Rapidly Rotating Convection and the Geodynamo

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Acknowledgements

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Quasi-Geostrophic Approximation, QGA

Rapidly rotating convection: flows with axial length scale much longer than transverse length scale

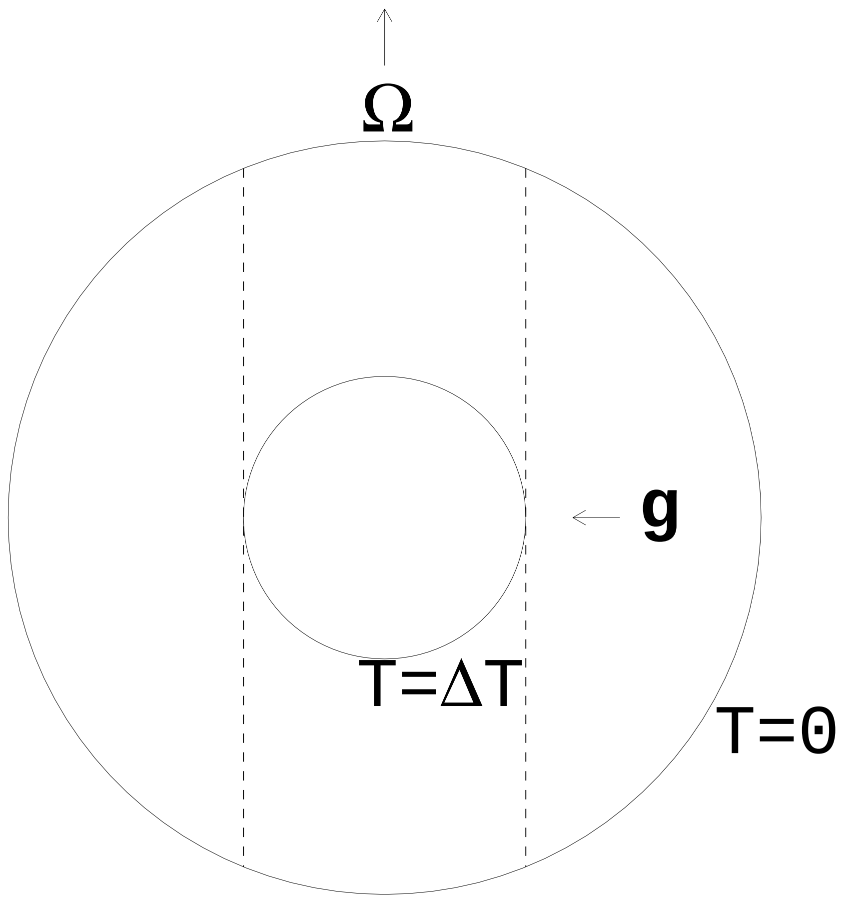
Cylindrical coordinates (s, ϕ, z)

$$\frac{\partial}{\partial s}, \ \frac{1}{s}\frac{\partial}{\partial \phi} \gg \frac{\partial}{\partial z}$$

Linear theory, sufficient to have $E = \nu/\Omega d^2 \ll 1$.

Nonlinear theory, need $Ro = U/\Omega d \ll 1$ also

Implies Rayleigh number, $R = g\alpha\Delta T d^3/\kappa\nu$, not too large



Spherical polar coordinates r, θ, ϕ

Cylindrical polar coordinates s, ϕ, z

$d = r_o - r_i$, r_o outer core radius, r_i inner core radius

QGA assumes $\partial\zeta/\partial z = 0$, $\zeta = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$. Considers only s component of gravity (cf experiments)

Axial vorticity independent of z

Justified rigorously only in Busse annulus model, where boundary slope $\eta \to 0$, $\Omega \to \infty$ so that $\eta\Omega$ is finite.

Not rigorously true even in **linear** theory, but outside TC numerical solution shows ζ has **weak** z-dependence

Follows that u_z is a linear function of z

Strength of QGA: Only requires 2D computing: able to resolve fine scales in s and ϕ

Able to reach very low E, explore parameter space

Weaknesses of QGA: (i) only works outside TC, because $\partial\zeta/\partial z \neq 0$ inside TC.

(ii) Temperature must be averaged over z. Cannot cope with thermal wind, which depends on temperature varying perpendicular to gravity. Zonal flow produced solely by Reynolds stresses. No meridional circulation

(iii) Consequence of QGA that latitudinal distribution of heat flux is specified ($F = F_T H/H_T$), $H(s)$ height of cylinder, H_T height of tangent cylinder.

Quasi-geostrophic equations

$$\frac{1}{P}\left[\frac{\partial\zeta}{\partial t}+\mathbf{u}\cdot\nabla\zeta\right]+\frac{2su_s}{E(r_o^2-s^2)}=\nabla_H^2\zeta-R\frac{\partial\theta}{\partial\phi}+\frac{2r_o}{E(r_o^2-s^2)}\mathbf{u}\cdot\hat{\mathbf{n}}|_{z\to H}$$

$$\nabla_H^2\psi=-\zeta,\quad u_s=\frac{1}{s}\frac{\partial\psi}{\partial\phi}$$

$$\frac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta=Q(s)\frac{\partial\psi}{\partial\phi}+\nabla_H^2\theta$$

ζ is the z-component of vorticity. Created by buoyancy and vortex stretching, dissipated by internal viscosity and boundary friction (Ekman suction $\mathbf{u}\cdot\hat{\mathbf{n}}$). $Q(s)$ depends on the heat sources

