-Serret curvature and torsion

3. Represent vorticity alignments with $S$ & $P$ as quaternions. These yield the Cayley-Klein parameters of $S\hat{\omega}$ & $P\hat{\omega}$

4. Recover dynamics of $S$-alignment $\zeta = [\alpha, \chi]$ driven by $P$-alignment $\zeta_p = [\alpha_p, \chi_p]$ in quaternionic form
Frame dynamics for $F$

Use $\hat{\omega}$ etc as row-vectors to define the $3 \times 3$ orthogonal frame-matrix

$$F(t, s) = \begin{pmatrix} \hat{\omega} \\ \hat{\chi} \\ \hat{\omega} \times \hat{\chi} \end{pmatrix}, \quad F^T = F^{-1}$$

The matrix $F(t, s)$ specifies the evolution in time $t$ of an orthonormal frame attached to any given Lagrangian label $s$ along the vortex line.

The previous frame dynamics may now be re-written using $F$ as,

$$\frac{DF}{Dt} = BF(t, s) \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 & -\chi \\ 0 & 0 & -c_1/\chi \\ \chi & c_1/\chi & 0 \end{pmatrix}$$

with $B_{ij} = \epsilon_{ijk}D_k$ for Darboux components $D_k$ and $\chi_p \cdot \hat{\chi} \times \hat{\omega} = -c_1$. 
Frenet-Serret equations

The unit tangent $\hat{\omega}$, normal $\hat{n}$ & binormal $\hat{b}$ of a vortex line define another $3 \times 3$ orthogonal frame-matrix $N(t, s)$ whose orientation varies with its arclength $s$ according to the Frenet-Serret equations,

$$N = \begin{pmatrix} \hat{\omega} \\ \hat{n} \\ \hat{b} \end{pmatrix}, \quad \frac{\partial N}{\partial s} = AN(t, s) \quad \text{where} \quad A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

Here $\kappa$ and $\tau$ are the curvature and torsion of the vortex line.

The matrix $N(t, s)$ is also orthogonal: $N^T = N^{-1}$.

The solution of Frenet-Serret for $N(t, s)$ determines an orthonormal frame at each point $s$ along the vortex line at a given time $t$. 
Frame dynamics for the Frenet-Serret matrix $N$

The frames $N$ & $F$ are related by a rotation $R(\phi)$ around the unit tangent vector $\hat{\omega}$ by an angle $\phi(t, s)$

$$N = R(\phi)F$$

where

$$\frac{DF}{Dt} = BF(t, s)$$

Consequently, the Frenet-Serret matrix $N$ satisfies

$$\frac{\partial N}{\partial s} = AN(t, s) \quad \text{and} \quad \frac{DN}{Dt} = BN(t, s) + \text{linear corex}$$

where the arclength derivative along a vortex line is defined as,

$$\frac{\partial}{\partial s} = \omega \cdot \nabla.$$
Evolving the curvature and torsion of a vortex line

Ertel’s Theorem tells us that the derivatives in $t$ and $s$ commute

$$\left[ \frac{D}{Dt}, \frac{\partial}{\partial s} \right] = 0.$$  

This commutation relation implies equality of cross derivatives of $N$. That is, $N_{ts} = N_{st}$. Hence,

$$\frac{DA}{Dt} = \frac{\partial B}{\partial s} - [A, B],$$

with $A = (\partial N/\partial s)N^{-1}$, $B = (DN/Dt)N^{-1}$ and $[A, B] = AB - BA$.

Hence,

$$-\kappa \frac{D\tau^{-1}}{Dt} = \chi + \text{Correx linear in } \chi$$

Look at the case of straight vortices $\kappa = 0$. 

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Alignment: cause and effect!

• Thus, swing rate $\chi \neq 0$ implies time-dependence of vortex torsion $\tau$. And $\kappa = 0$ implies $\chi = 0$, so straight vortices don’t swing!

