Lagrangian evolution and non-Gaussianity in intermittent hydrodynamic turbulence

Yi Li, Carlos Rosales & Charles Meneveau

Thanks to Dr. Laurent Chevillard, Prof. Greg Eyink,
Shiyi Chen & Ethan Vishniac for many useful discussions

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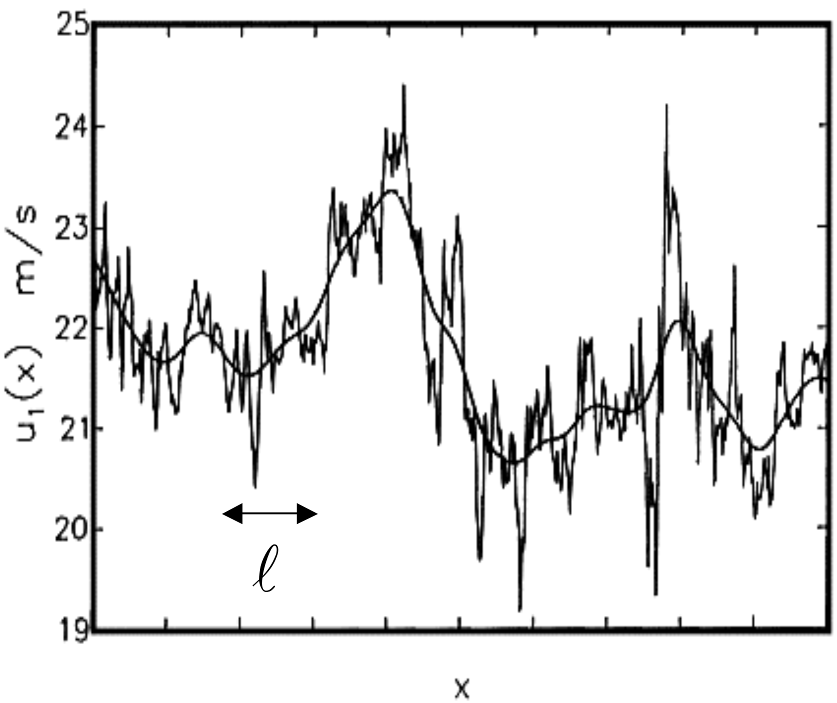
Mechanical Engineering

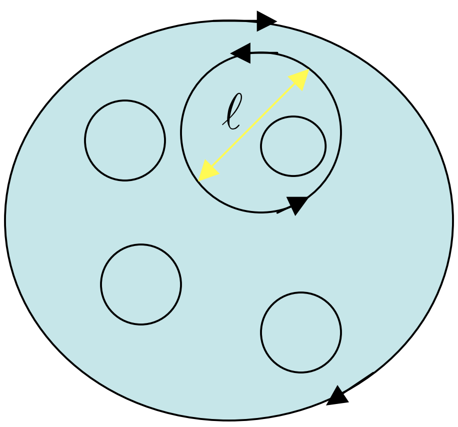
Center for Environmental & Applied Fluid Mechanics

Turbulent flow: multiscale (here: hydrodyamics, no MHD)

Characterize velocity field at particular scale
(filter out larger-scale advection): use velocity increments

$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$

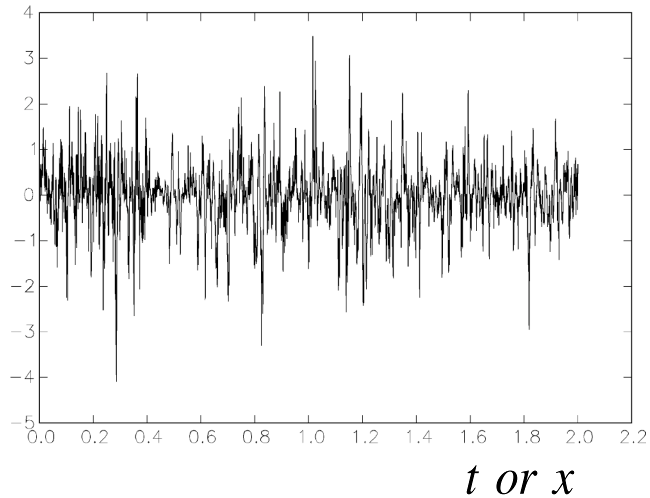




Velocity increment statistics:

$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$

$\delta u(\ell)$

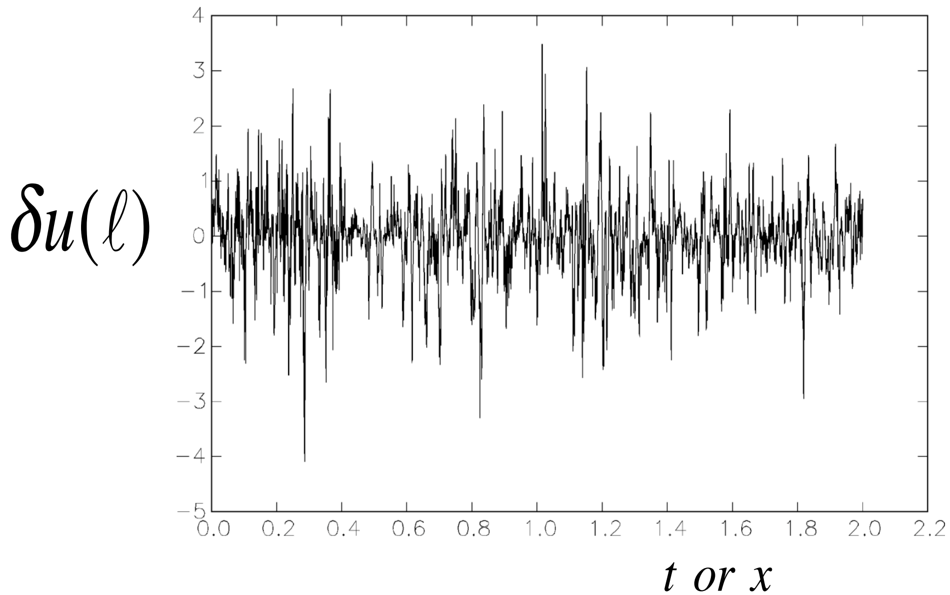


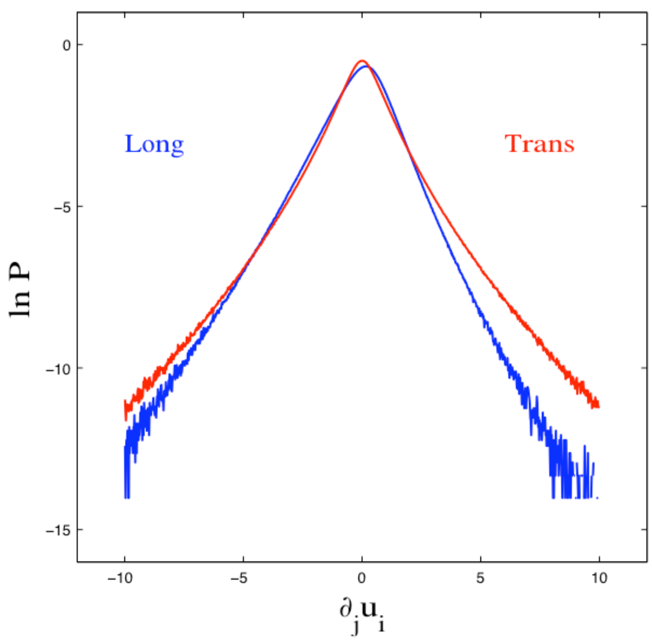
Mean: stationarity, homogeneity : $\langle \delta u(\ell) \rangle = 0$

Variance: Kolmogorov 1941 $\langle \delta u(\ell)^2 \rangle = C_2 \varepsilon^{2/3} \ell^{2/3}$

Velocity increment statistics: Intermittency

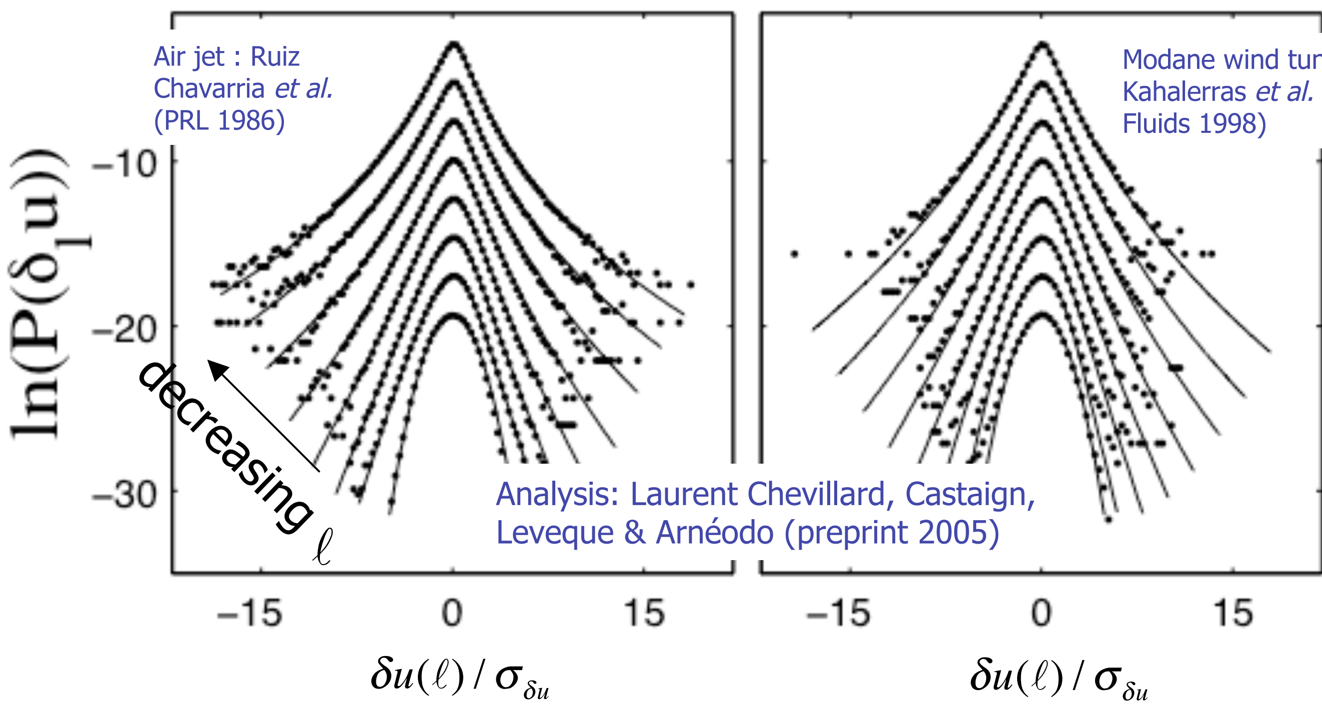
$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$





Small-scale intermittency = **Elongated tails in PDFs** + **Anomalous scaling (clustering and non-trivial dependence on length-scale)**

Elongated tails in PDFs: effect of scale



Wide body of literature, e.g.:

Anselmet *et al.*, JFM **140**, 1984

Kailasnath *et al.*, PRL **68**, 1992

Frisch, *Turbulence*, CUP 1995

Tabeling *et al.*, PRE **53**, 1995

Sreenivasan, Rev. Mod Phys. **71**, 1999

Zeff et al. Nature **421**, 2003

….

….