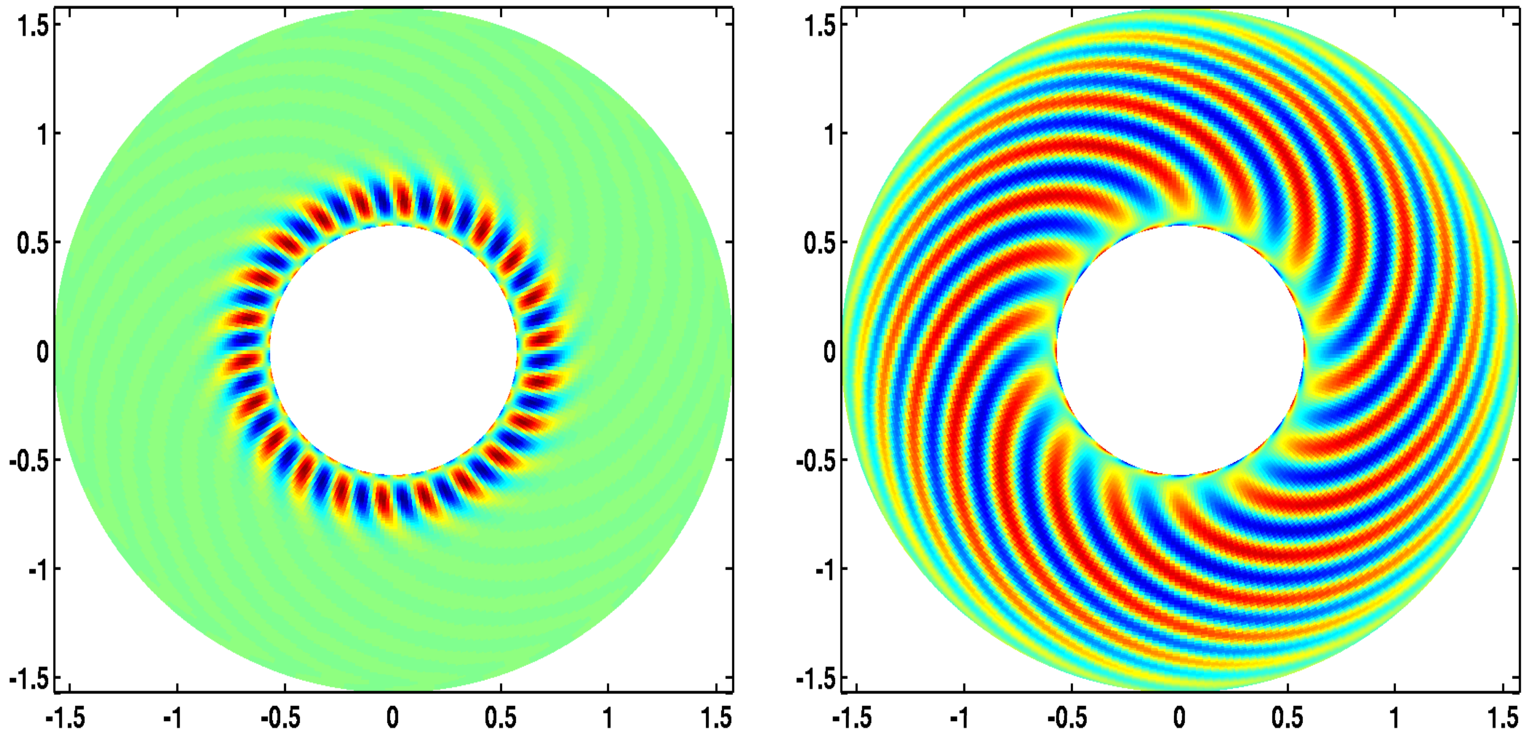
Linear theory

QGA linear theory gives results very close to fully 3D spherical case

Prandtl number dependence very similar

Disturbances $\sim \exp(im\phi)$

$R_{crit} \sim E^{-4/3}$, $m_{crit} \sim E^{-1/3}$



(a) (b)

Snapshot of the vorticity in the equatorial plane at the onset of convection, viewed from above.

(a) $P = 7.0$, $E = 6.5 \times 10^{-6}$, $R_c = 1.92 \times 10^7$, $m_c = 22$, $\omega_c = 1932$

(b) $P = 0.025$, $E = 1.95 \times 10^{-6}$, $R_c = 6.8 \times 10^6$, $m_c = 14$, $\omega_c = 318$.

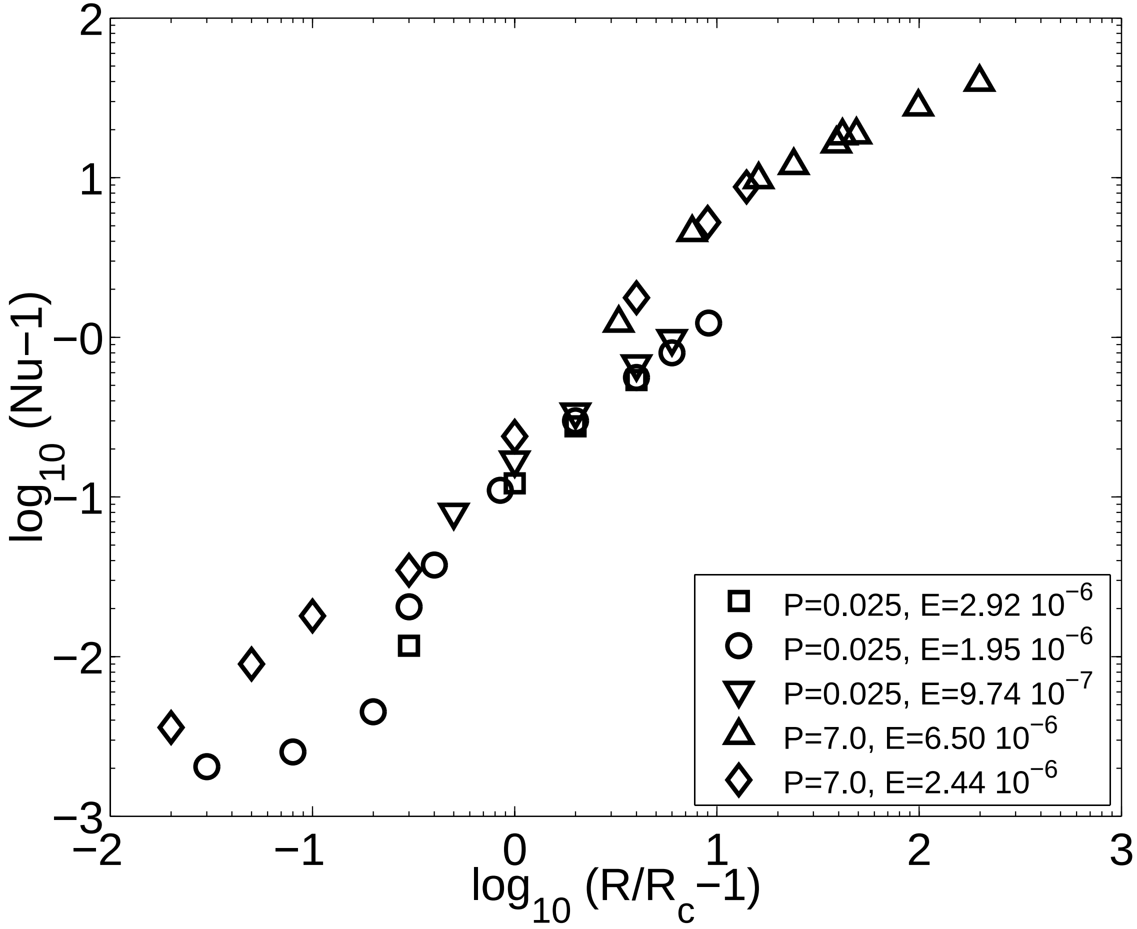
Nusselt number

Nu is (Convective + Conducted) / (Conducted) heat flux

Onset of convection delayed by rotation, but Nu grows almost linearly with R until thermal boundary layers develop.