• We now seek alignment-parameter dynamics of growth rate ($\alpha$) and swing rate ($\chi$) for a combined scalar and vector quantity denoted $\zeta = [\alpha, \chi]$.

\[ S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega} \]

We rewrite this as quaternionic multiplication:

\[ [0, S\hat{\omega}] = [\alpha, \chi] \otimes [0, \hat{\omega}] \]

which expresses parallel & perpendicular decomposition of $S\hat{\omega}$. 
What about using quaternions? (Hamilton 1843)

Quaternions combine scalar \( q \) & 3-vector \( q \) into a tetrad \( q = [q, q] \) as

\[
q = [q, q] = qI - q \cdot \sigma, \quad \text{with} \quad q \cdot \sigma = \sum_{i=1}^{3} q_i \sigma_i
\]

The Pauli spin matrices \( \sigma \) obey the relations \( \sigma_i \sigma_j = -\delta_{ij} - \epsilon_{ijk} \sigma_k \)

\[
\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

By this definition tetrads obey the multiplication rule denoted \( \odot \)

\[
p \odot q = [pq - p \cdot q, pq + qp + p \times q]
\]

Vorticity dynamics suggests alignment tetrads \( \zeta = [\alpha, \chi] \), \( \zeta_p = [\alpha_p, \chi_p] \)

which satisfy \( [0, S\hat{\omega}] = \zeta \odot [0, \hat{\omega}] \) and \( [0, P\hat{\omega}] = \zeta_p \odot [0, \hat{\omega}] \)

as parallel and perpendicular decompositions.
Are quaternions a good idea?

Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way. – Lord Kelvin (William Thompson)


Hamilton was vindicated – quaternions are now used in the robotics and avionics industries to track objects undergoing a sequence of tumbling rotations and are also heavily used in graphics.

- Visualizing quaternions, by Andrew J. Hanson, MK-Elsevier, 2006.
Quaternions & Cayley-Klein parameters I

The dot product of two quaternions \( p := [p, p] \) and \( q := [q, q] \) is defined as
\[
p \cdot q := pq + p \cdot q
\]
The magnitude of quaternion \( q \) is
\[
|q| := (q \cdot q)^{1/2} = (q^2 + q \cdot q)^{1/2}
\]
One defines the conjugate of \( q := [q, q] \) as \( q^* = [q, -q] \)
So, product \( q \odot q^* = (q \cdot q) e \), where \( e = [1, 0] \) is the identity.
Hence
\[
q^{-1} := q^*/(q \cdot q) \quad \text{is the inverse of quaternion } q
\]
under \( \odot \) product. (Recall that vectors don’t have inverses.)
Quaternions & Cayley-Klein parameters II

Consider the map under the quaternionic product (which is associative)

\[ \mathbf{r} \rightarrow \mathbf{r}' = \hat{p} \ast \mathbf{r} \ast \hat{p}^* \]

where \( \hat{p} \) is a unit quaternion, \( \hat{p} \cdot \hat{p} = 1 \), so \( \hat{p} \ast \hat{p}^* = e = [1, 0] \)

The inverse map is

\[ \mathbf{r} = \hat{p}^* \ast \mathbf{r}' \ast \hat{p} \]

If \( \mathbf{r} = [0, \mathbf{r}] \) then \( \mathbf{r}' = [0, \mathbf{r}'] = [0, \mathbf{r} + 2\hat{p}(\mathbf{p} \times \mathbf{r}) + 2\mathbf{p} \times (\mathbf{p} \times \mathbf{r})] \)

For \( \hat{p} := \pm [\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{n}] \), this is a rotation of \( \mathbf{r} \) by angle \( \theta \) about \( \hat{n} \).

In \( \hat{p} = [\mathbf{p}, \mathbf{p}], \mathbf{p} \) & \( \mathbf{p} \) are the Cayley-Klein parameters of the rotation.

\[ \therefore \text{Composition of rotations} \simeq \text{Multiplication of (±) unit quaternions} \]
Alignment $S\hat{\omega}$ vs $\hat{\omega}$ & Cayley-Klein parameters