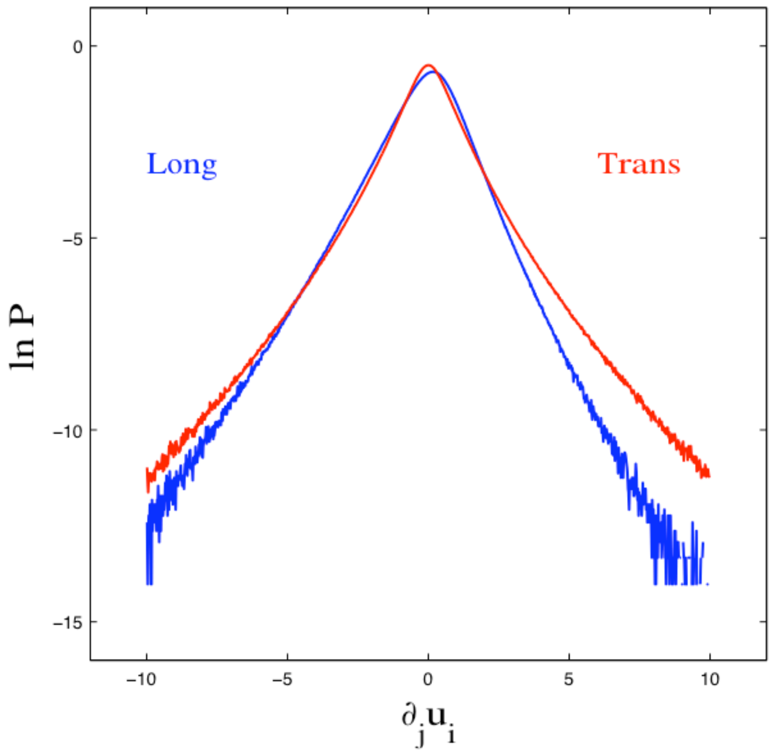
Measured intermittency trends:

A. Longitudinal velocity is skewed:

Well-known Kolmogorov equation:

$$\left\langle \delta u(\ell)^3 \right\rangle = -\frac{4}{5}\varepsilon\ell$$

256^3 DNS



1024^3 DNS
(Gotoh et al. 2002)

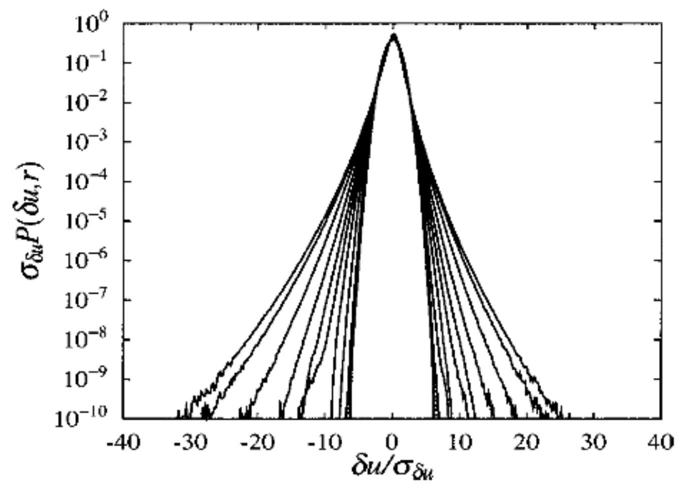
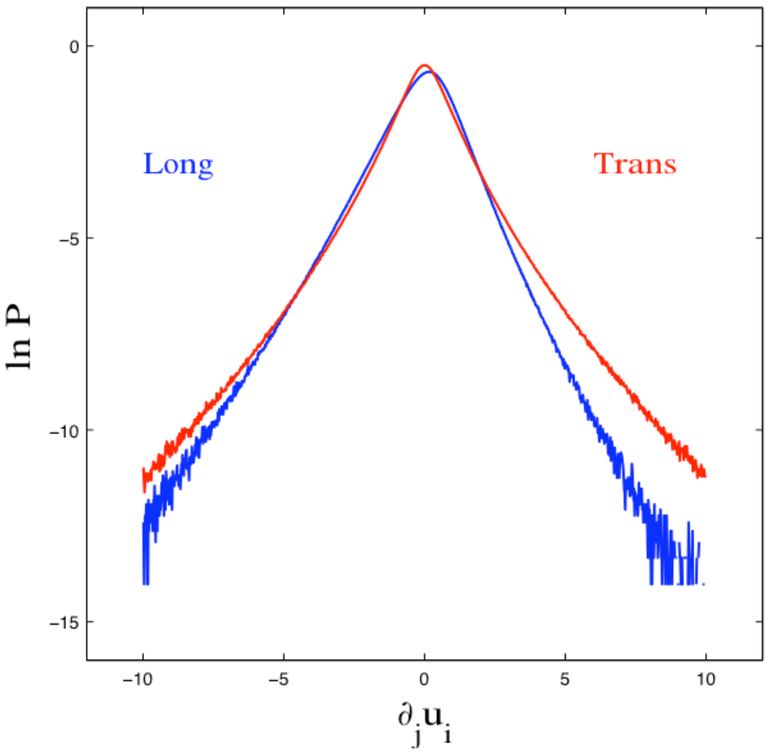


FIG. 15. Variation of the δu_r PDF with r for $R_\lambda = 381$. From the outermost curve, $r_n/\eta = 2^{n-1} dx/\eta = 2.38 \times 2^{n-1}$, $n = 1, \ldots, 10$, where $dx = 2\pi/1024$. The inertial range corresponds to $n = 6, 7, 8$. Dotted line: Gaussian.

Measured intermittency trends:

B. Transverse velocity is more intermittent

256^3 DNS



1024^3 DNS
(Gotoh et al. 2002)

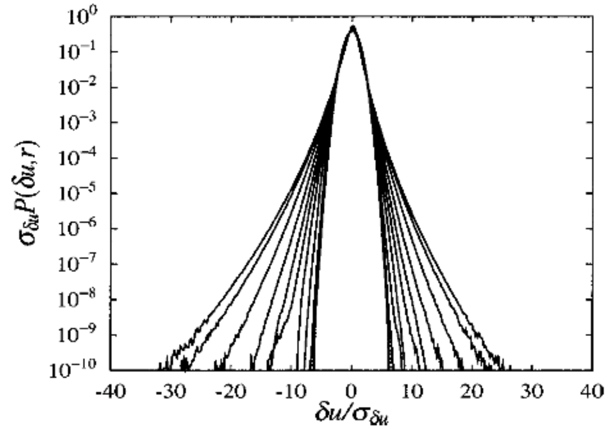


FIG. 15. Variation of the δu_r PDF with r for $R_\lambda = 381$. From the outermost curve, $r_n/\eta = 2^{n-1}dx/\eta = 2.38 \times 2^{n-1}$, $n = 1,...,10$, where $dx = 2\pi/1024$. The inertial range corresponds to $n = 6, 7, 8$. Dotted line: Gaussian.

Gotoh, Fukayama, and Nakano

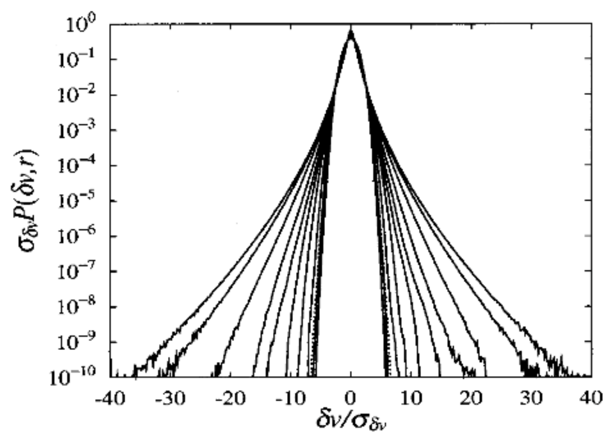


FIG. 16. Variation of PDF for δv_r with r at $R_\lambda = 381$. The classification of curves is the same as in Fig. 17.

Measured intermittency trends:

C. No intermittency in 2-D turbulence for velocity increments

J. Paret and P. Tabeling 3127

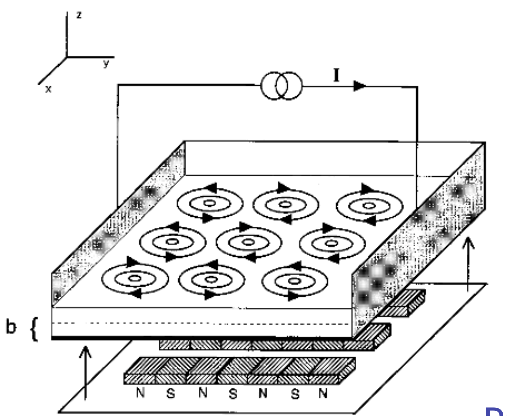


FIG. 1. The experimental set-up.

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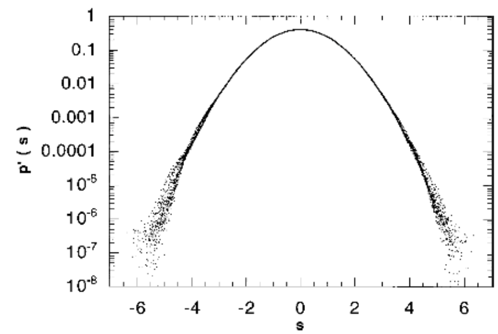


FIG. 12. Rescaled PDF of longitudinal velocity increments for 7 different separations in the inertial range. $s=\delta v/\langle \delta v^2\rangle^{1/2}$.

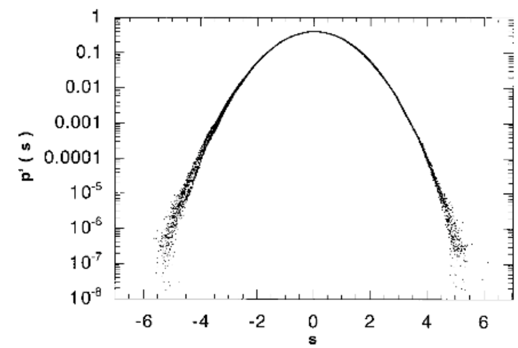


FIG. 15. Rescaled PDF of transverse velocity increments for 7 different separations in the inertial range. $s=\delta v/\langle \delta v^2\rangle^{1/2}$.

Paret & Tabeling, Phys. Fluids, 1998

DNS:

Bofetta et al.
Phys. Rev. E, 2000

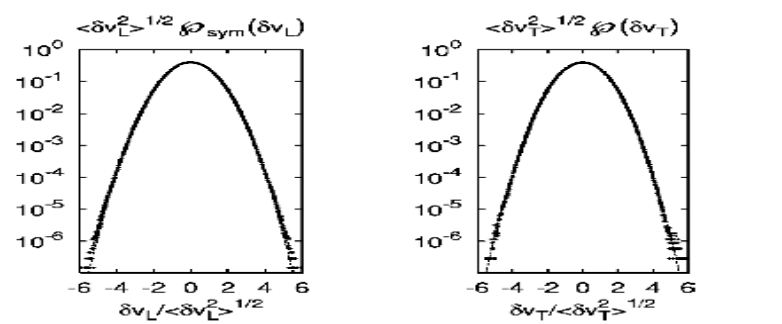
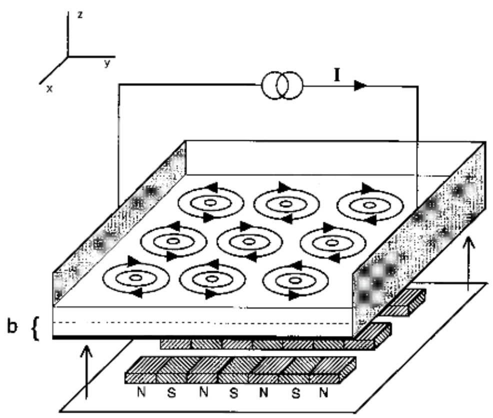


FIG. 6. Left: symmetric part of the longitudinal velocity difference PDF. Right: PDF of transverse velocity differences. The forcing is restricted to a band of wave numbers. Gaussian distributions are shown as solid lines.

Measured intermittency trends:

D. Intermittency in 2-D turbulence for vorticity increments

J. Paret and P. Tabeling 3127



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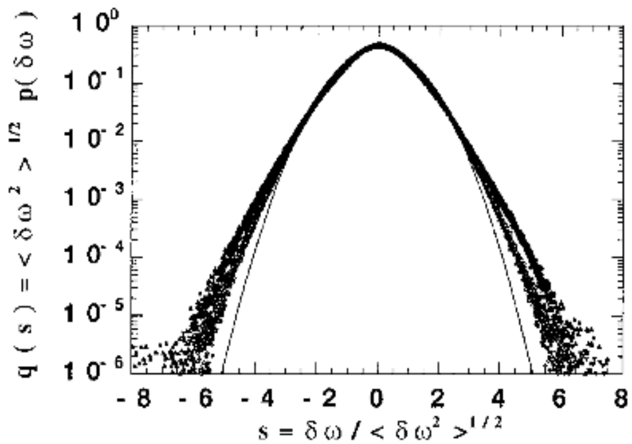


FIG. 5. Normalized distributions of vorticity increments, for five separations of r: 2, 3, 5, 7, and 9 cm.