dNu/dR initially much less at small P, as predicted by weakly nonlinear theory: due to rapid growth of zonal flow at small P

However, at larger R zonal wind changes form and Nu grows rapidly even at low P



$Nu - 1$ as a function of R/R_c for various P and E

Local Peclet number

Distinguish between convective velocity, $\hat{U}_c$, dominated by wavenumbers of order m_{crit}, and the $m = 0$ zonal flow $\hat{U}_\phi$. $\hat{U}_c$ is root mean square u_s

Length scale of convection velocity at large P does not change much from the critical wavelength $\ell_c = 2\pi r_i/m$

Local Peclet number defined as $Pe_\ell = \hat{U}_c \ell_c/\kappa$

Weakly nonlinear theory gives

$$Pe_\ell = 2\pi\sqrt{3}p^{1/2}, \quad p = \frac{RNu}{R_c} - 1$$

Flux Rayleigh number $R_Q = R(Nu - 1)$

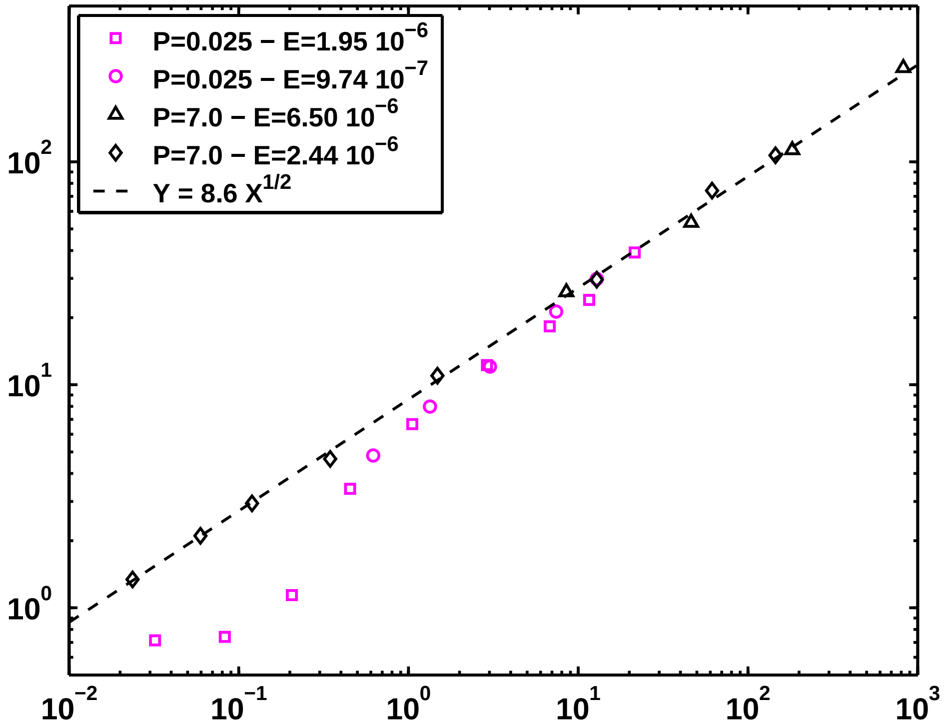
Surprisingly,

$$Pe_\ell = 2\pi\sqrt{3}p^{1/2}, \quad p = \frac{RNu}{R_c} - 1$$

gives a very good fit to the numerical data even at $R = 50R_{crit}$

At fixed E and P, this means $\hat{U}_c \sim R_q^{1/2}$

Time-dependence from weakly nonlinear theory is exactly periodic, while numeric give highly chaotic time-dependence at $50R_{crit}$



Scaling of Local Peclet number $Pe_\ell = \hat{U}_c \ell_c$, as a function of $p = RNu/R_c - 1$ for various E and P. Points are from the QGA calculations

Inertial scaling

Now we don't fix the convective length scale ℓ

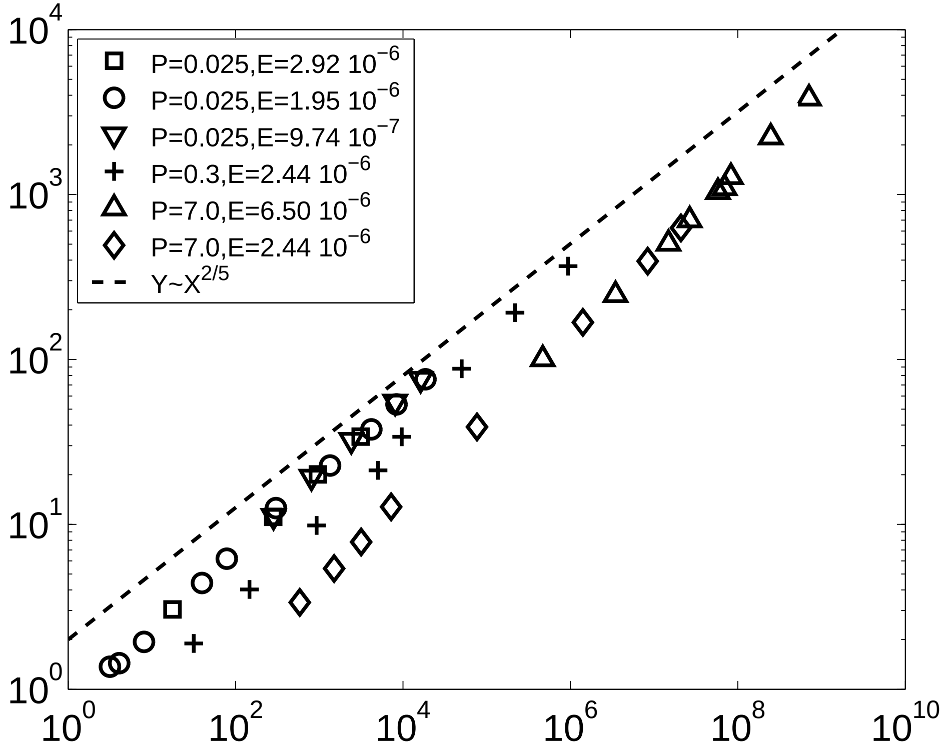
$$\frac{1}{P}\mathbf{u}\cdot\nabla\zeta \sim \frac{2su_s}{E(r_o^2 - s^2)} \sim R\frac{\partial\theta}{\partial\phi}; \quad Nu - 1 \sim \hat{U}_c\hat{\Theta}_c$$