Consider the unit quaternion relation with $\hat{p} := \pm [\cos\frac{\theta}{2}, \sin\frac{\theta}{2} \hat{\chi}]$,

$$|S\hat{\omega}|^{-1}[0, S\hat{\omega}] = \hat{p} \mathbin{\ast} [0, \hat{\omega}] \mathbin{\ast} \hat{p}^* = [0, \cos \theta \hat{\omega} + \sin \theta \hat{\chi} \times \hat{\omega}]$$

$$= |S\hat{\omega}|^{-1} \zeta \mathbin{\ast} [0, \hat{\omega}] = (\alpha^2 + \chi^2)^{-1/2} [0, \alpha \hat{\omega} + \chi \times \hat{\omega}]$$

where $\zeta = [\alpha, \chi]$. Thus, the unit vector $|S\hat{\omega}|^{-1} S\hat{\omega}$ is a rotation of $\hat{\omega}$ by angle $\theta$ around $\hat{\chi}$ with

$$\cos \theta = \frac{\alpha}{(\alpha^2 + \chi^2)^{1/2}} \quad \text{and} \quad \sin \theta = \frac{\chi}{(\alpha^2 + \chi^2)^{1/2}}$$

Hence,

Alignment parameters $\alpha$ and $\chi$ define $S\hat{\omega}$ as a stretching of $\hat{\omega}$ by $(\alpha^2 + \chi^2)^{1/2}$ & rotation of $\hat{\omega}$ by $\theta = \tan^{-1} \chi/\alpha$ about $\hat{\chi}$.

Likewise for $P\hat{\omega}$ and its alignment parameters $\alpha_p$ and $\chi_p$ relative to $\hat{\omega}$.

The angle $\theta$ is the **misalignment** between $S\hat{\omega}$ & $\hat{\omega}$.
The Euler equations in quaternionic form

Define velocity & pressure tetrads $\mathcal{U}$ & $\Pi$ and the 4-derivative $\nabla$ as

$$\mathcal{U} = [0, u] \quad \Pi = [p, 0] \quad \nabla = [0, \nabla]$$

Then Euler’s fluid equation is written in quaternionic form as

$$\frac{D\mathcal{U}}{Dt} = -\nabla \otimes \Pi$$

The vorticity tetrad $\Omega$ is formed from

$$\nabla \otimes \mathcal{U} = [-\text{div } u, \text{curl } u] = [0, \omega] =: \Omega$$

Operating with $\nabla \otimes$ on Euler’s equation above produces

$$[\Delta p, 0] = \left[ -|\nabla u|^2, \frac{D\omega}{Dt} - S\omega \right]$$

Identifying terms yields $\Delta p = -|\nabla u|^2$ and Euler’s vorticity equation.
**Theorem:** The vorticity tetrad $\Omega(x, t) = [0, \omega]$ satisfies

$$\frac{D\Omega}{Dt} = \zeta \otimes \Omega$$

(Frozen-in tetrad field)

$$\frac{D^2\Omega}{Dt^2} + \zeta_p \otimes \Omega = 0$$

(Ohkitani’s relation)

where $\zeta = [\alpha, \chi]$ and $\zeta_p = [\alpha_p, \chi_p]$.

Consequently, the growth & swing rate tetrad $\zeta(x, t) = [\alpha, \chi]$ satisfies

$$\frac{D\zeta}{Dt} + \zeta \otimes \zeta + \zeta_p = 0$$

**Remark:** The $\zeta$-equation is a **Riccati equation** driven by $\zeta_p$ which, in turn, depends on the other variables through the pressure Hessian $P$.

The growth/swing rate tetrad $\zeta(x, t) = [\alpha, \chi]$ evolves by quadratic nonlinearity and is driven by the $P$—alignment tetrad $\zeta_p = [\alpha_p, \chi_p]$. 
Proof:

\[
\frac{D\Omega}{Dt} = [0, \underbrace{\alpha\omega + \chi \times \omega}_{S\omega}] = [\alpha, \chi] \otimes [0, \omega] = \zeta \otimes \Omega.
\]

\[
P\omega = \alpha_p \omega + \chi_p \times \omega \Rightarrow [0, P\omega] = \zeta_p \otimes \Omega
\]

Use Ertel’s Theorem to express Ohkitani’s relation as

\[
\frac{D^2\Omega}{Dt^2} = \frac{D}{Dt} [0, S\omega] = -[0, P\omega] = -\zeta_p \otimes \Omega
\]