Paret & Tabeling, Phys. Rev. Lett., 1999

Measured intermittency trends:

E. Increased intermittency in 4-D turbulence for velocity increments

DNS of 128^4 turbulence

Suzuki et al., Phys. Fluids, 2005

081702-4 Suzuki *et al.*

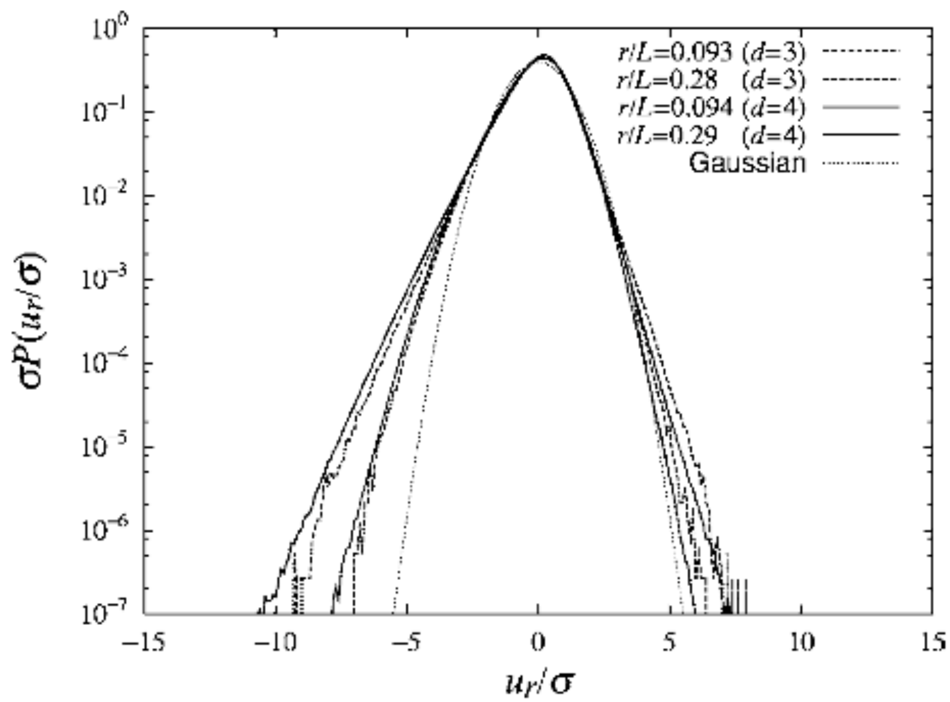


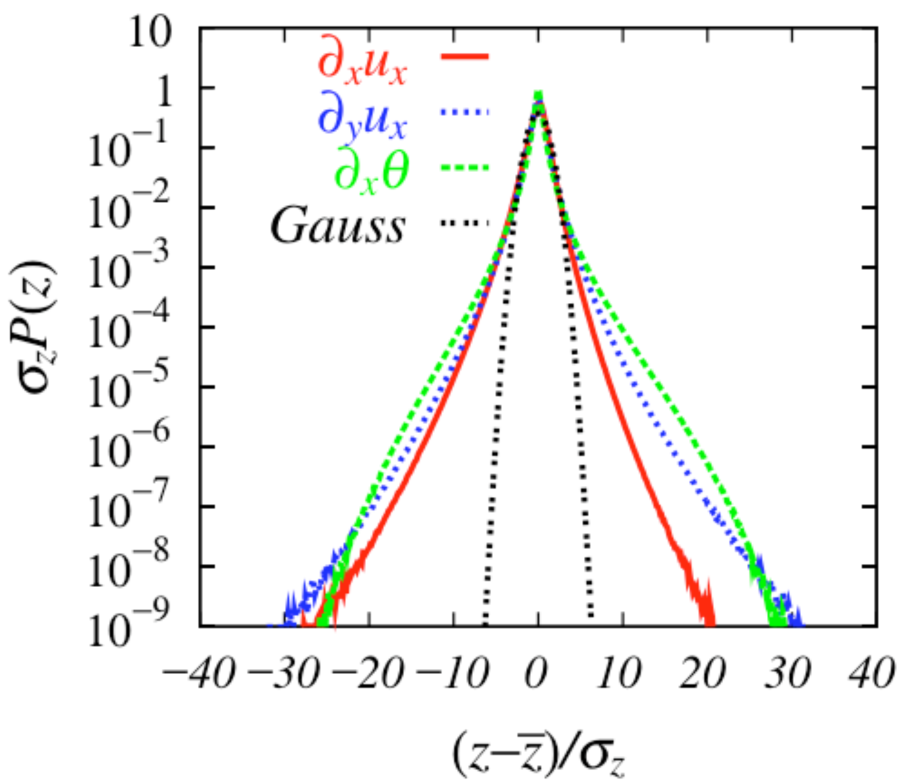
FIG. 6. The PDF of the longitudinal velocity differences u_r for run C and run F, where the abscissa is normalized in terms of the corresponding standard deviation.

Measured intermittency trends:

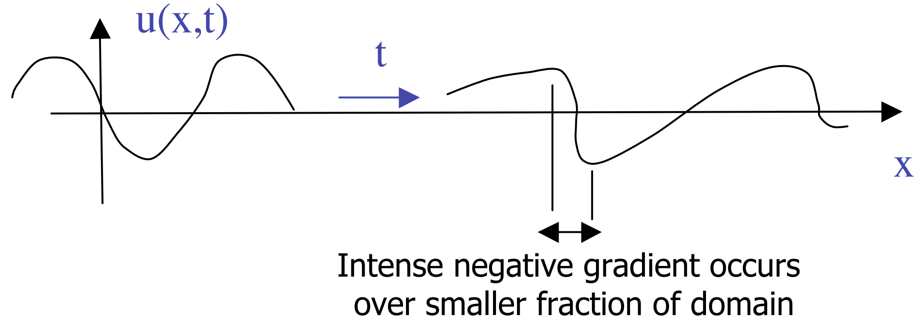
F. Back to 3-D: Passive scalar transport: $\delta T(\ell) = T(\mathbf{x} + \ell \mathbf{e}_L) - T(\mathbf{x})$

Larger intermittency for scalar increments than for velocity increments

e.g. Antonia et al. Phys. Rev. A 1984, and
Watanabe & Gotoh, NJP, 2004:



Trivial "intermittency" trend:



1-D Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

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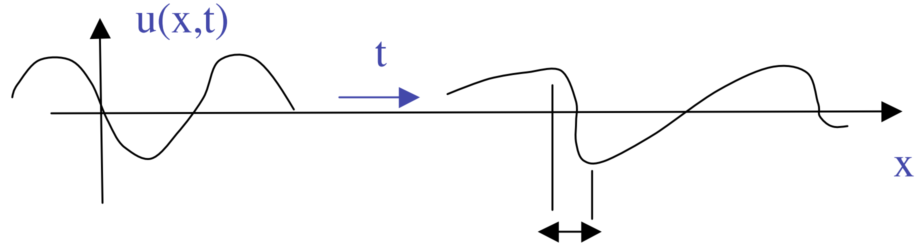


$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

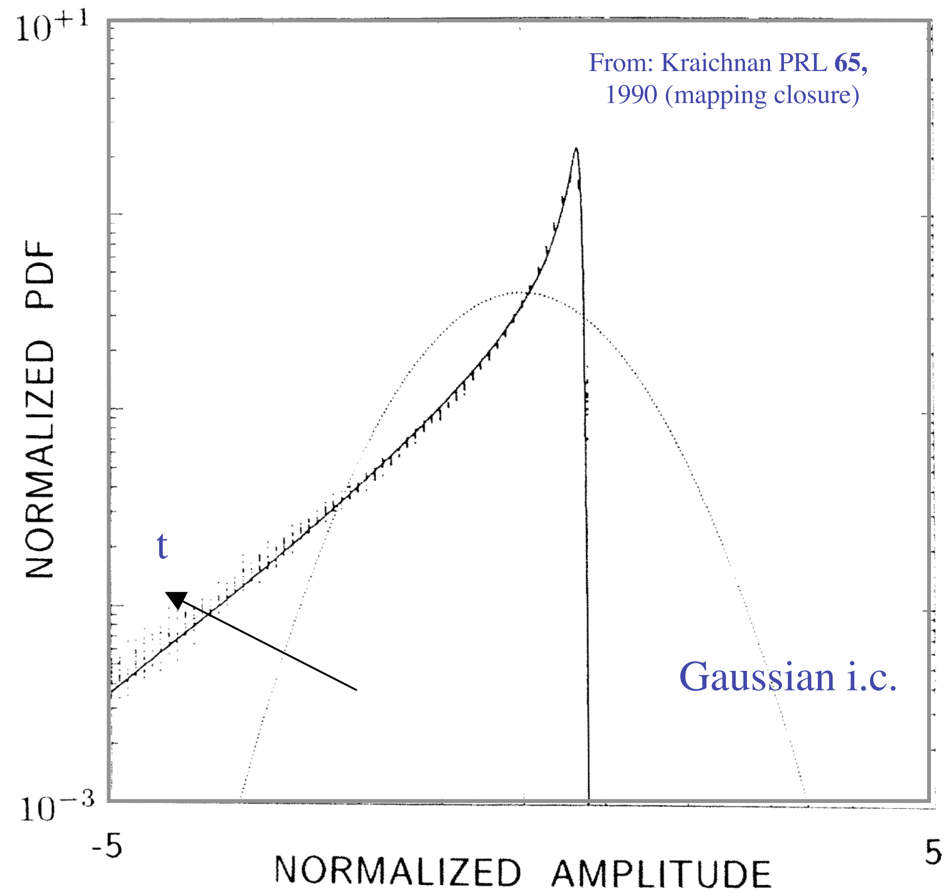
$$A = \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + AA$$

$$\frac{dA}{dt} = -A^2$$

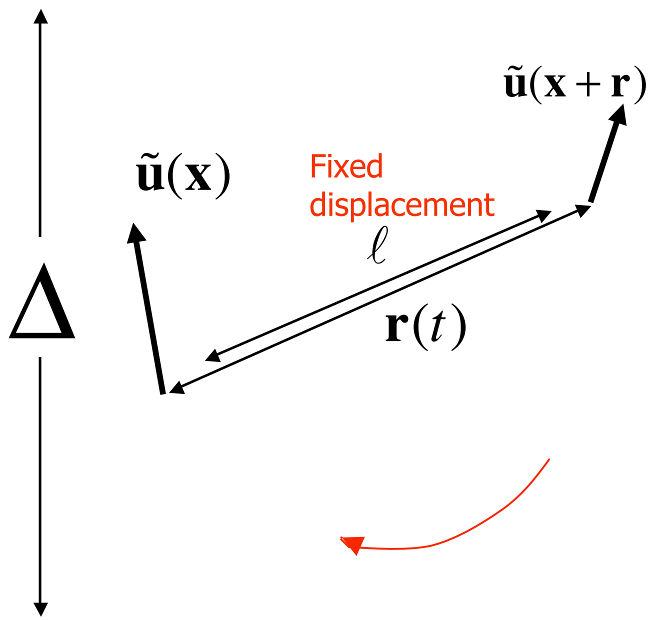
$$\delta u(\ell) \equiv A\ell$$

$$\boxed{\frac{d\delta u}{dt} = -\delta u^2 \ell^{-1}}$$



Here is a "trivial" picture of origin of non-Gaussian tails and skewness: free particle motion. What is the 3-D analogue of this? Problems: which direction? Multiple velocity components..