Nu is Nusselt number and $\zeta \sim \hat{U}_c/\ell$

$$\frac{\hat{U}_c^2}{P\ell^2} \sim \frac{\hat{U}_c}{E} \sim \frac{R\hat{\Theta}_c}{\ell} \rightarrow \hat{U}_c \sim (EP)^{1/5} R_Q^{2/5}$$

$$\ell \sim E^{3/5}P^{-2/5}R_Q^{1/5} \quad \text{where } R_Q = R(Nu - 1)$$

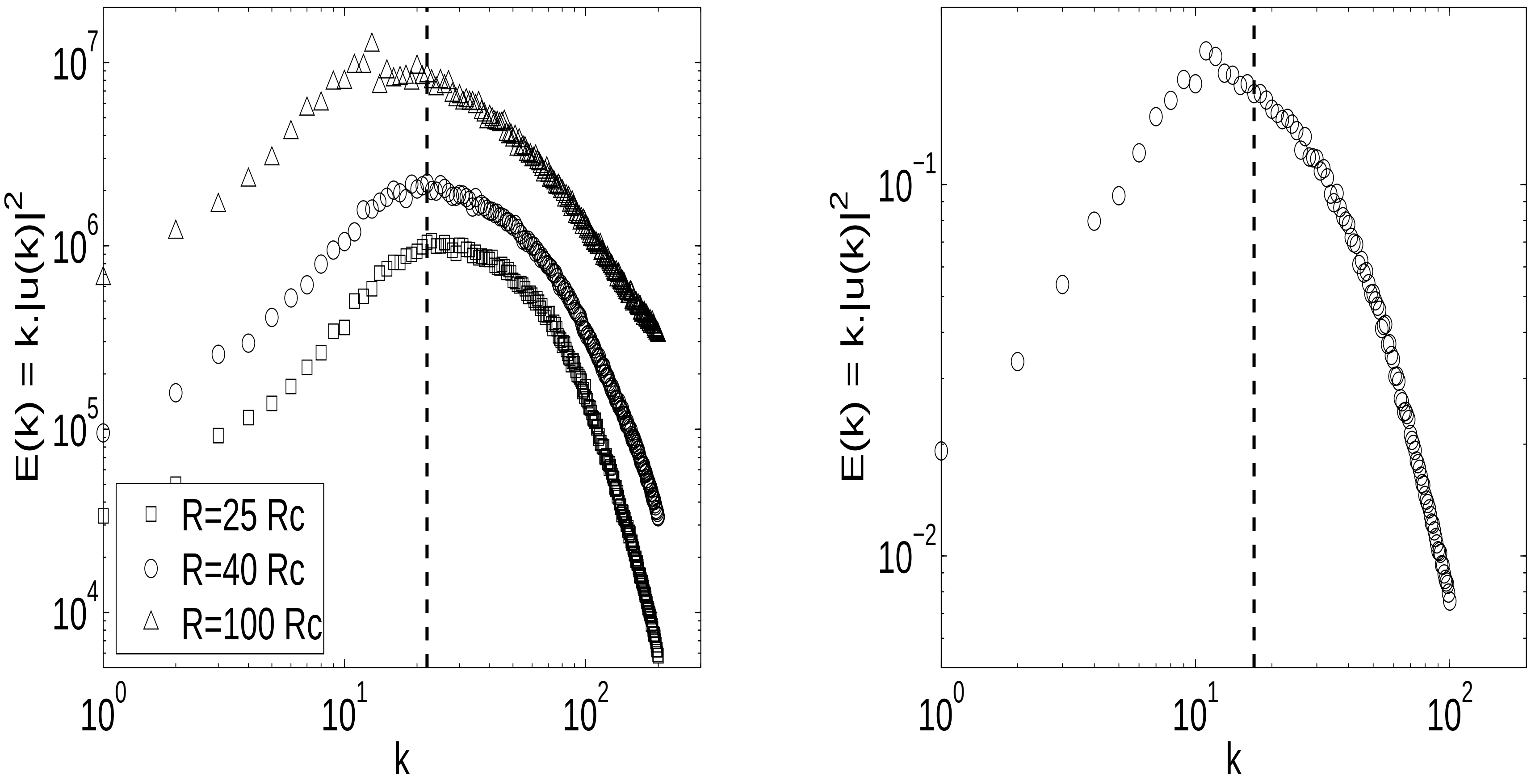
ℓ is the Rhines scale. Simulations don't support this very well; keeping ℓ at its linear critical value does better!



Test of the inertial scaling.

$\hat{U}_c$ plotted against $R_Q(EP)^{1/2}$.

Points are from the QGA calculations



Spectrum of the kinetic energy $E(k)$

with the linear critical k marked as dashed and dotted lines.

(a) $P = 7.0$, $E = 6.5 \times 10^{-6}$, $R = 25R_c$, $40R_c$ and $100R_c$;