Compare this relation with \( \frac{D^2\Omega}{Dt^2} = \frac{D}{Dt}(\zeta \otimes \Omega) \) to find

\[
0 = \frac{D\zeta}{Dt} \otimes \Omega + \zeta \otimes (\zeta \otimes \Omega) + \zeta_p \otimes \Omega
\]

The equation for \( \zeta \) follows, because \( \otimes \) is associative.
Quaternion alignment dynamics in components

The alignment equation for tetrads $\zeta = [\alpha, \chi]$ with $\zeta_p = [\alpha_p, \chi_p]$ is

$$\frac{D\zeta}{Dt} + \zeta \otimes \zeta + \zeta_p = 0$$

Recall the components of the tetrad multiplication rule

$$p \otimes q = [pq - p \cdot q, pq + qp + p \times q]$$

So $\zeta \otimes \zeta = [\alpha^2 - \chi^2, 2\alpha\chi]$ in components & the alignment variables $\alpha, \chi$ are driven by $\alpha_p, \chi_p$ according to

$$\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 + \alpha_p = 0 \quad \text{and} \quad \frac{D\chi}{Dt} + 2\alpha\chi + \chi_p = 0$$

where $S\hat{\omega} = \alpha \hat{\omega} + \chi \times \hat{\omega}$ and $P\hat{\omega} = \alpha_p \hat{\omega} + \chi_p \times \hat{\omega}$

$$\frac{D\omega}{Dt} = S\omega \quad \& \quad \omega = \text{curl} u, \quad \frac{DS\omega}{Dt} = -P\omega \quad \& \quad \text{tr} P = -|\nabla u|^2$$
Alignment dynamics in polar coordinates

In polar coordinates given by the stretching rate along $\hat{\omega}$ as the radius $r = (\alpha^2 + \chi^2)^{1/2} = |S\hat{\omega}|$ and the angle $\theta = \tan^{-1} \chi/\alpha$ of rotation about the comoving $\hat{\chi}$ axis from $\hat{\omega}$ to $S\hat{\omega}$, the alignment dynamics derived from

$$\frac{DS\omega}{Dt} = - P\omega$$

becomes, upon using

$$S\hat{\omega} = \alpha \hat{\omega} + \chi \hat{\chi} \times \hat{\omega} = r (\cos \theta \hat{\omega} + \sin \theta \hat{\chi} \times \hat{\omega})$$

the $2 \times 2$ system in polar coordinates,

$$\frac{D}{Dt} \frac{\sin \theta}{r} + \cos 2\theta = \frac{\alpha_p}{r^2}$$

$$\frac{D}{Dt} \frac{\cos \theta}{r} - \sin 2\theta = \frac{\hat{\chi} \cdot \chi_p}{r^2}$$

where one recalls that $\hat{\chi} \cdot \chi_p = -c_2$ and $\theta = 0$ is perfect alignment.
A simple solution: the Burgers vortex

The most elementary Burgers vortex solution is (with $\gamma_0 = \text{const}$)

$$
\mathbf{u} = \left( -\frac{1}{2} \gamma_0 x + \psi_y, -\frac{1}{2} \gamma_0 y - \psi_x, z \gamma_0 \right) \implies \mathbf{\omega} = (0, 0, \omega_3)
$$

$$
\omega_3(r, t) = e^{\gamma_0 t} \omega_0 \left( r e^{\frac{1}{2} \gamma_0 t} \right) \quad \text{(note exponential growth)}
$$

Thus, for the Burgers vortex one computes

$$
\alpha = \gamma_0, \quad \chi = 0, \quad \alpha_p = -\gamma_0^2
$$

$$
\zeta = [\gamma_0, 0] \quad \zeta_p = -[\gamma_0^2, 0]
$$

Conclusions: Burgers tubes/sheets are scalar objects: they don’t swing.

(In fact, they are steady solutions of the $\zeta$-equation.)

When tubes & sheets bend then $\chi \neq 0$ and $\zeta$ becomes a full tetrad driven by $\zeta_p$ which is coupled back through the pressure Hessian $P$. 

When do $[\alpha, \chi]$ tetrad equations arise in fluids?