Velocity increments: Lagrangian evolution



$$\tilde{u}_i(\mathbf{x}+\mathbf{r}) - \tilde{u}_i(\mathbf{x}) = \tilde{A}_{ki} r_k + O(r^2)$$

$$\tilde{A}_{ji} = \frac{\partial \tilde{u}_i}{\partial x_j}$$

Longitudinal

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r}$$

$$\delta u(t) \equiv \tilde{\mathbf{A}}(t) : (\hat{\mathbf{r}}(t)\hat{\mathbf{r}}(t))\ \ell = \tilde{A}_{rr} \ell$$

Transverse

$$\delta v^2 \equiv \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \tilde{A}_{kj} r_k \frac{\ell}{r} \right]^2$$

See: Yi & Meneveau, Phys. Rev. Lett. **95**, 164502, Oct. 2005

Velocity increments: Lagrangian evolution

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r}$$

Rate of change of longitudinal velocity increment, following the flow (both end-points in linearized flow):

$$\frac{d}{dt}\delta u = \frac{d}{dt}\left(\tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r} \right)$$

Velocity increments: Lagrangian evolution

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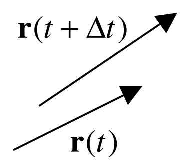
$$\frac{d}{dt}\delta u = \frac{d}{dt}\left(\tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r} \right) = \frac{d\tilde{A}_{ki}}{dt} \frac{r_k r_i}{r} \frac{\ell}{r} + \tilde{A}_{ki} \frac{dr_k}{dt} \frac{r_i}{r} \frac{\ell}{r} + \tilde{A}_{ki} \frac{dr_i}{dt} \frac{r_k}{r} \frac{\ell}{r} - 2\tilde{A}_{ki} \frac{r_k r_i}{r^3} \frac{dr}{dt} \ell$$



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$$\frac{d\tilde{A}_{ij}}{dt} = -(\tilde{A}_{ik}\tilde{A}_{kj} - \tfrac{1}{D}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) + H_{ij}$$

$$\frac{dr_i}{dt} = \frac{\partial \tilde{u}_i}{\partial x_m} r_m = \tilde{A}_{mi} r_m$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k} \left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu \tilde{S}_{kn} \right] \right)^{anisotropic}$$

Velocity increments: Lagrangian evolution

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Velocity increments: Lagrangian evolution

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$$\delta v^2 \equiv \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\tilde{A}_{kj} r_k \frac{\ell}{r}\right]^2$$

Rate of change of longitudinal velocity increment, following the flow (both end-points in linearized flow):

$$\frac{d}{dt}\delta u = \frac{d}{dt}\left(\tilde{A}_{ki} r_k \frac{r_i}{r}\frac{\ell}{r}\right) = \frac{d\tilde{A}_{ki}}{dt}\frac{r_k r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\frac{dr_k}{dt}\frac{r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\frac{dr_i}{dt}\frac{r_k}{r}\frac{\ell}{r} - 2\tilde{A}_{ki}\frac{r_k r_i}{r^3}\frac{dr}{dt}\ell$$

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$$\frac{d}{dt}\delta u = \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\tilde{A}_{kj} r_k \frac{\ell}{r}\right]^2 \frac{1}{\ell} - \left(\tilde{A}_{ki} r_k \frac{r_i}{r}\frac{\ell}{r}\right)^2 \frac{1}{\ell} + \tfrac{1}{D}\tilde{A}_{pq}\tilde{A}_{qp}\ell + H_{mn}\frac{r_m r_n}{r^2}\ell$$

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Velocity increments: Lagrangian evolution

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Rate of change of longitudinal velocity increment, following the flow (both end-points in linearized flow):

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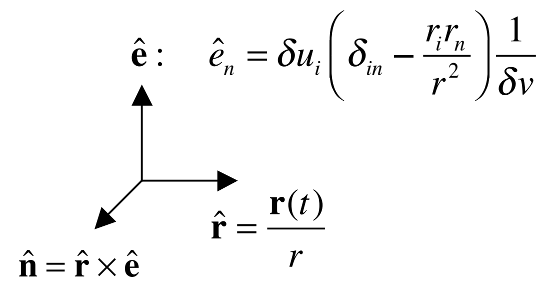
$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) + \frac{1}{D}\tilde{A}_{pq}\tilde{A}_{qp}\ell + H_{mn}\frac{r_m r_i}{r^2}\ell$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau^{SGS}_{kn} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic}$$

Velocity increments: Lagrangian evolution

$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) + \frac{1}{D}\underbrace{\tilde{A}_{pq}\tilde{A}_{qp}}\ell + H_{mn}\frac{r_m r_i}{r^2}\ell$$



Tensor invariant (Q)
Write A in frame formed by:

$$\begin{bmatrix} A_{rr} & A_{re} & A_{rn} \\ A_{er} & A_{ee} & A_{en} \\ A_{nr} & A_{ne} & -(A_{rr}+E_{ee}) \end{bmatrix} = \begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\frac{1}{\ell}$$

$$\tilde{A}_{pq}\tilde{A}_{qp}\ell = \begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\begin{bmatrix} \delta u & \delta v & 0 \\ ? & ? & ? \\ ? & ? & -(\delta u + ?) \end{bmatrix}\frac{1}{\ell} = \left(\delta u^2 + [\delta u + ?]^2 + ?+\ldots\right)\frac{1}{\ell} = 2\delta u^2\frac{1}{\ell} + ?+\ldots$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic} = 0$$

and ? =0

$$\frac{d}{dt}\delta u = -\left(1 - \frac{2}{D}\right)\delta u^2 + \delta v^2$$

Velocity increments: Lagrangian evolution

From a similar derivation for δv:

$$\frac{d}{dt}\delta v = -\frac{2}{\ell}\delta u \delta v$$

From a similar derivation for δT, neglecting diffusion and SGS fluxes:

$$\frac{d}{dt}\delta T = -\frac{1}{\ell}\delta u \delta T$$

In 2D, vorticity is passive scalar, so for 2D:

$$\frac{d}{dt}\delta \omega_z = -\frac{1}{\ell}\delta u \delta \omega_z$$

Velocity increments: Lagrangian evolution

Set of equations:

$$
\begin{cases}
\dfrac{d}{dt}\delta u = -\left(1-\dfrac{2}{D}\right)\delta u^2 + \delta v^2 \\[2ex]
\dfrac{d}{dt}\delta v = -2\delta u \delta v \\[2ex]
\dfrac{d}{dt}\delta T = -\delta u \delta T \\[2ex]
\dfrac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)}
\end{cases}
$$

$D = \infty$ (Burgers)

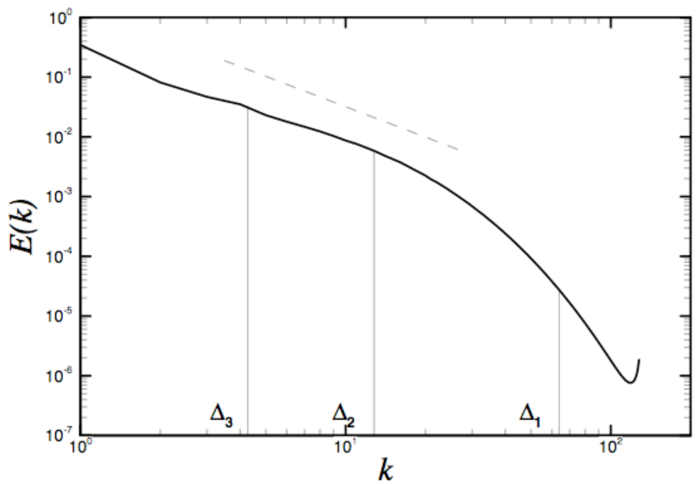
$z = \delta u + i\delta v, \qquad \dfrac{dz}{dt} = -z^2$

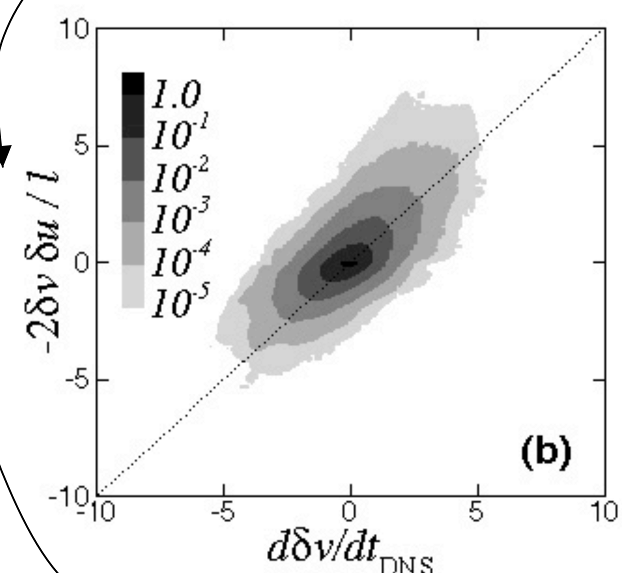
"Advected delta-vee system"

Comparison with DNS,
Lagrangian rate of change of velocity increments:

256^3 DNS, filtered at 40η, $\Delta=40\,\eta$, evaluated δu, δv, and their Lagrangian rate of change of velocity increments numerically

$$\begin{cases} \dfrac{d}{dt}\delta u = \left(-\dfrac{1}{3}\delta u^2 + \delta v^2\right)\dfrac{1}{\ell} & \rho = 0.51 \\ \dfrac{d}{dt}\delta v = -\dfrac{2}{\ell}\,\delta u\,\delta v \end{cases}$$





$\rho = 0.61$

Evolution from Gaussian initial conditions:

Initial condition:

δu = Gaussian zero mean, unit variance

δv_k = Gaussian zero mean, unit variance, k=1,2

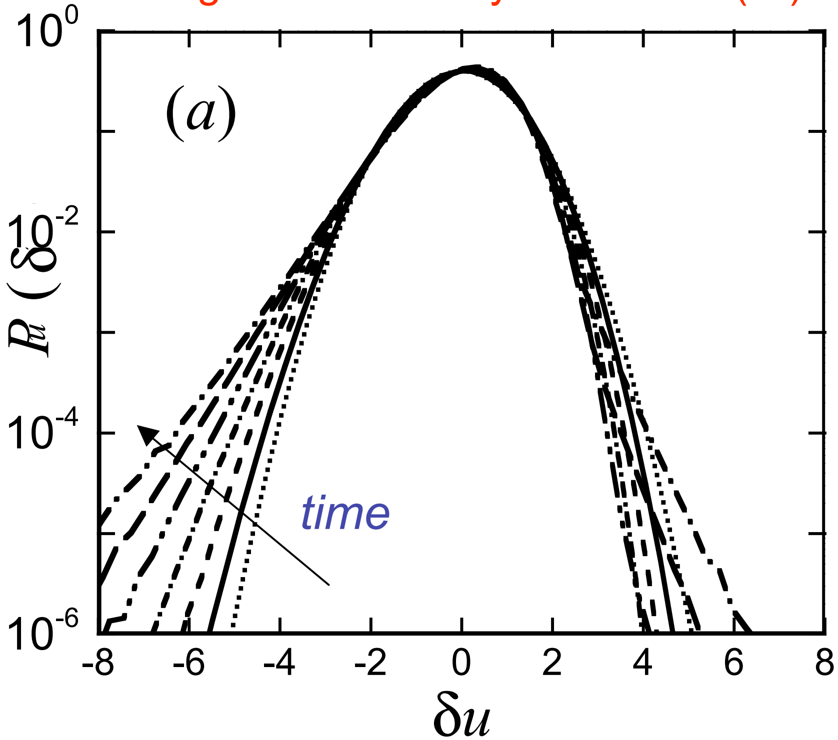
$$\delta v = \sqrt{\delta v_1^2 + \delta v_2^2}$$

δT = Gaussian zero mean, unit variance

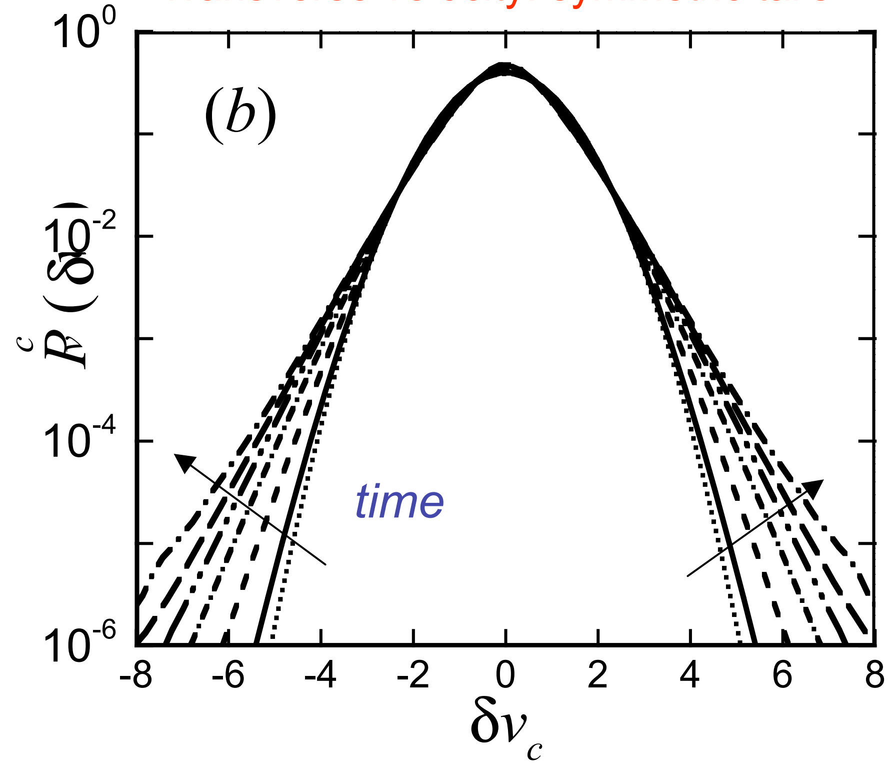
set $\ell = 1$

Numerical Results: PDFs in 3D

Longitudinal velocity: skewness (A.)