(b) $P = 0.025$, $E = 9.74 \times 10^{-7}$, $R = 3.5R_c$

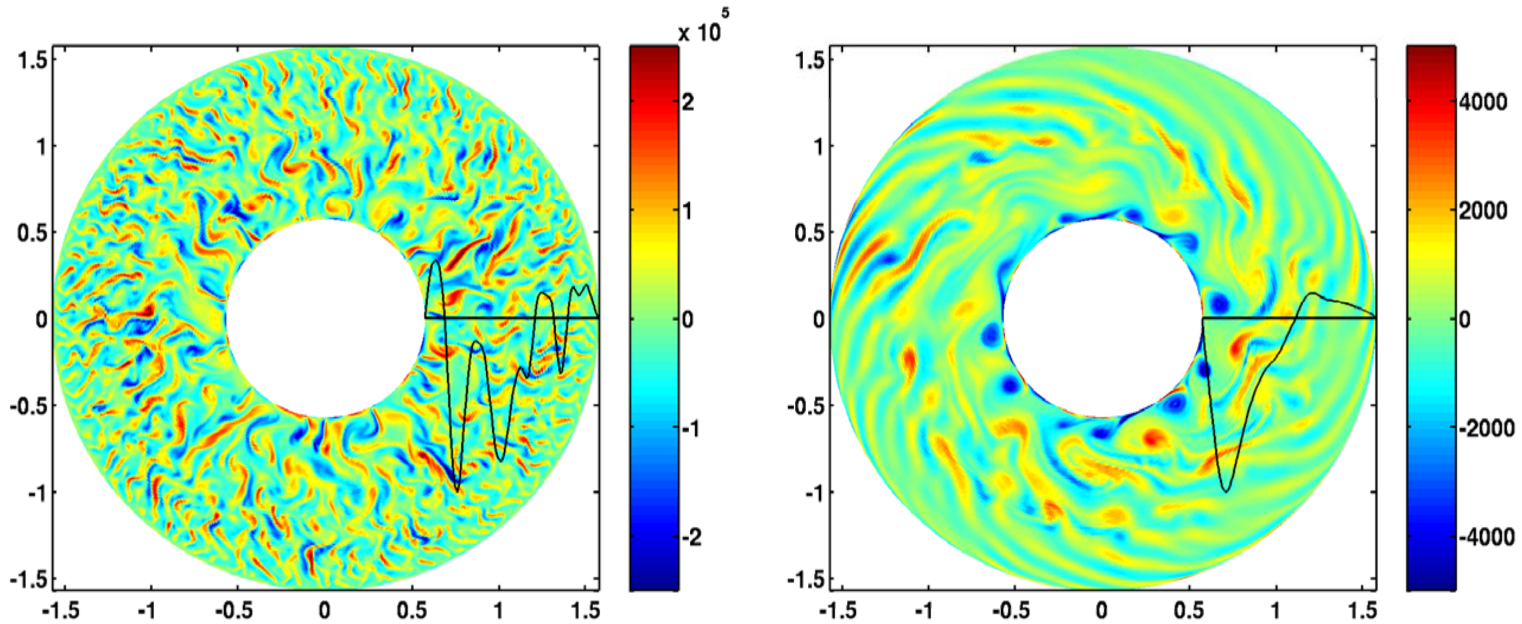
Nonlinear development: zonal flow

Spiralling convection produces a Reynolds stress; drives an axisymmetric zonal flow

$$\frac{1}{P}\frac{\partial \bar{u}_\phi}{\partial t} = -\frac{1}{s^2}\frac{\partial}{\partial s}(s^2 u'_s u'_\phi) + \left(\nabla^2 \bar{u}_\phi - \frac{\bar{u}_\phi}{s^2}\right) - \frac{E^{-1/2} r_o^{1/2}}{H^{3/2}}\bar{u}_\phi$$

Prograde spiralling leads to a differential rotation accelerating the equator, as in the solar differential rotation

Linear theory can only give this simple pattern; multiple jet formation is a strongly nonlinear process



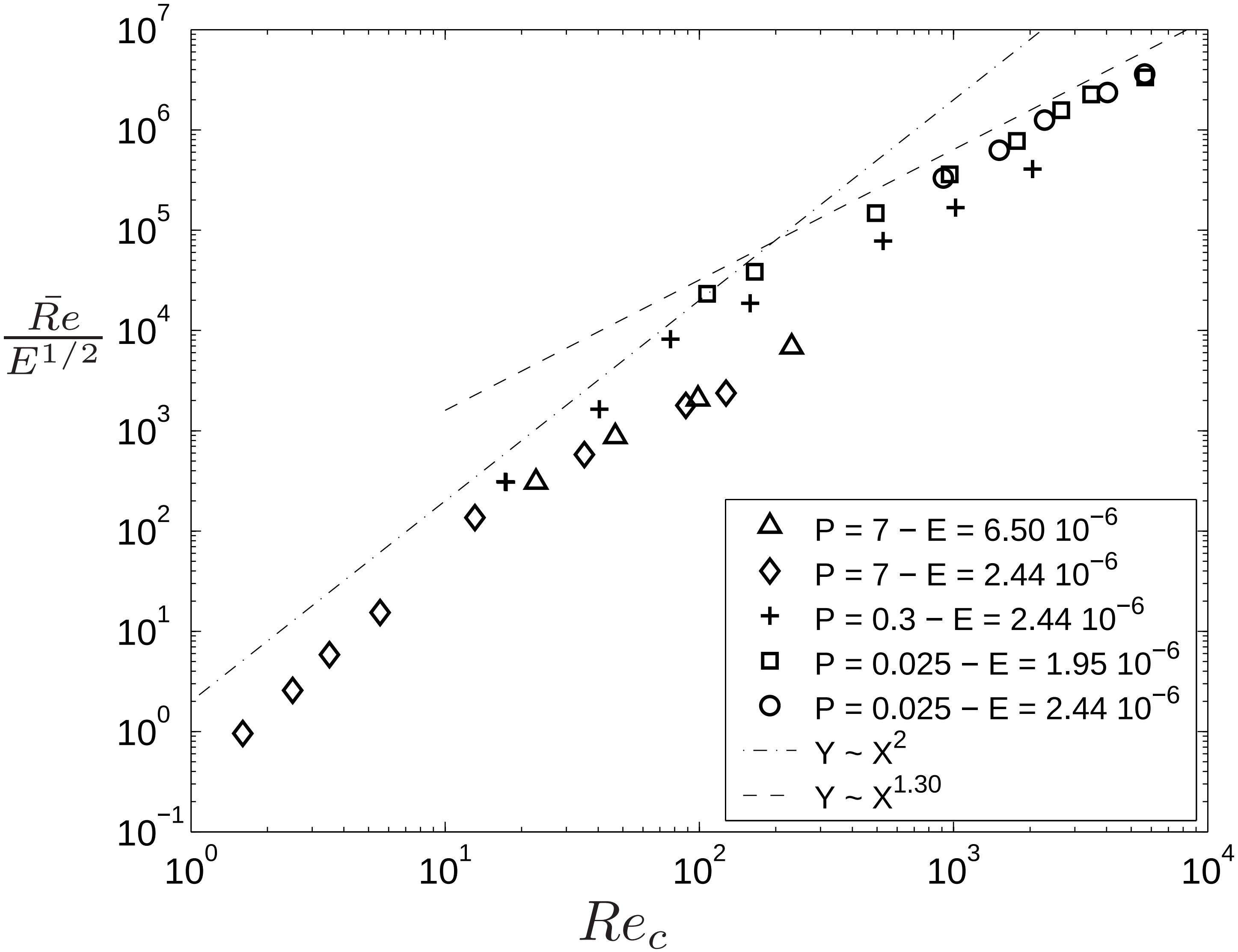
(a) (b)

Vorticity snapshots with the zonal flow profile added.

(a) $P = 7.0$, $E = 6.5 \times 10^{-6}$, $R = 42.7 R_c$

(b) $P = 0.025$, $E = 1.95 \times 10^{-6}$, $R = 3.0 R_c$

Zonal flow as a function of convective Reynolds number



What changes in convection driven dynamos?

Based on fully 3D numerical dynamo simulations, so dynamo generated field adds Lorentz force

- Zonal flows are significantly reduced by magnetic field

- Nonaxisymmetric convection is not affected as much as might be expected from magnetoconvection studies.