- First, we need a **Frozen-in Vector Field**, $\varpi \cdot \nabla$, so that
  \[ \frac{Du}{Dt} = \mathcal{F} \quad \text{implies} \quad \frac{D\varpi}{Dt} = \varpi \cdot \nabla u \quad \text{for} \quad \varpi = Q_{op} u \]

- This will produce an **Ertel Theorem** and **Ohkitani relation**
  \[ \left[ \frac{D}{Dt}, \varpi \cdot \nabla \right] = 0, \quad \text{so} \quad \frac{D^2 \varpi}{Dt^2} = \frac{D}{Dt} (\varpi \cdot \nabla u) = \varpi \cdot \nabla \mathcal{F} \]

- In turn these will produce orthonormal **Frame Dynamics** for $\hat{\varpi}$, whose alignment parameters will satisfy **Quaternion equations**.

- **Other examples:**
  1. Lagrangian Averaged Euler-alpha (LAE$-\alpha$) equations
  2. MP97mod2’ as Euler-Poincaré equations
  3. Ideal MHD and LAMHD-alpha.
Lagrangian Averaged Euler-alpha (LAE−α) model

Lagrangian averaging preserves Kelvin’s circulation theorem, which leads to a frozen-in vector field and thereby produces Ertel’s theorem.

The LAE−α motion equation is

\[ \frac{Dw}{Dt} + \nabla u^T \cdot w = -\nabla p \quad \text{for} \quad w = u - \alpha^2 \Delta u \quad \text{and} \quad \nabla \cdot u = 0 \]

or, in Kelvin circulation form,

\[ \frac{D}{Dt}(w \cdot dx) = -dp \quad \text{along} \quad \frac{Dx}{Dt} = u \]

Stokes-ing (or taking d) and \( \nabla \cdot u = 0 \) yield a Frozen-in Vector Field

\[ \frac{D\varpi}{Dt} = \varpi \cdot \nabla u \quad \text{for} \quad \varpi = \nabla \times w \]
Ertel Theorem & Ohkitani relation for LAE$-\alpha$

The LAE$-\alpha$ motion equation may also be written using $u = G \ast w$ as

$$\frac{Du}{Dt} = \mathcal{F} = -G \ast (\nabla p + 4\alpha^2 \nabla \cdot \Omega S)$$

where $2\Omega = \nabla u - \nabla u^T$ and $G\ast = (1 - \alpha^2 \Delta)^{-1}$ denotes convolution with the Greens function for the Helmholtz operator.

The Ertel Theorem and Ohkitani relation for LAE$-\alpha$ are then

$$\left[ \frac{D}{Dt}, \varpi \cdot \nabla \right] = 0, \quad \text{and} \quad \frac{D}{Dt}(\varpi \cdot \nabla u) = \frac{D^2 \varpi}{Dt^2} = \varpi \cdot \nabla \mathcal{F}$$

where $\varpi = \nabla \times w$ and $w = (1 - \alpha^2 \Delta)u$

The rest (Dynamics of Vorticity Frames and Quaternionic Alignment Parameters) follows the pattern of Euler fluids.
Ertel Theorem & Ohkitani relation for \textbf{MP97mod2’}

The MP97mod2’ motion equation may be written as

\[
\frac{\tilde{D}\tilde{U}}{\tilde{D}t} = -\nabla(p + 2q) + \nabla \cdot 2\mathbf{\tilde{\omega}} \otimes \mathbf{\tilde{\omega}} =: \tilde{F},
\]

where \(\nabla \cdot \tilde{U} = 0\) and \(\mathbf{l} \cdot \nabla \tilde{U} \cdot \mathbf{l} = 0\) determine \(p\) & \(q\), and

\[
\frac{\partial \mathbf{l}}{\partial t} = \text{curl} (\tilde{U} \times \mathbf{l}), \quad \mathbf{\hat{\omega}} = \mathbf{l}/|\mathbf{l}|, \quad |\mathbf{l}|^2 = 1 \quad \text{with} \quad \nabla \cdot \mathbf{l} = 0
\]