Transverse velocity: symmetric tails



$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

D =3

$$\begin{cases} \frac{d}{dt}\delta u = -\frac{1}{3}\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

Technical point #1:
Individual component θ is random, uniformly distributed in $[0, 2\pi)$

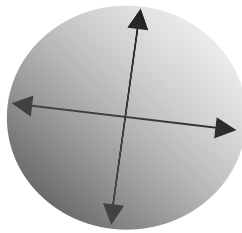
$$\delta v_c = |\delta v| \cos\theta$$

$$P_v^c(\delta v_c) = \frac{1}{\pi}\int_{|\delta v_c|}^{\infty} \frac{P_v(\delta v)}{\sqrt{\delta v^2 - \delta v_c^2}}\, d\delta v$$

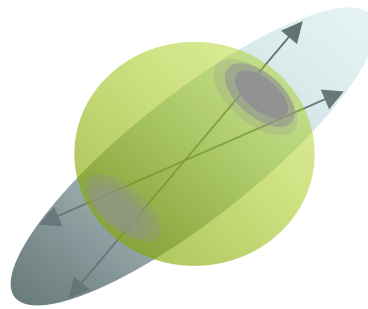
Technical point #2: Alignment bias correction factor

(thanks to Greg Eyink for pointing out the need for a correction)

$\mathbf{r}(0), \quad |\mathrm{r}(0)| = \ell$



$\mathbf{r}(t)$



$$\ell^D d\Omega_0 = r(t)^D d\Omega(t) \qquad \frac{d\Omega(t)}{d\Omega_0} = \left(\frac{\ell}{r(t)}\right)^D$$

$$\frac{dr}{dt} = \delta u(r,t) = \delta u \cdot \left(\frac{r}{\ell}\right)$$

$$\frac{d}{dt}\ln(r/\ell) = \delta u/\ell$$

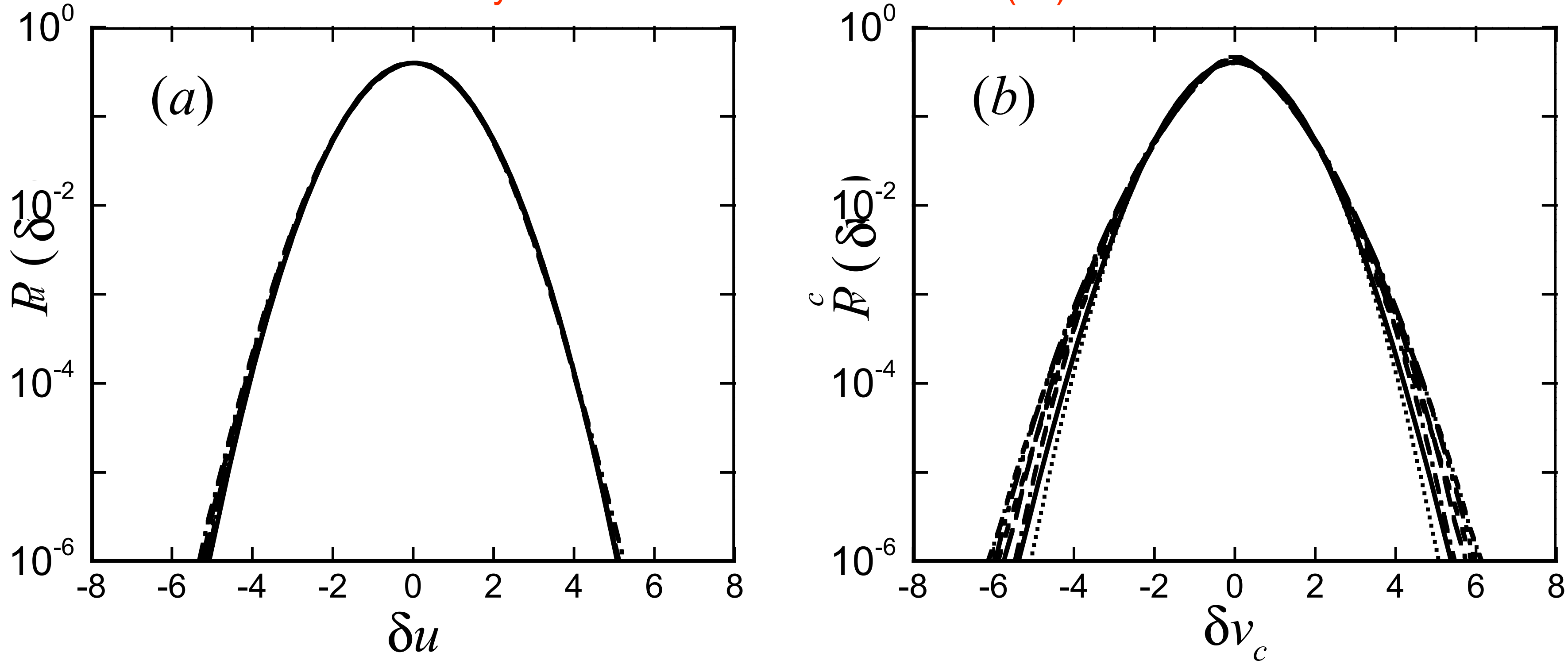
See: Yi & Meneveau,
Phys. Rev. Lett. **95**, 164502,

$$\frac{d\Omega(t)}{d\Omega_0} = \exp\left(-D\ell^{-1}\int_0^t \delta u(t')dt'\right) \qquad P \rightarrow P\frac{d\Omega(t)}{d\Omega_0}$$

Can be evaluated from advected delta-vee system

Numerical Results: PDFs in 2D

2-D turbulence: velocity increments NOT intermittent (C.)



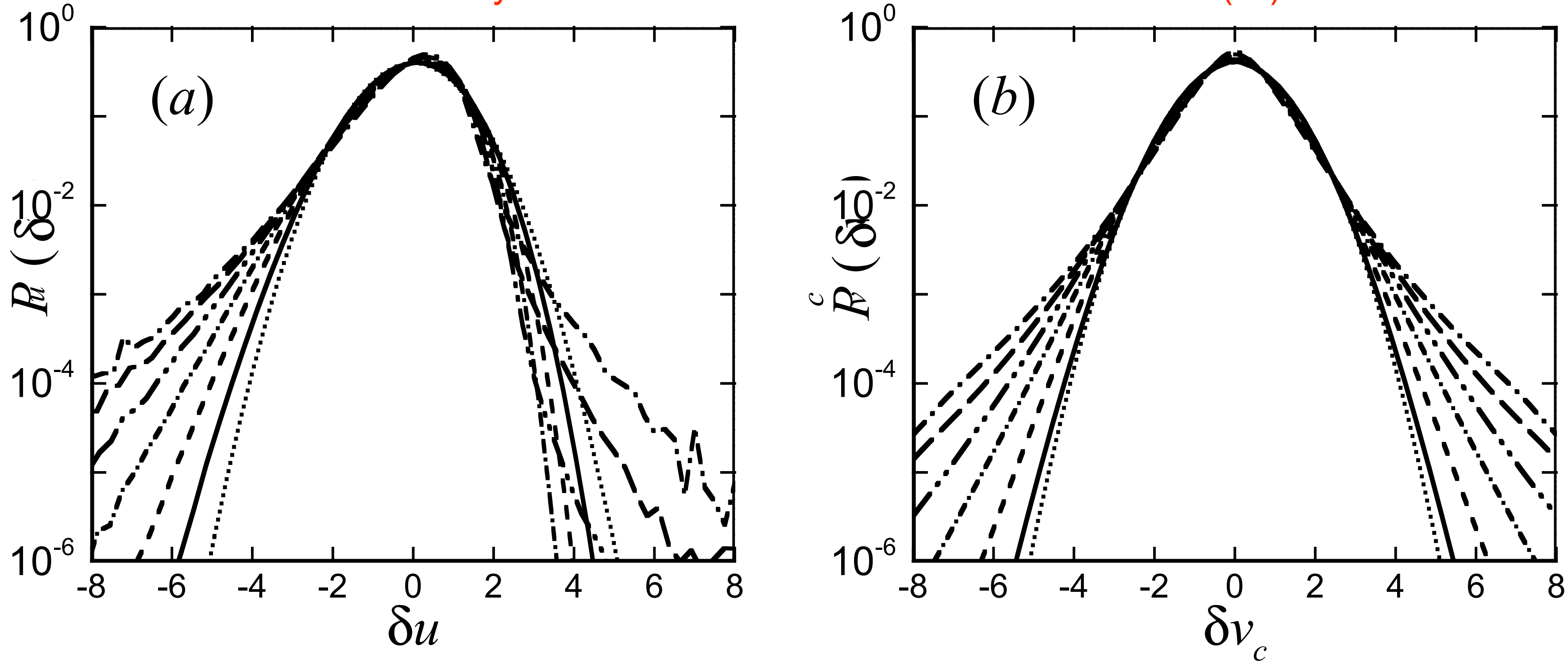
$$\begin{cases} \dfrac{d}{dt}\delta u = -\left(1-\dfrac{2}{D}\right)\delta u^2 + \delta v^2 \\ \dfrac{d}{dt}\delta v = -2\delta u \delta v \\ \dfrac{d}{dt}\delta T = -\delta u \delta T \\ \dfrac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

$D = 2$

$$\begin{cases} \dfrac{d}{dt}\delta u = \delta v^2 \\ \dfrac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

Numerical Results: PDFs in 4D

4D Turbulence: velocity increments MORE intermittent than in 3D (E.)



$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

D =4

$$\begin{cases} \frac{d}{dt}\delta u = -\frac{1}{2}\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

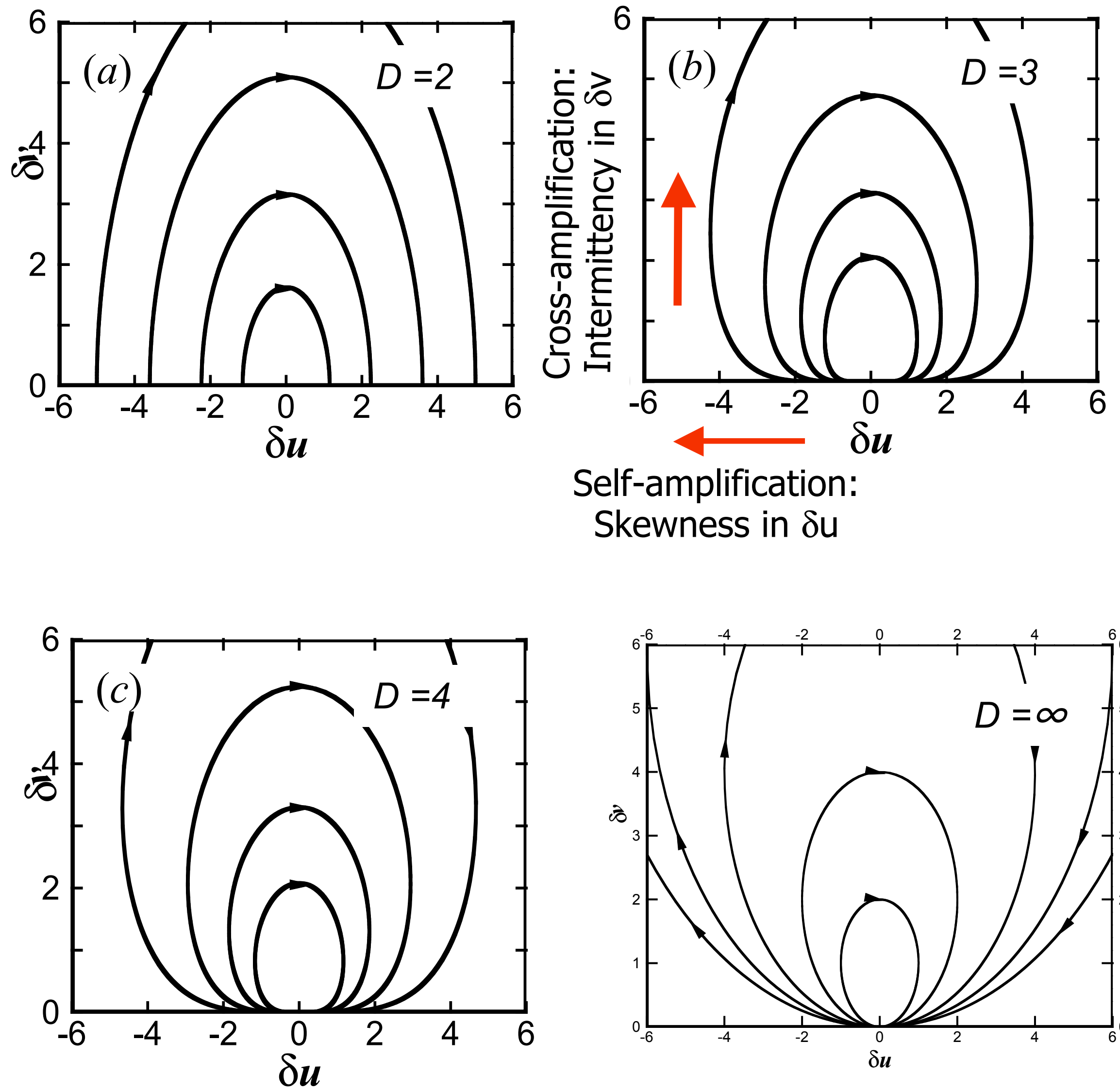
Phase portraits in (δu, δv) phase space:

$$\begin{cases} \dfrac{d}{dt}\delta u = -\left(1-\dfrac{2}{D}\right)\delta u^2 + \delta v^2 \\ \dfrac{d}{dt}\delta v = -2\delta u \delta v \end{cases}$$