- Dominant azimuthal wavenumber fairly close to non-axisymmetric linear critical (non-magnetic) m

Small E, low Pm dynamo

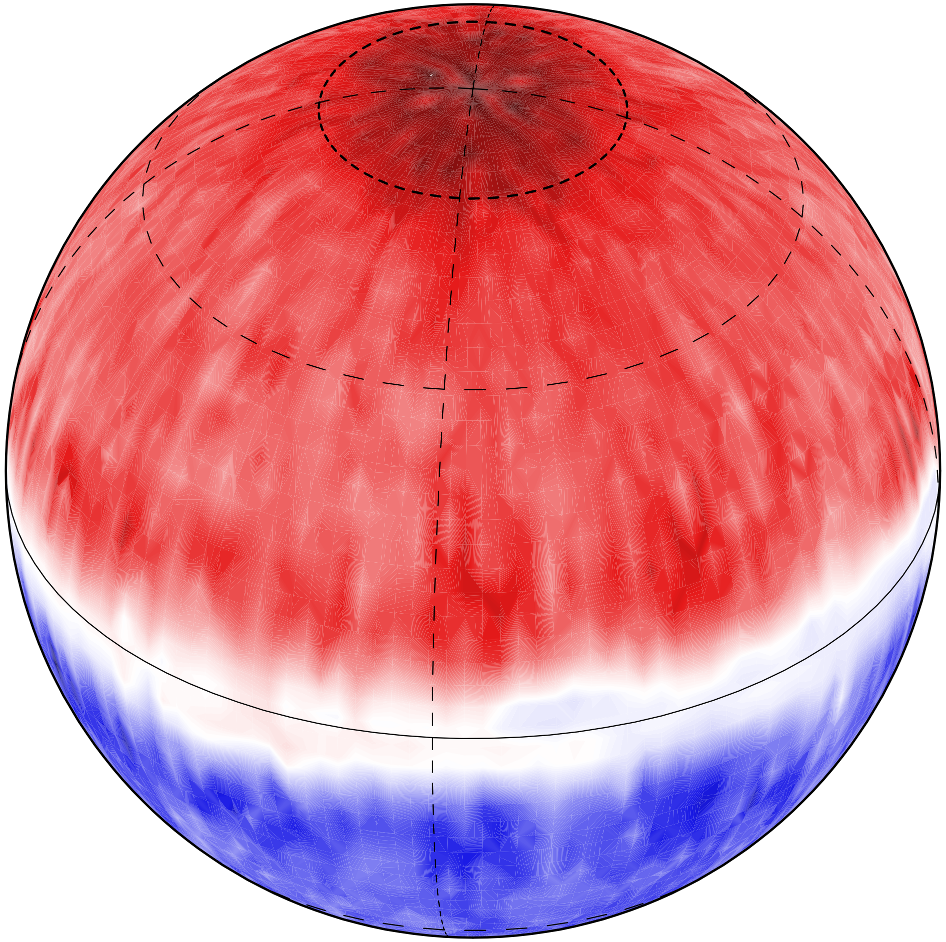
$E = 3 \times 10^{-6}$, $Pr = \nu/\kappa = 1$, $Pm = \nu/\eta = 0.1$
$R \approx 50 R_{crit}$

Field is strongly dipolar: $R_m \approx 125$, Elsasser number $\Lambda = B^2/\rho\mu\Omega\eta \approx 0.5$

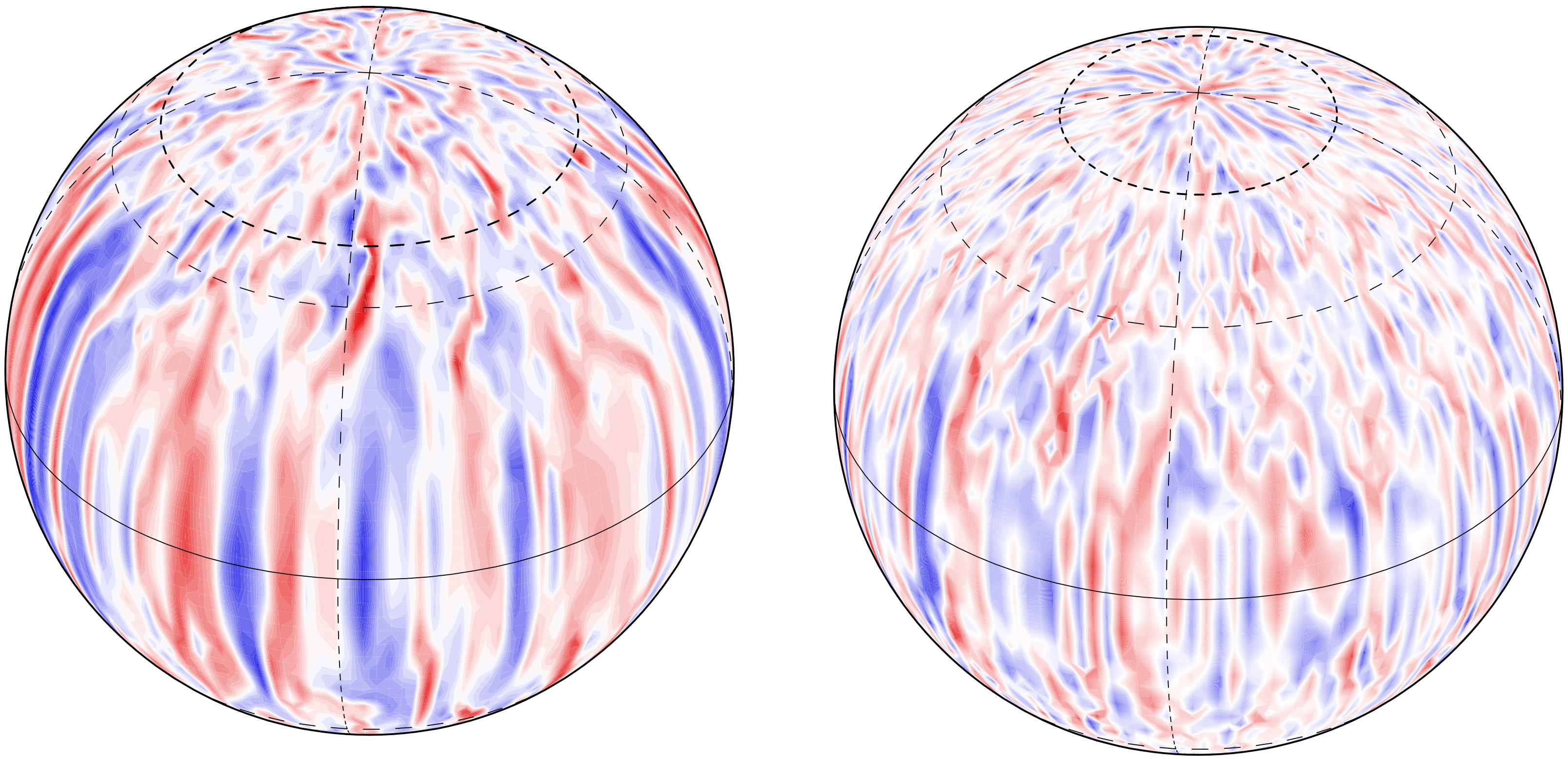
Columnar structure is evident in the dynamo simulations, particularly near the tangent cylinder, and this seems to be related to the dipole dominance. These dynamos won't reverse.