The Ertel Theorem and Ohkitani relation for MP97mod2’ are then

\[
\left[ \frac{D}{Dt}, \mathbf{l} \cdot \nabla \right] = 0, \quad \text{so} \quad \frac{D\mathbf{l}}{Dt} = \mathbf{l} \cdot \nabla \tilde{U} \quad \text{and} \quad \frac{D}{Dt}(\mathbf{l} \cdot \nabla \tilde{U}) = \frac{D^2\mathbf{l}}{Dt^2} = \mathbf{l} \cdot \nabla \tilde{F}
\]

Together, Ertel and Ohkitani conveniently deliver

\[
\frac{D}{Dt}(\mathbf{l} \cdot \nabla \tilde{U} \cdot \mathbf{l}) = \mathbf{l} \cdot \nabla \tilde{F} \cdot \mathbf{l} + |\mathbf{l} \cdot \nabla \tilde{U}|^2
\]
The equation system for Lagrange multipliers $p$ & $q$

Preservation of $\nabla \cdot \tilde{U} = 0$ and $l \cdot \nabla \tilde{U} \cdot l = 0$ determines Lagrange multipliers $p$ & $q$ from the system

$$0 = \frac{\partial}{\partial t} (\nabla \cdot \tilde{U}) = - |\nabla \tilde{U}|^2 + \nabla \cdot \tilde{F}$$

$$0 = \frac{\partial}{\partial t} (l \cdot \nabla \tilde{U} \cdot l) = - \tilde{U} \cdot \nabla (l \cdot \nabla \tilde{U} \cdot l) + |l \cdot \nabla \tilde{U}|^2 + l \cdot \nabla \tilde{F} \cdot l$$

where the MP97mod2' force $\tilde{F}$ depends linearly on $p$ & $q$ as

$$\frac{\tilde{D}}{\tilde{D}t} \tilde{U} = - \nabla (p + 2q) + \nabla \cdot (2q l \otimes l) =: \tilde{F}$$

The rest (Dynamics of $\hat{\omega}$ Frames and Quaternionic Alignment Parameters) follows for MP97mod2' as for Euler fluids, provided the $p$, $q$ system may be solved at each time step.
Ertel Theorem & Ohkitani relation for **Ideal MHD**

\[
\frac{Du}{Dt} = B \cdot \nabla B - \nabla p, \quad \frac{DB}{Dt} = B \cdot \nabla u,
\]

and \(\text{div} \ u = 0 = \text{div} \ B\). Notice that \(B\) is a Frozen-in Vector Field.

The Elsasser variables \& \((\pm)\) material derivatives are defined as

\[
w^\pm = u \pm B; \quad \frac{D^\pm}{Dt} = \frac{\partial}{\partial t} + w^\pm \cdot \nabla
\]

The magnetic field \(B\) and \(w^\pm\) with \(\text{div} \ w^\pm = 0\) satisfy (note \(\pm\) vs \(\mp\))

\[
\frac{D^\pm w^\mp}{Dt} = -\nabla p \quad \text{and} \quad \frac{D^\pm B}{Dt} = B \cdot \nabla w^\pm
\]

Ertel’s Theorem and Ohkitani’s relation for ideal MHD are then

\[
\left[ \frac{D^\pm}{Dt}, B \cdot \nabla \right] = 0, \quad \text{and} \quad \frac{D^\pm}{Dt} (B \cdot \nabla w^\mp) = \frac{D^\pm D^\mp}{Dt \, Dt} B = -PB
\]
Definition of $\alpha^\pm$ and $\chi^\pm$ in Elsasser variables

The stretching rates $\alpha^\pm$ & swing rates $\chi^\pm$ for evolving magnetic field $B = B\mathbf{\hat{B}}$ along the $\pm$ characteristics are given by

$$\frac{D^\pm B}{Dt} = \alpha^\pm B, \quad \frac{D^\pm \mathbf{\hat{B}}}{Dt} = \chi^\pm \times \mathbf{\hat{B}}$$

where

$$\alpha^\pm = \mathbf{\hat{B}} \cdot (\mathbf{\hat{B}} \cdot \nabla w^\pm) \quad \chi^\pm = \mathbf{\hat{B}} \times (\mathbf{\hat{B}} \cdot \nabla w^\pm)$$

As Moffatt (1985) suggested, $B$ in ideal MHD is analogous to vorticity $\omega$ in Euler fluids – except MHD has two $\pm$ characteristic velocities!