Invariant:

$$U = \left(\delta u^2 + \frac{D}{D+2}\delta v^2\right)\delta v^{2/D-1}$$

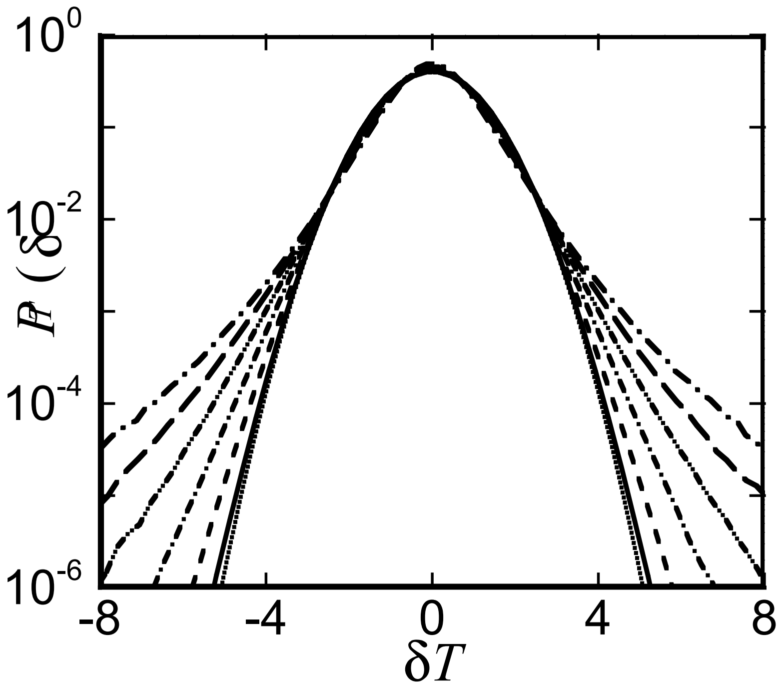
"For small initial δv (particles moving directly towards each other), gradient can become arbitrarily large at later times"

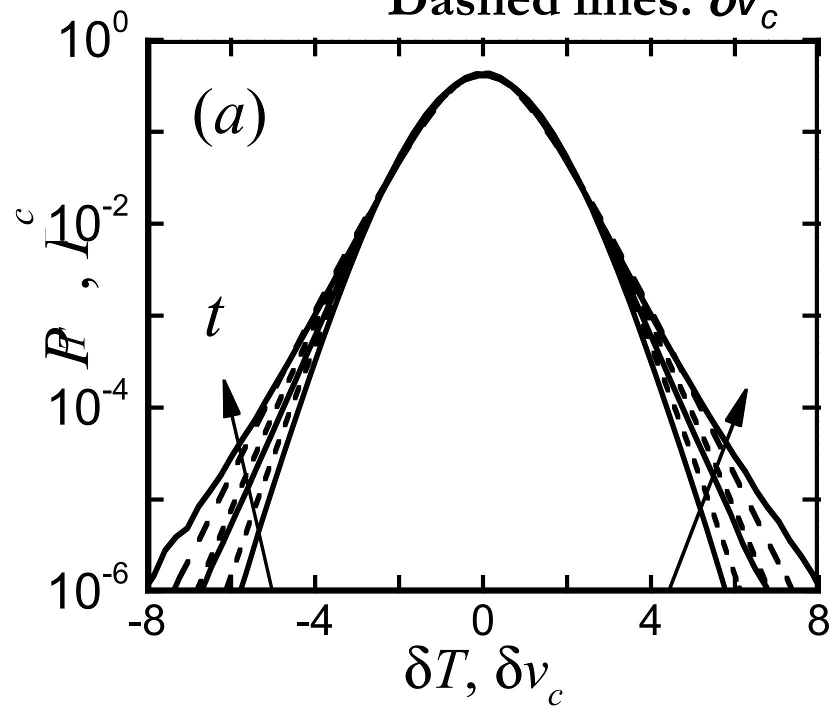


Passive scalar

Solid lines: δT

Dashed lines: δv_c





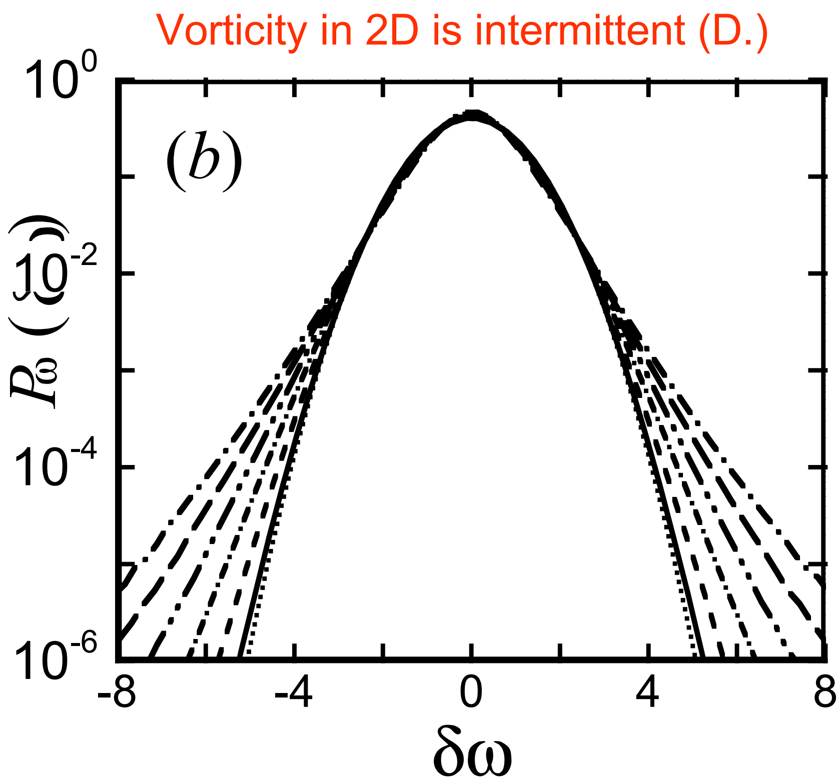
$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

$D = 3$

$$\begin{cases} \frac{d}{dt}\delta u = -\frac{1}{3}\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \end{cases}$$

Scalar increments MORE intermittent than transverse velocity (F.), after initial transient

Vorticity in 2D

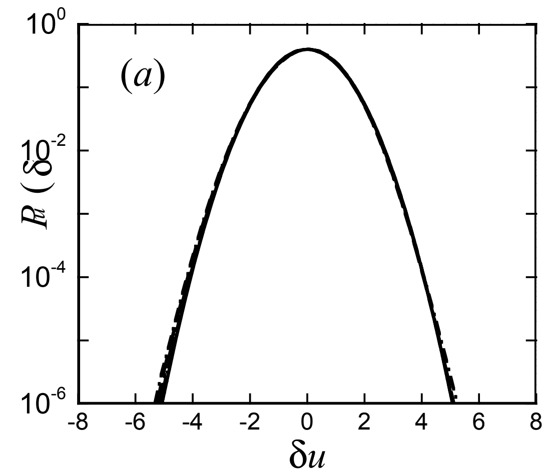


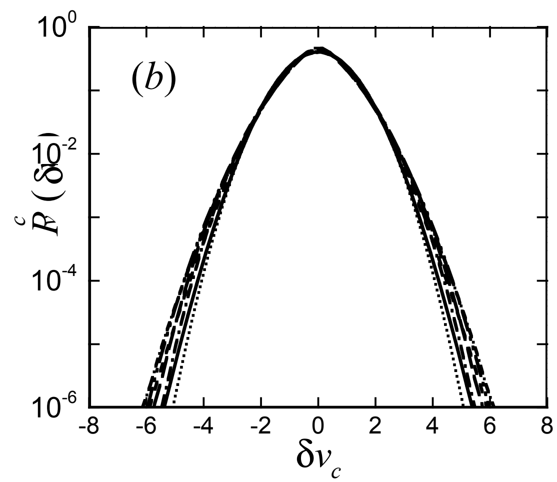
$$\begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \quad \text{(only for D=2)} \end{cases}$$

***D* =2**

$$\begin{cases} \frac{d}{dt}\delta u = \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta \omega_z = -\delta u \delta \omega_z \end{cases}$$