For this run, Kinetic and Magnetic energies similar:

K.E. $\approx$ M.E., and also ohmic dissipation similar to viscous dissipation.

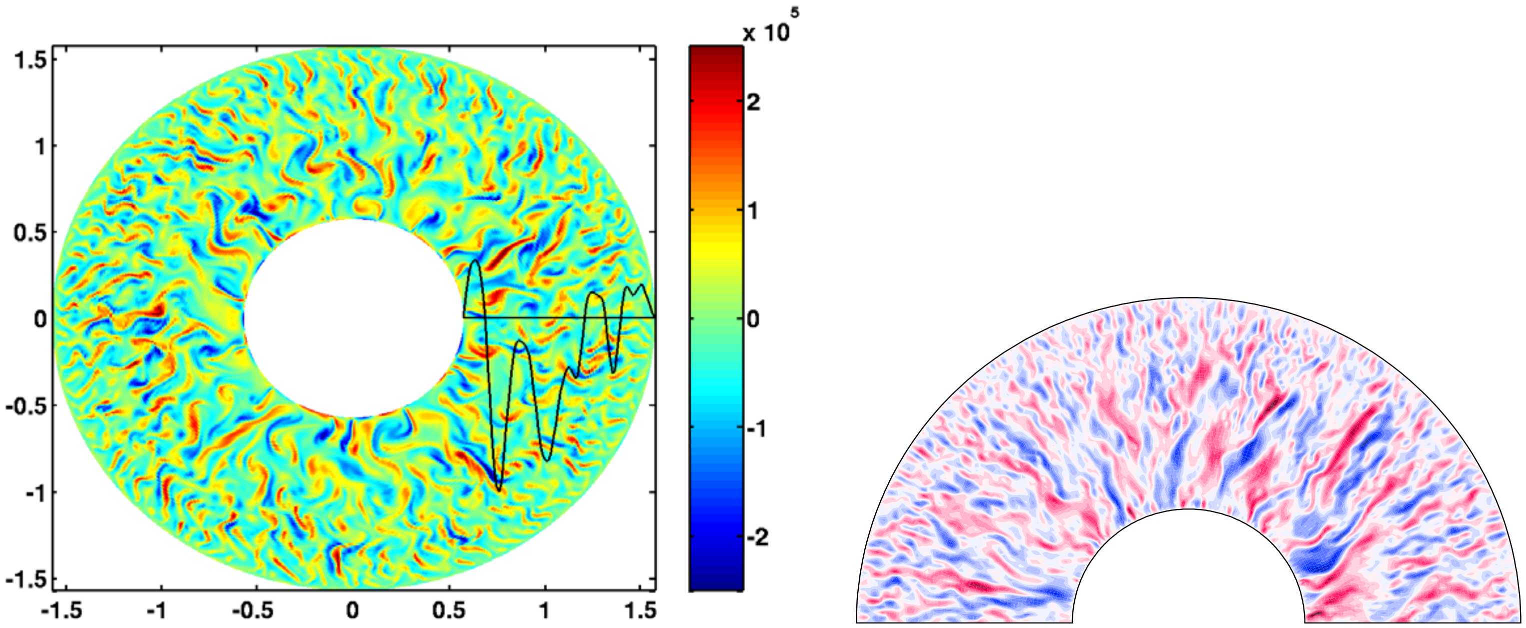


Field at the CMB



(a) (b)

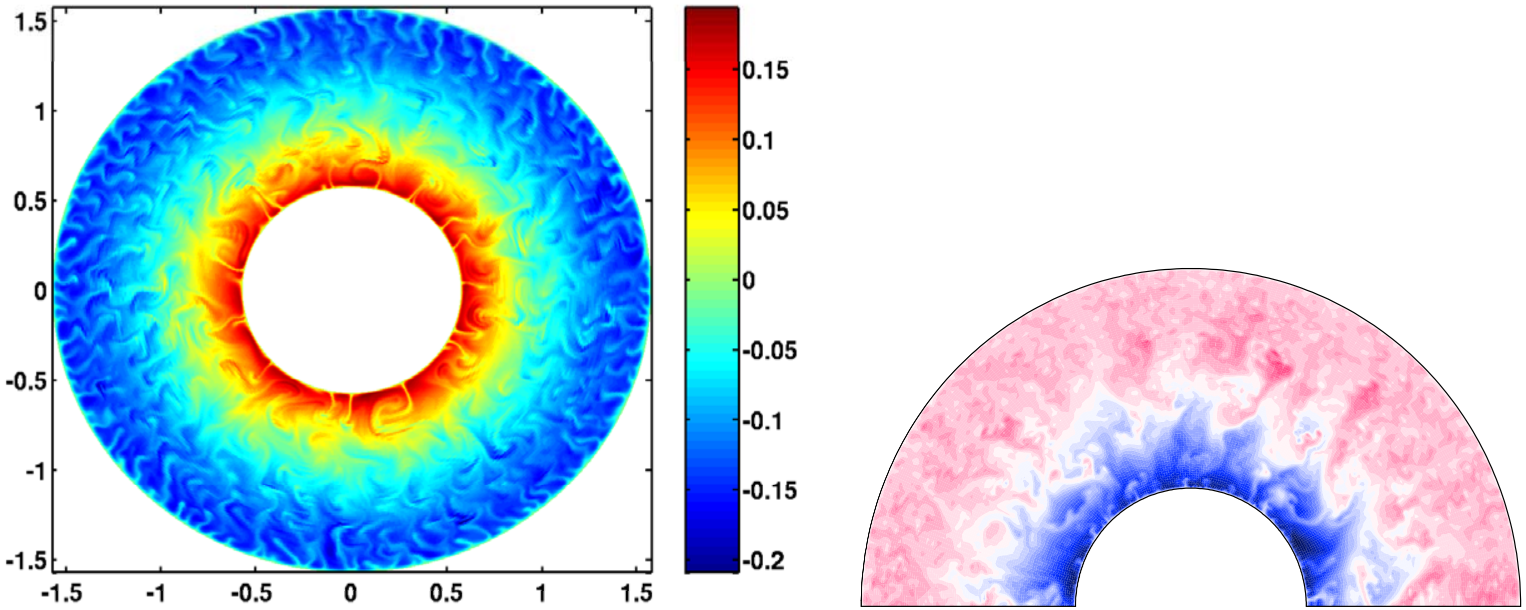
(a) Contours of u_r at $r = r_i + 0.5d$
(b) Contours of u_r at $r = r_i + 0.8d$



(a) (b)

(a) Non-magnetic vorticity snapshot.