Per Moffatt’s suggestion, we introduce the MHD analogs of $\alpha_p$ & $\chi_p$,

$$\alpha_{pb} = \mathbf{\hat{B}} \cdot P \mathbf{\hat{B}} \quad \chi_{pb} = \mathbf{\hat{B}} \times P \mathbf{\hat{B}}$$
Lagrangian frame dynamics for ideal MHD

The 2 sets of orthonormal vectors \( \{\hat{B}, (\hat{B} \times \hat{\chi}^\pm), \hat{\chi}^\pm\} \) acted on by their opposite Lagrangian time derivatives are found to obey

\[
\begin{align*}
\frac{D^\pm \hat{B}}{Dt} &= \mathcal{D}^\pm \times \hat{B}, \\
\frac{D^\pm (\hat{B} \times \hat{\chi}^\pm)}{Dt} &= \mathcal{D}^\pm \times (\hat{B} \times \hat{\chi}^\pm), \\
\frac{D^\pm \hat{\chi}^\pm}{Dt} &= \mathcal{D}^\pm \times \hat{\chi}^\pm
\end{align*}
\]

where the \((\mp)\) pair of Darboux vectors \( \mathcal{D}^\mp \) are defined as

\[
\mathcal{D}^\mp = \chi^\mp - \frac{c_1^\mp}{\chi^\mp} \hat{B}, \quad c_1^\mp = \hat{B} \cdot [\hat{\chi}^\mp \times (\chi_{pb}^\mp + \alpha^\mp \chi^\mp)].
\]

The corresponding Frenet-Serret frames and their Lagrangian parameter evolution may again be found, by using Ertel’s theorem for ideal MHD.
Quaternionic alignment dynamics for ideal MHD

Tetrads $\Omega_B = [0, B]$, $\zeta^\pm = [\alpha^\pm, \chi^\pm]$ and $\zeta_{pb} = [\alpha_{pb}, \chi_{pb}]$ are used to express the following

**Theorem:** The magnetic field tetrad $\Omega_B$ satisfies the two relations

$$\frac{D^\pm \Omega_B}{Dt} = \zeta^\pm \otimes \Omega_B, \quad \text{(Frozen-in tetrads)}$$

$$\frac{D^\pm}{Dt} \left( \frac{D^\pm \Omega_B}{Dt} \right) + \zeta_{pb} \otimes \Omega_B = 0, \quad \text{(Ohkitani relations)}$$

Consequently, the tetrads $\zeta^\pm$ satisfy the coupled *Riccati equations*

$$\frac{D^\mp \zeta^\pm}{Dt} + \zeta^\pm \otimes \zeta^\mp + \zeta_{pb} = 0$$
Quaternionic MHD alignment eqns in components

Alignment dynamics of tetrads $\zeta^\pm = [\alpha^\pm, \chi^\pm]$ with $\zeta_{pb} = [\alpha_{pb}, \chi_{pb}]$ is

$$\frac{D^\pm \zeta^\pm}{Dt} + \zeta^\pm \odot \zeta^\mp + \zeta_{pb} = 0$$

Recall the components of the tetrad multiplication rule

$$p \odot q = [pq - p \cdot q, pq + qp + p \times q]$$

So $\zeta^\pm \odot \zeta^\mp = [\alpha^\pm \alpha^\mp - \chi^\pm \cdot \chi^\mp, \alpha^\pm \chi^\mp + \alpha^\mp \chi^\pm + \chi^\pm \times \chi^\mp]$ in components & the alignment variables $[\alpha^\pm, \chi^\pm]$ with $\zeta_{pb} = [\alpha_{pb}, \chi_{pb}]$ evolve by

$$\frac{D^\pm \alpha^\pm}{Dt} + \alpha^\pm \alpha^\mp - \chi^\pm \cdot \chi^\mp = -\alpha_{pb}$$

and

$$\frac{D^\pm \chi^\pm}{Dt} + \alpha^\pm \chi^\mp + \alpha^\mp \chi^\pm + \chi^\pm \times \chi^\mp = -\chi_{pb}$$

This is quaternionic alignment dynamics for ideal MHD.
References