Recall: velocity in 2D

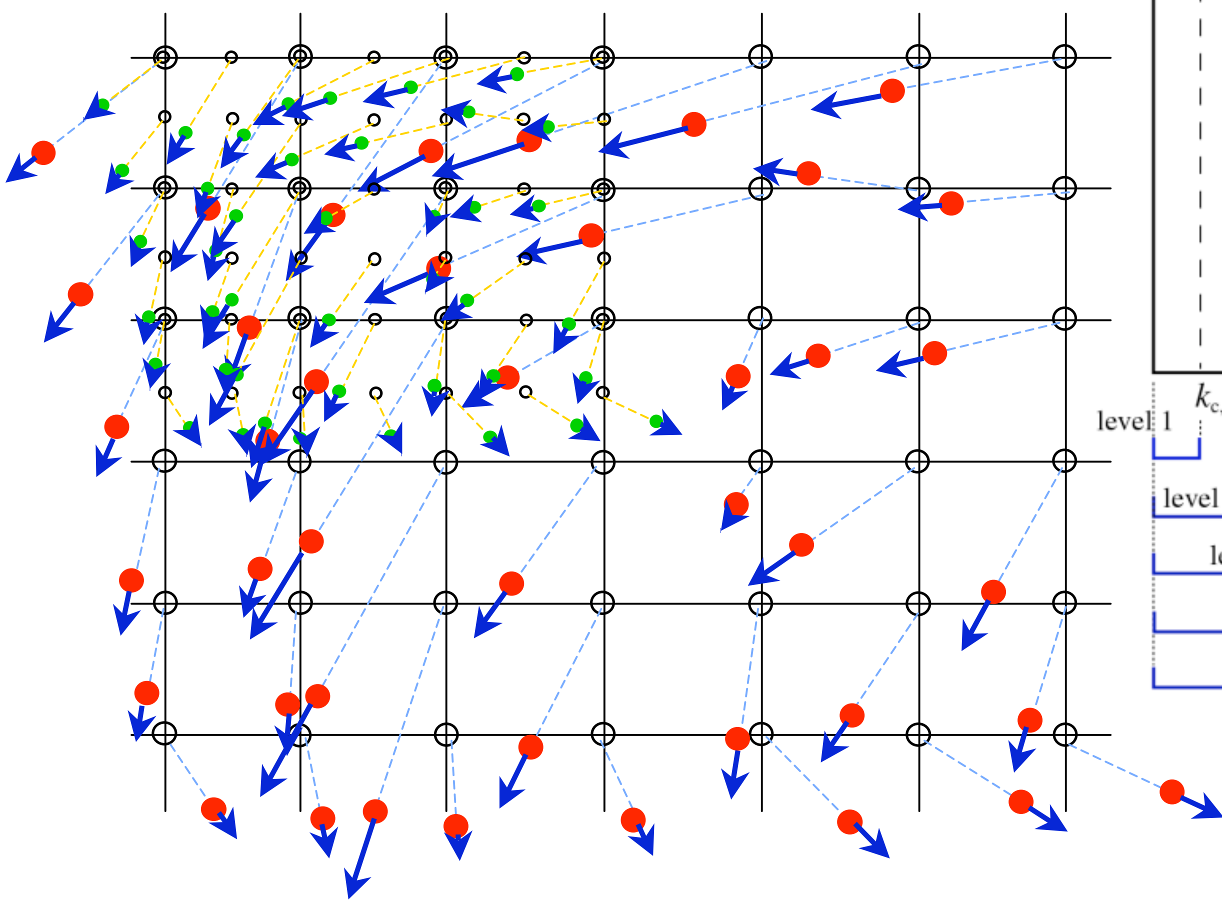


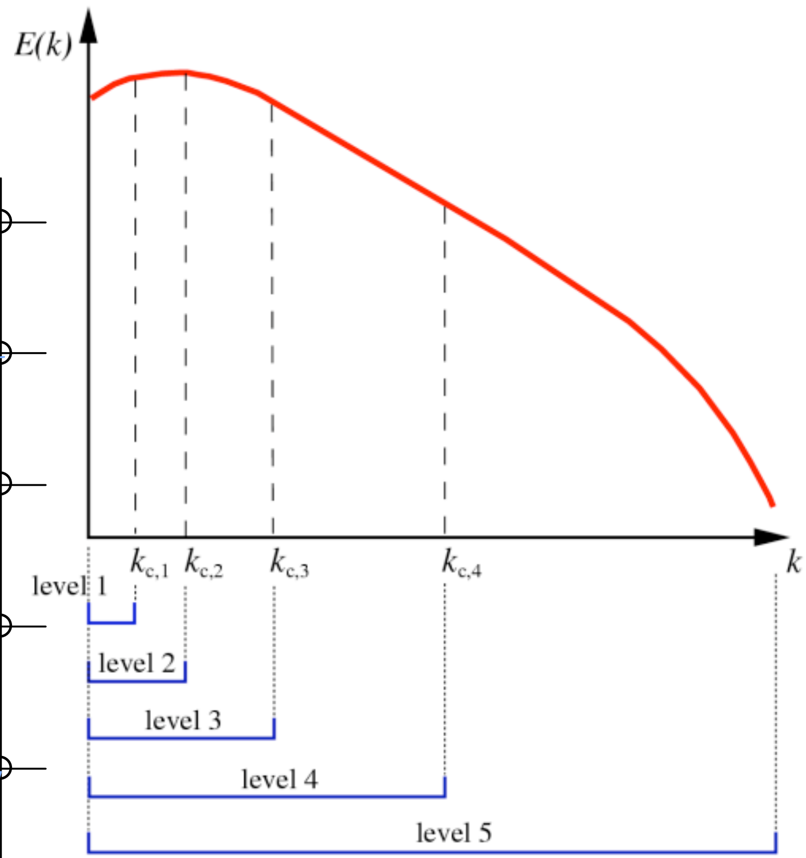


Practical applications:

- **Synthetic turbulence:** Generation of initial and inlet turbulence with realistic non-Gaussian statistics (with Carlos Rosales):

Lagrangian map is applied at several spatial scales, on filtered velocity fields obtained from the originally random field





Multiscale Minimal Lagrangian Map (MMLM):

- “Ballistic evolution” of fluid particles at all scales”
- Projection onto div-free field (in Fourier space)
- Spectral rescaling to E(k)

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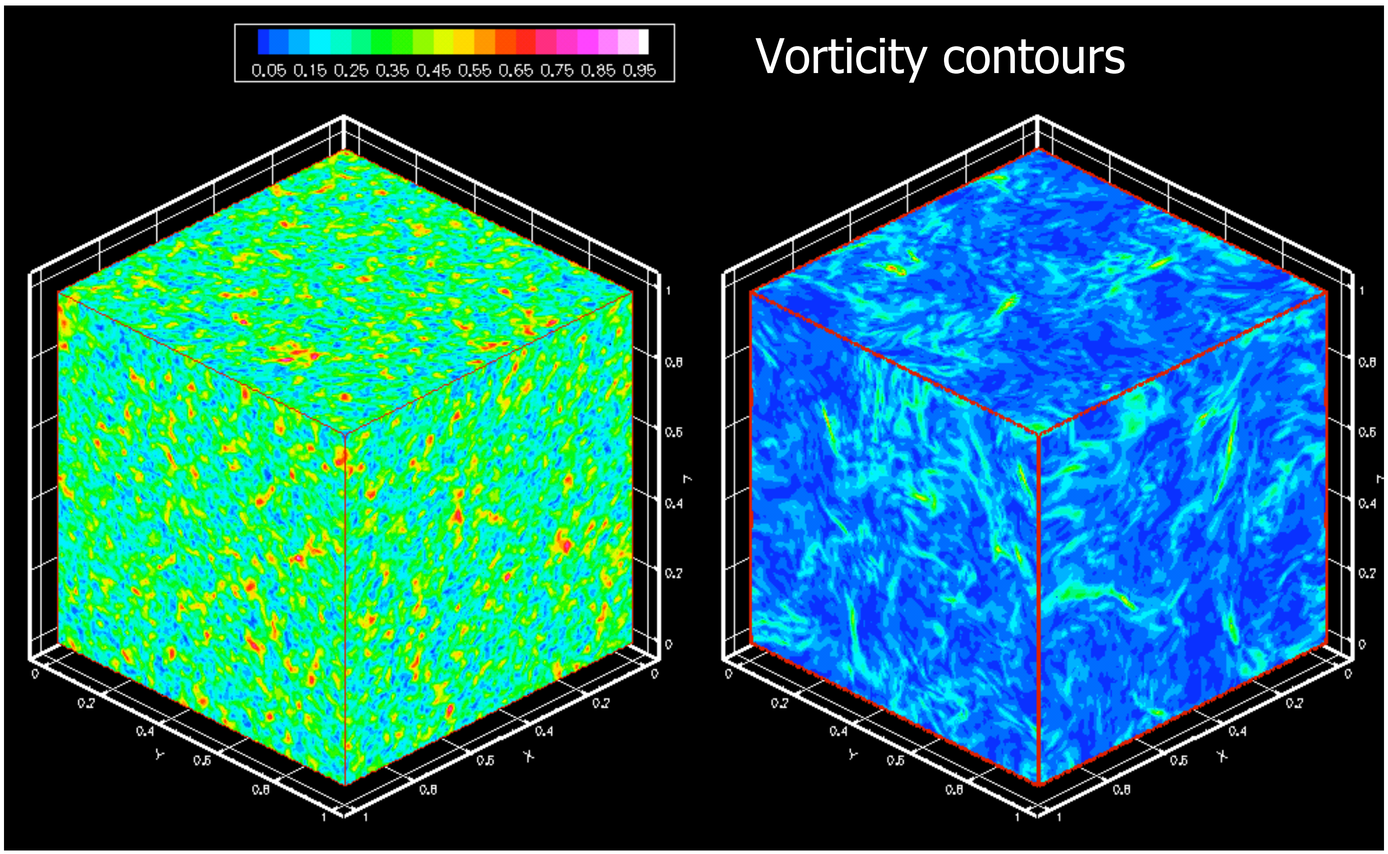


Gaussian 3D field

MMLM synthetic field (non-Gaussian)

We construct the velocity field $\hat{\mathbf{u}}_M$ by the recursive multiscale process

$$\widehat{\mathbf{u}}_n = \mathbf{S}\big(\mathbf{T}(\widehat{\mathbf{u}}_{n-1})\big) \quad ; \quad n = 1, \ldots, M$$

with the operators

$$\mathbf{T}(\widehat{\mathbf{u}}_{n-1}) = \mathbf{P} \cdot \mathcal{F} \left\{ \frac{\displaystyle\sum_{|\mathbf{r}_n(\mathbf{a}_n) - \mathbf{x}_n| \le R} W\big(\mathbf{x}_n, \mathbf{r}_n(\mathbf{a}_n)\big) \mathcal{F}^{-1}\{\widehat{G}_n \widehat{\mathbf{u}}_{n-1}\}(\mathbf{a}_n)}{\displaystyle\sum_{|\mathbf{r}_n(\mathbf{a}_n) - \mathbf{x}_n| \le R} W\big(\mathbf{x}_n, \mathbf{r}_n(\mathbf{a}_n)\big)} \right\} + \left(1 - \widehat{G}_n\right) \widehat{\mathbf{u}}_{n-1}$$

$$\mathbf{S}\big(\mathbf{z}(\mathbf{k})\big) = \left[E(k) \Big/ \sum_{|\mathbf{p}| = k} \mathbf{z}(\mathbf{p}) \cdot \mathbf{z}^*(\mathbf{p}) \right]^{1/2} \mathbf{z}(\mathbf{k})$$

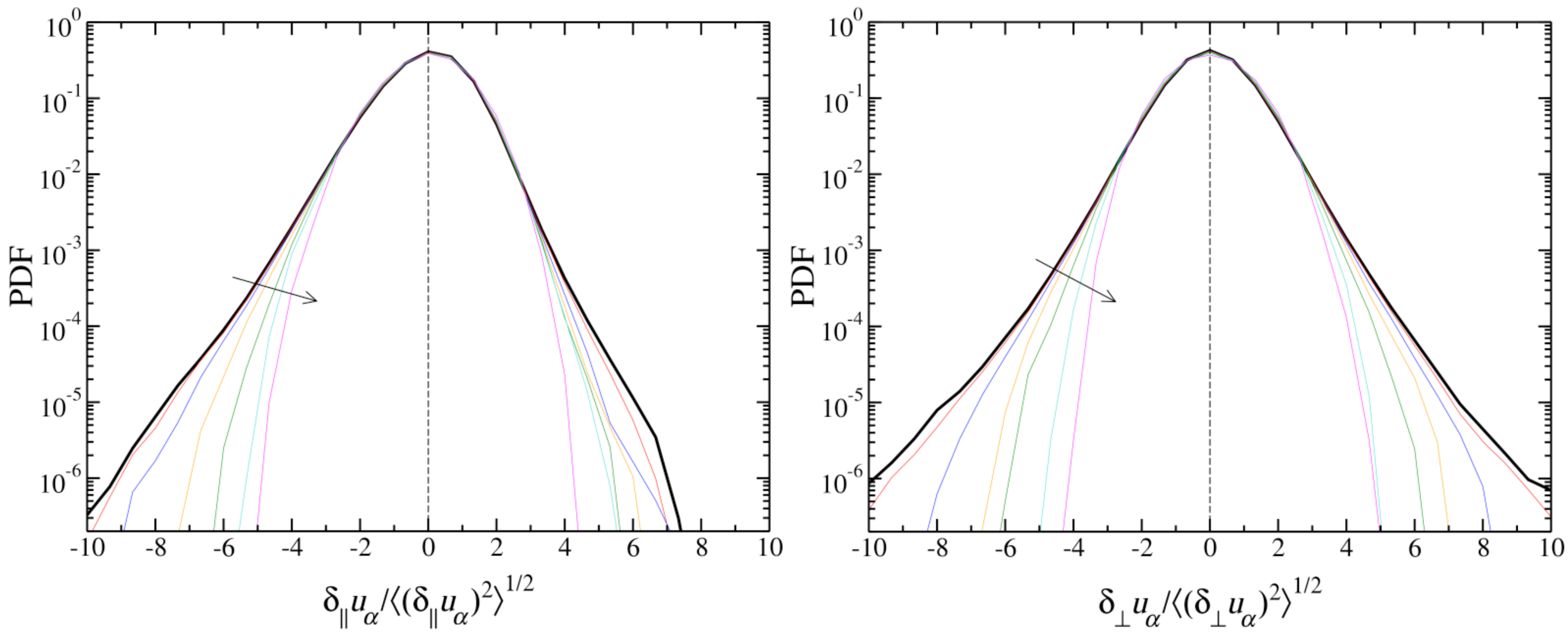
PDF of velocity increments

$$\delta_{\|} u_\alpha(r) = u_\alpha(\mathbf{x} + r\mathbf{b}^{(\alpha)}) - u_\alpha(\mathbf{x})$$

$$\delta_{\perp} u_\alpha(r) = u_\alpha(\mathbf{x} + r\mathbf{b}^{(\beta)}) - u_\alpha(\mathbf{x}); \quad \beta \neq \alpha$$

