Scaling laws for dynamos in rotating spherical shells and applications to planetary magnetic fields

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Outline of geodynamo models

Solve equations of thermal / compositional convection and magnetic induction in a rotating and electrically conducting spherical shell

- Fundamental MHD equations
- Some parameters not Earth-like
- Differences between models:
  - parameter values
  - boundary conditions
  - mode of driving convection
Successes of geodynamo modeling

Earth’s radial magnetic field at CMB

Dynamo model field at full resolution and filtered to $\ell < 13$

Tilt of dipole axis vs. time in dynamo model

$Ra = 35 \cdot Ra_{crit}$

1 million years
### Control parameters

<table>
<thead>
<tr>
<th></th>
<th>Name</th>
<th>Force balance</th>
<th>Earth value</th>
<th>Model values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ra*</td>
<td>Rayleigh number</td>
<td>Buoyancy</td>
<td>5000 x critical</td>
<td>&lt; 50 x critical</td>
</tr>
<tr>
<td>E</td>
<td>Ekman number</td>
<td>Viscosity Coriolis force</td>
<td>$10^{-14}$</td>
<td>$\geq 3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Pr</td>
<td>Prandtl number</td>
<td>Viscosity Thermal diffusion</td>
<td>0.1 - 1</td>
<td>0.1 - 10</td>
</tr>
<tr>
<td>Pm</td>
<td>Magnetic Prandtl #</td>
<td>Viscosity Magnetic diffusion</td>
<td>$10^{-6}$</td>
<td>0.06 - 10</td>
</tr>
</tbody>
</table>
Models all wrong?

- Geodynamo models successfully reproduced the properties of the geomagnetic field.
- This is surprising, because several control parameters are far from Earth values (viscosity and thermal diffusivity too large).
- Pessimistic view: Models give right answer for wrong reasons.
- Optimistic assumption: Models are already in regime where diffusion plays no first-order role.
Scaling laws

• Use many dynamo models, covering decent range in control parameters.

• Try to find laws that relate the characteristic velocity, magnetic field strength and heat flow to the control parameters.

• If diffusion plays no role, these properties must be characterised by non-dimensional numbers unrelated to any diffusivity.
Control parameters

Three diffusivities enter into the dynamo problem:
\( \nu \): viscous \hspace{1cm} \kappa \): thermal \hspace{1cm} \lambda \): magnetic

Parameters describing diffusive effects:

\( E = \frac{\nu}{\Omega D^2} \) \hspace{1cm} Ekman number
\( Pr = \frac{\nu}{\kappa} \) \hspace{1cm} Prandtl number
\( Pm = \frac{\nu}{\lambda} \) \hspace{1cm} magnetic Prandtl number

Control parameter independent of any diffusivity:
\( Ra^* = \alpha g \Delta T/(\Omega^2D) \) \hspace{1cm} modified Rayleigh number

Choose basic scales for time \((\Omega^{-1})\) and magnetic field \([\rho \mu^{1/2} \Omega D]\) that do not depend on diffusivities
Boussinesq equations

\[
\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + 2\vec{e}_z \times \vec{u} + \nabla P = E \nabla^2 \vec{u} + Ra* \frac{\vec{r}}{r_o} T + (\nabla \times \vec{B}) \times \vec{B}
\]

- Inertia
- Coriolis
- Viscosity
- Buoyancy
- Lorentz

\[
\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = E_\kappa \nabla^2 T \quad (E_\kappa = \frac{E}{Pr})
\]

- Advection
- Diffusion

\[
\frac{\partial \vec{B}}{\partial t} + \vec{u} \cdot \nabla \vec{B} = \vec{B} \cdot \nabla \vec{u} + E_\lambda \nabla^2 \vec{B} \quad (E_\lambda = \frac{E}{Pm})
\]

- Advection
- Induction
- Diffusion
Output parameters

- \( \text{Ro} = \frac{U}{(D\Omega)} \)  
  Rossby number  
  non-dim velocity
- \( \text{Lo} = \frac{B}{[\rho\mu]^{1/2}\Omega D} \)  
  Lorentz number  
  non-dim field strength
- \( \text{Nu}^* = \frac{\gamma Q_{adv}}{(\rho c_p \Delta T \Omega D^3)} \)  
  Modified Nusselt number

Relations to conventional parameters

- Rayleigh number \( \text{Ra} \)  
  \( \text{Ra}^* = \text{Ra} \frac{E^2}{\text{Pr}} \)
- Nusselt number \( \text{Nu} \)  
  \( \text{Nu}^* = (\text{Nu}-1) \frac{E}{\text{Pr}} \)
- Elsasser number \( \Lambda \)  
  \( \text{Lo} = (\Lambda \frac{E}{\text{Pm}})^{1/2} \)
- magn. Reynolds # \( \text{Rm} \)  
  \( \text{Ro} = \text{Rm} \frac{E}{\text{Pm}} \)
Data basis and case selection

- Total of 162 model cases (127 dynamos, 35 failed dynamos)
- Driven by fixed $\Delta T$, no-slip boundaries
- All parameters varied by at least two orders of magnitude
- Each run for at least 50 advection times (some much longer)
- Symmetry in longitude assumed for $E \leq 10^{-5}$