(b) u_r in equatorial plane from dynamo simulation



(a) (b)

(a) Non-magnetic temperature snapshot.
(b) Temperature snapshot from dynamo simulation

Some relevant length scales

Magnetic length

$$L_B = \left[\int \mathbf{B}^2 \, dv / \int |\nabla \times \mathbf{B}|^2 \, dv \right]^{1/2} \sim 0.03d$$

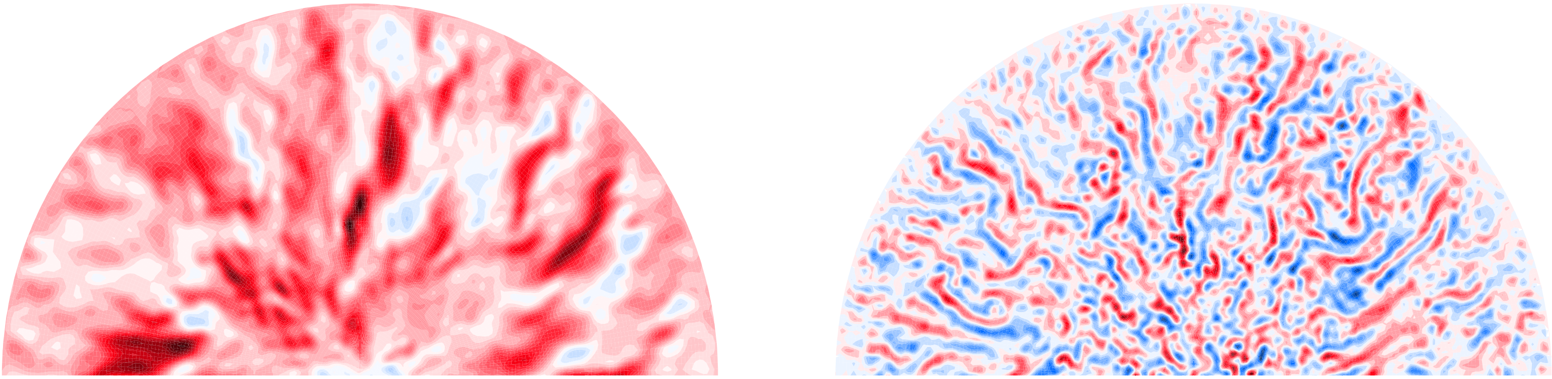
Velocity length

$$L_V = \left[\int \mathbf{u}^2 \, dv / \int |\nabla \times \mathbf{u}|^2 \, dv \right]^{1/2} \sim 0.01d$$

Linear onset length $L_{crit} = d \left(\frac{E(1+Pr)}{Pr} \right)^{1/3} \sim 0.02d$

Rhines length $L_R = \left(\frac{\hat{U}_c d}{\Omega} \right)^{1/2} \sim 0.05d$

All fairly similar! There is though evidence that $L_B > L_V$ in the plots. In geodynamo expect $L_{crit} \ll L_B$



(a) (b)

(a) Constant $z = 0.9d$ section: B_s
(b) Constant $z = 0.9d$ section: u_r

Note $Pm = \nu/\eta = 0.1$

Convective velocity

$$-2(\Omega \cdot \nabla)\mathbf{u} = \nabla \times g\alpha\theta\hat{\mathbf{r}} + \frac{1}{\rho}\nabla \times (\mathbf{j} \times \mathbf{B}), \quad \text{vorticity eqn}$$

Giving $\qquad 2\Omega\hat{U}_c/L_z \sim g\alpha\hat{\Theta}_c/L_x$

Convective heat flux per square metre $F \sim \rho c_p \hat{U}_c \hat{\Theta}_c$, eliminate $\hat{\Theta}_c$ to give

$$\hat{U}_c \sim \left(\frac{g\alpha F}{\rho c_p \Omega}\right)^{1/2} \left(\frac{L_z}{2L_x}\right)^{1/2}$$

where L_z is axial length scale and L_x is convective length scale. Inertial model gives

$$L_x/L_z \sim \left(\frac{g\alpha F}{\rho c_p \Omega^3 d^2}\right)^{1/5}$$

Giving

$$\hat{U}_c \sim \Omega d \left(\frac{g\alpha F}{\rho c_p \Omega^3 d^2}\right)^{2/5}$$

Christensen and Aubert find this is a good fit to the **magnetic** dynamo simulations

Starchenko and Jones suggested that L_x/L_z would be constant, giving

$$\hat{U}_c \sim \left(\frac{g\alpha F}{\rho c_p \Omega} \right)^{1/2}$$

which is equivalent to $\hat{U}_c \sim R_Q^{1/2}$, the other plausible scaling for non-magnetic convection.

Kinetic/Magnetic energy balance

Energy input from buoyancy per unit volume balances ohmic and viscous dissipation. If f_{ohm} is fraction of total dissipation that is ohmic

$$f_{ohm}\frac{g\alpha F}{c_p} \sim \frac{\eta|\nabla \times \mathbf{B}|^2}{\mu}$$

To get magnetic energy, need a relation between the field and the current i.e. the magnetic length scale.

Flux-rope arguments, and analysis of dynamo models (Christensen and Tilgner) suggest $L_B \sim dRm^{-1/2}$. This gives, assuming $f_{ohm} = 1$,

$$B \sim \mu^{1/2} d^{1/2} \left(\frac{g\alpha F}{c_p}\right)^{1/2} \frac{1}{\hat{U}_c^{1/2}}$$

This gives

$$\frac{K.E.}{M.E.} \sim \frac{L_z}{2L_x}\left(\frac{\hat{U}_c}{\Omega d}\right)$$

In the core, $\hat{U}_c/\Omega d \approx 3 \times 10^{-6}$. The velocity estimate suggests $L_z/2L_x \approx 100$ for a convective flux 2TW (thermal + compositional) so expect M.E. will dominate K.E. Then expect $\mathbf{j} \times \mathbf{B}$ to dominate $\mathbf{u} \cdot \nabla \mathbf{u}$, confirming ohmic > viscous dissipation.

Work done by Lorentz force balances buoyancy work

$$\mathbf{u} \cdot \mathbf{j} \times \mathbf{B} \sim \frac{g\alpha F}{c_p}$$

Above estimates give,

$$\hat{U}_c j B \sim \frac{d}{L_b} \frac{g\alpha F}{c_p}$$

too big!

Two possible ways out of this:

(i) systematic alignments of $\mathbf{u}$, $\mathbf{j}$ and $\mathbf{B}$, or

(ii) $|\mathbf{j}|$ is larger than $|\mathbf{B}|/L_B$, but current only significant over a small filling fraction $O(R_m^{-1/2})$.

Possibly some combination of the above holds!