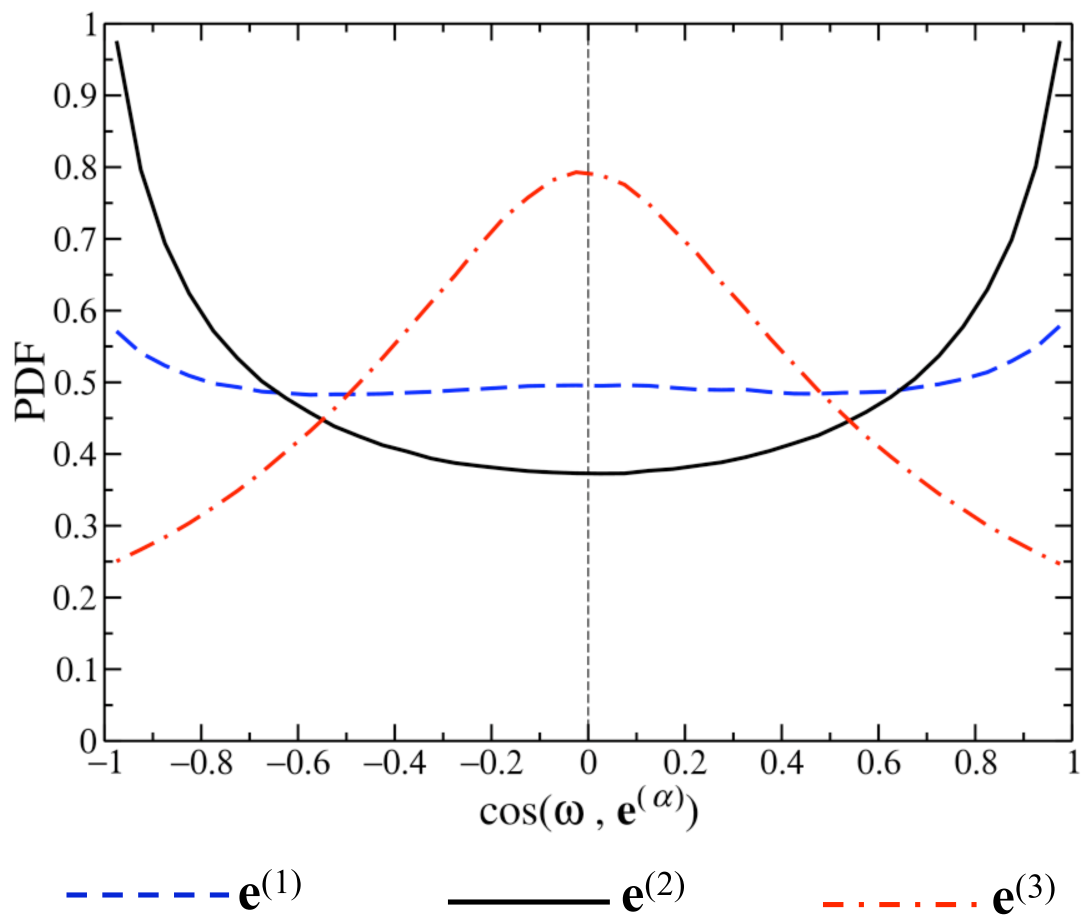
Thick line: $r/h = 1$

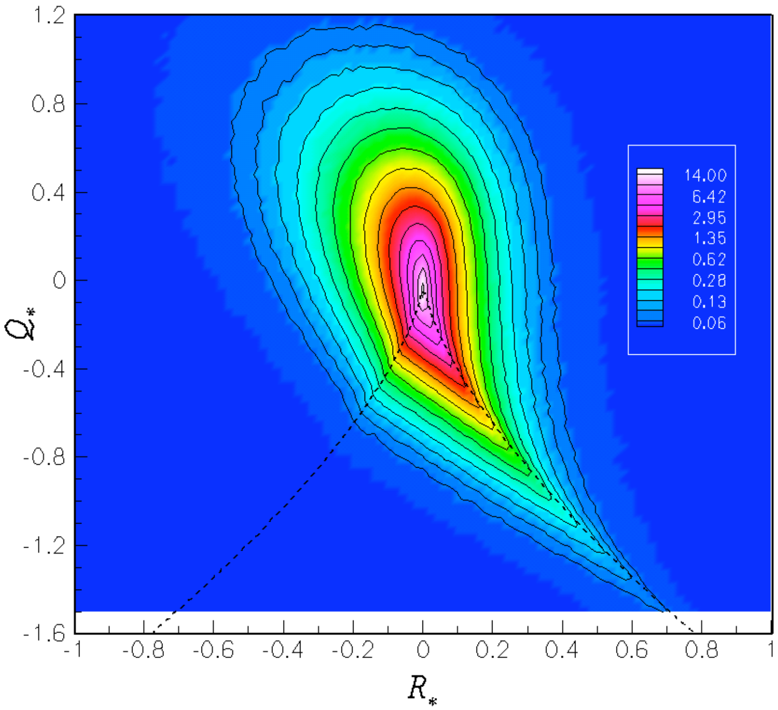
Thin lines: $r/h = 2, 4, 8, 16, 32, 64$ in the direction of the arrow



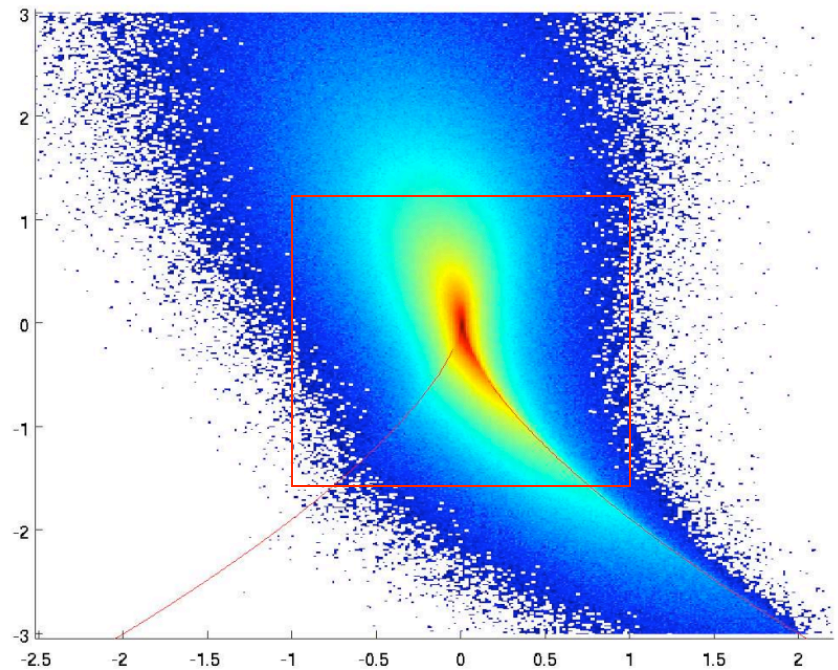
PDFs of cosines between $\boldsymbol{\omega}$ and eigenvectors $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\}$

(eigenvectors of $\mathbf{S}$ ordered according to $\lambda_1 < \lambda_2 < \lambda_3$)

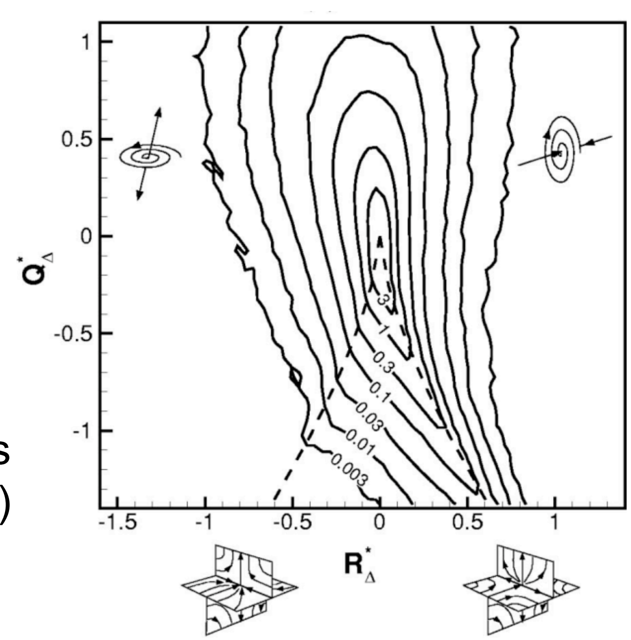




Synthetic field



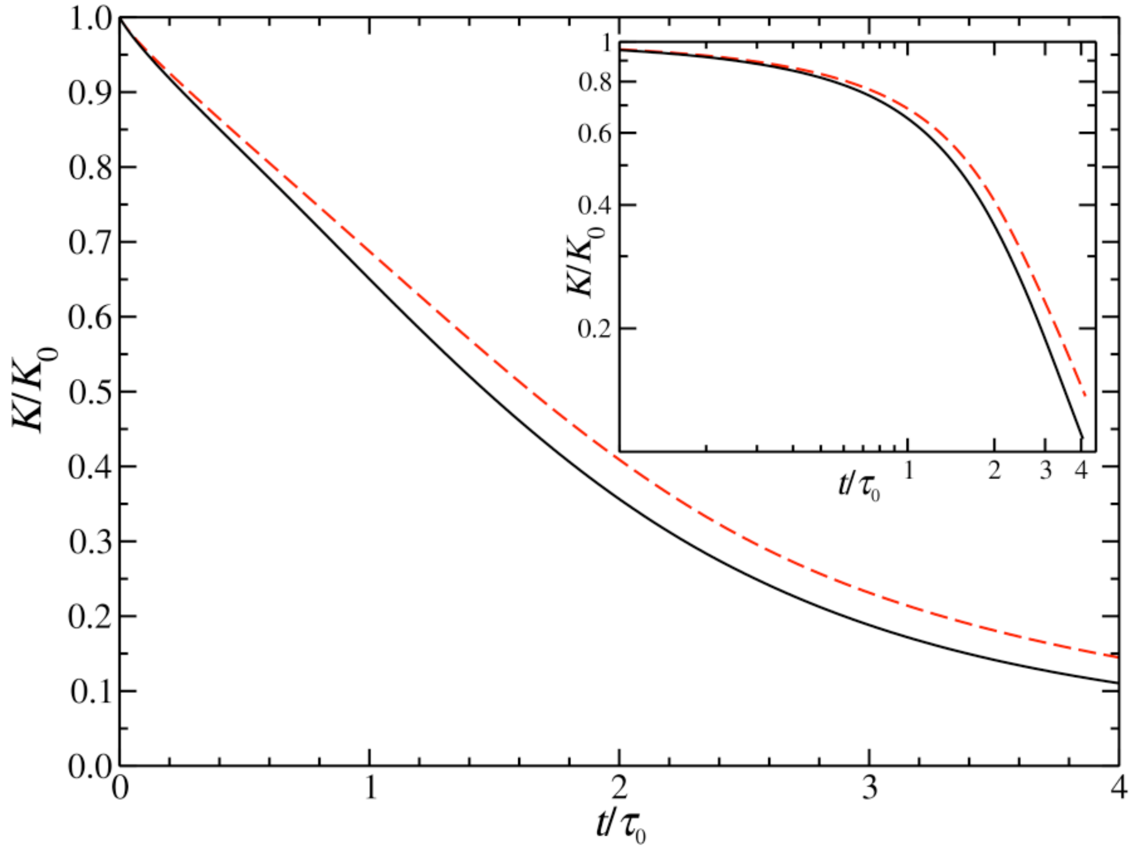
DNS (Yi Li) ↑



→ Experimental measurements
(van der Bos, Tao, Meneveau and Katz, 2002)

Tests in DNS of Decaying Turbulence

- standard pesudo-spectral method, AB2 time integration (and exact integration of viscous term), CFL ≤ 0.1 (Δt adjusted with RK2).



Time evolution of kinetic energy.

Starting from the non-Gaussian MMLM field

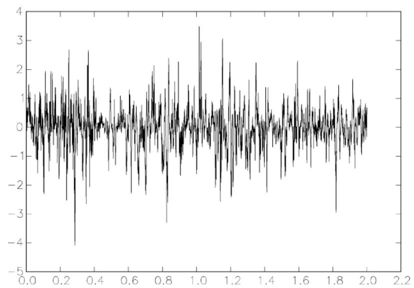
Starting from the Gaussian field

Conclusions:

- We have found a higher-dimensional variant of the Burgers' 1-D gradient-steepening equation:

$$\frac{d}{dt}\delta u = -\delta u^2 \longrightarrow \begin{cases} \frac{d}{dt}\delta u = -\left(1-\frac{2}{D}\right)\delta u^2 + \delta v^2 \\ \frac{d}{dt}\delta v = -2\delta u \delta v \\ \frac{d}{dt}\delta T = -\delta u \delta T \end{cases}$$

- Yi & Meneveau, Phys. Rev. Lett. **95**, 164502, Oct. 2005
- Yi & Meneveau JFM **558**, p. 133 (2006)



- Intermittency caused by self-stretching of larger scales, no need for exotic mechanisms…
- Quantitative predictions - stationary PDFs as function of scale: need to take into account the effects of neglected terms: pressure Hessian, subgrid force gradients, viscous force gradients. Preliminary analysis of DNS shows these effects are highly "non-trivial"…
- MHD (but passive vector field): Velocity increments parallel and transverse to **B**: same eqs as δu and δv
- δB equation: so far we have not found "simple" expressions due to more complex nature of equation for grad(**B**).