Selection criteria for scaling analysis:

- Self-sustained dynamo
- Dipole-dominated magnetic field ($f_{\text{dip}} > 0.35$)
- $E \leq 3 \times 10^{-4}$
- Fully developed convection ($\text{Nusselt}-\# > 2.0$)

$\rightarrow$ 70 model cases pass these criteria
As the Ekman number is lowered, dipolar dynamos occupy a broader region and are found at lower magnetic Prandtl #
Minimum magnetic Prandtl number

\[ P_{m\text{min}} \approx 450 \ E^{3/4} \]

Earth values:
\[ E \approx 10^{-14} \rightarrow P_{m\text{m}} \approx 2 \times 10^{-8} \]
The minimum magnetic Reynolds number for self-sustained dynamos is \( \approx 40 \), irrespective of the value of the magnetic Prandtl number.
Dynamo regimes \((0.1 \leq \text{Pr} \leq 10)\)

Inertial vs. Coriolis force:

Local Rossby number \(\text{Ro}_L\) calculated with mean length scale \(\ell\) in the kinetic energy spectrum

\[
\text{Ro}_L = \frac{U}{(\ell\Omega)}
\]

When inertia dominates the dipolar regime breaks down
Flux-based Rayleigh number

Modified Rayleigh # based on buoyancy flux:

$$Ra_Q^* = \frac{\gamma g_0 Q_{buoy}}{(\rho c_p \Omega^3 D^2)}$$

$$Ra_Q^* = Ra^* Nu^* = Ra (Nu-1) E^3 Pr^{-2}$$

To very good approximation, $Ra_Q^*$ is equivalent to the power generated by buoyancy forces.

$$\gamma = \frac{(1-\eta^2)}{4\pi \eta} : \text{geometry factor}$$
Scaling of Nusselt number

Use of modified "diffusionless" parameters allows to collapse the data and express the dependence by a single power-law. Compared to non-rotating convection, the exponent is very large ($\approx 0.5$).
Dimensional heat flow

For an exponent of 0.5:

\[ Q_{\text{adv}} \sim \gamma \alpha g \rho c_p D^2 \Delta T^2/ \Omega \]

Advected heat flow is independent of thermal conductivity.

Earth‘s core: \( Q_{\text{adv}} = 2 \text{ TW} \quad \rightarrow \quad \Delta T_{\text{superadiab}} \approx 1 \text{ mK} \)

Requirement for validity is probably that the thermal boundary layer thickness is larger than the Ekman layer thickness, which in terms of non-dimensional parameters requires \( Ra^*_Q < 400 \text{ E Pr}^{-2} \). Satisfied in the numerical models and perhaps for thermal convection in the Earth‘s core.
Dependence on magnetic Prandtl number?

Velocity: Rossby number

Ro ~ Ra_Q^{2/5}

(Rossby number vs. Rayleigh number)

- E=3x10^{-6}
- E=1x10^{-5}
- E=3x10^{-5}
- E=1x10^{-4}
- E=3x10^{-4}

- Slope = 0.414
- Const = 0.846
- Q = 0.179
- Corr = 0.9874
Velocity scaling II

\[ \text{Ro} \sim \text{Ra}^* Q^{0.43} \text{Pm}^{-0.13} \]

Two-parameter regression reduces misfit by factor 2.5.

But is dependence on the magnetic Prandtl number definitely required?
Scaling ohmic dissipation time

\[ \tau_{\text{diss}} = \frac{\text{Magnetic energy}}{\text{Ohmic dissipation}} \]

\[ \tau_{\text{diss}} \sim Rm^{-1} \]

\[ \tau_{\text{diss}} \sim Rm^{-7/6} Pm^{1/6} \]

Which law is right?

Christensen & Tilgner, 2004
Karlsruhe laboratory dynamo

Magnetic Prandtl number
≈ 1 in numerical models
≈ $10^{-5}$ for liquid sodium
Scaling ohmic dissipation time

\[ \tau_{\text{diss}} \sim R_m^{-1} \]

Better agreement with the simple scaling law

\[ \tau_{\text{diss}} \sim R_m^{-7/6} \quad P_m^{1/6} \]
Velocity Scaling: stay simple

Let us assume that in general the magnetic Prandtl number has no influence (at least in the limit $Pm \ll 1$)

\[ Ro = 0.85 \left(\frac{Ra^*}{Q}\right)^{2/5} \]
Rayleigh number of the core

Core velocity estimated from secular variation:
Large-scale flow $\sim 0.5$ mm/sec  Total rms-flow $\sim 1$ mm/sec

$Ro \sim 6 \times 10^{-6}$

Core Rayleigh number $\rightarrow Ra^*_Q \sim 3 \times 10^{-13}$

The conventional Rayleigh number is $10^{23}$ ($5000 \times Ra_c$)
Light element flux and inner core growth rate

\[ Ra^*_Q \sim 3 \times 10^{-13} \rightarrow Q_{buoy} \sim 3 \times 10^4 \text{ kg s}^{-1} \]

If convection is driven entirely by compositional flux associated with inner core growth, the growth rate is obtained as

\[ \frac{dR_{ic}}{dt} = \frac{Q_{buoy}}{4\pi r_i^2 \Delta \rho_{ic}}. \]

\[ \rightarrow \frac{dR_{ic}}{dt} \sim 0.1 \text{ mm/yr} \]

Implication: Inner core is old:

3.5 ± 1.5 Gyr
What controls the strength of the magnetic field?

Magnetostrophic balance often associated with an Elsasser number $\Lambda = B^2/\mu \eta \rho \Omega \sim O(1)$

In the numerical models, the Elsasser number varies widely.

→ Force balance not magnetostrophic, or $\Lambda$ not good measure.

Alternative scaling: Magnetic field strength based on available power?
Power-controlled field strength

Power driving convection $\sim Ra_Q^*$

Magnetic energy density $\sim 1/2 \ Lo^2$

Ratio magn. energy / dissipation $= \tau_{diss}$

**Prediction:**

If $Ro \sim Ra_Q^{*2/5}$ and $\tau_{diss} \sim Rm^{-1} \sim Pm (Ro E)^{-1}$ then

$$Lo / \sqrt{f_{ohm}} \sim Ra_Q^{*0.3}$$

where $f_{ohm}$ is the relative fraction of energy dissipated by ohmic losses (30 – 85% in the models)
Magnetic Field Scaling

\[ L_0 \sim R_{\alpha Q}^{0.34} \]

Empirical fit is close to predicted dependence
Assume $f_{\text{ohm}} \approx 1$ in the core

In dimensional form the magnetic field strength is

$$B \sim \mu^{1/2} \rho^{1/6} \left( g Q_{\text{buoy}} / D \right)^{1/3}$$

$B$ independent of conductivity and rotation rate
Strength of core magnetic field

With the estimated buoyancy flux of $3.5 \times 10^4$ kg/s the predicted magnetic field strength in the core is

$$B \approx 1.2 \text{ mT}$$

Compare to:
- „Observed“ field at CMB: 0.39 mT ($\ell < 13$)
- $B_s \approx 0.4$ mT inside core from torsional oscillations
  (Zatman & Bloxham, 1997)
How robust are the scaling laws?

Tests:

• Exclude cases with $E \geq 10^{-4}$ from fit
  → Results virtually unchanged

• Exclude cases with $Ro_\ell > 0.05$ (high inertia) from fit
  → Results virtually unchanged

• Attempt general least-squares fit of the form:
  $$Y = A \ Ra_Q^{*\alpha} \ Pm^\beta \ E^\gamma \ Pr^\delta$$
  where $Y$ stands for $Nu^*$, $Ro$, or $Lo/\sqrt{f_{ohm}}$
  → Exponents for $E$ and $Pr$ very small ($<0.03$)
Other planets: Jupiter

For the observed excess heat flow (5.4 W/m²) and reasonable estimates for other relevant parameters, the magnetic field in the dynamo region is $B \approx 8 \text{ mT}$.

This is in agreement with Jupiter’s surface field strength roughly 10 times the Earth value.
Mercury

Surface field strength $\approx 1/100$ Earth value.

Cannot be explained by low buoyancy flux, because it would imply magnetic Reynold number below critical.
Conclusions

• Numerical dynamo models reproduce observed properties of the geomagnetic field, even though viscous and thermal diffusivity are far too large.

• Scaling laws that are independent of diffusivities fit the model results well and lead to reasonable predictions for the Earth.

• This supports the view that the models are already in a regime similar to that of the Earth’s core.

• The range of validity of laws must be explored. Is there a dependence on the magnetic Prandtl number? Laboratory dynamo experiments will help to decide.
Scaling of local Rossby number

Decent fit possible, but involves all four control parameters

\[ \text{Ro}_L \sim \text{Ra}_Q^{1/2}E^{-1/3}\text{Pr}^{1/5}\text{Pm}^{-1/5} \]

Predicted Earth value is \( \text{Ro}_L \approx 0.1 - 0.2 \), very close to the transition point between dipolar and non-dipolar dynamos.
Enstrophy balance

Vorticity $\omega$

Enstrophy $= \omega \cdot \omega$ „Energy of vorticity“

$$\frac{\partial \omega^2}{\partial t} =$$

$$- \nabla \times [\omega \times \vec{u}] \cdot \omega - 2 \frac{\partial \tilde{\omega}}{\partial z} \cdot \omega + E \nabla^2 \omega \cdot \omega + \frac{Ra^*}{r_o} \nabla \times \tilde{T} \cdot \omega + \nabla \times [(\nabla \times \vec{B}) \times \vec{B}] \cdot \omega$$

Inertia   Coriolis   Viscosity   Buoyancy   Lorentz
Sources & sinks of enstrophy

Buoyancy is main source and Lorentz force and viscosity is main sink
Enstrophy budget

Coriolis ∼ buoyancy
Lorentz variable
Inertia large in non-dipolar case