Scaling laws for dynamos in rotating spherical shells and applications to planetary magnetic fields

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Outline of geodynamo models

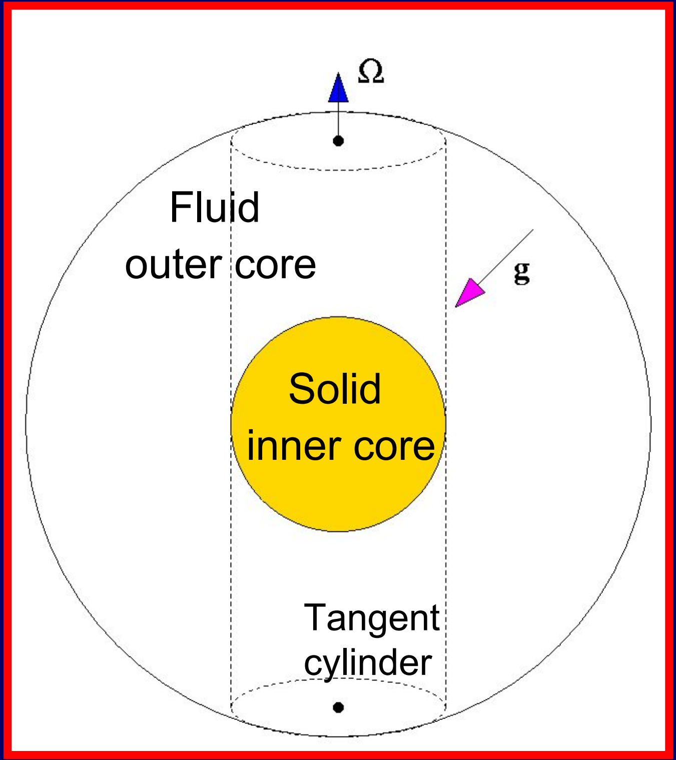
Solve equations of thermal / compositional convection and magnetic induction in a rotating and electrically conducting spherical shell

- **Fundamental MHD equations**
- **Some parameters not Earth-like**
- **Differences between models:**

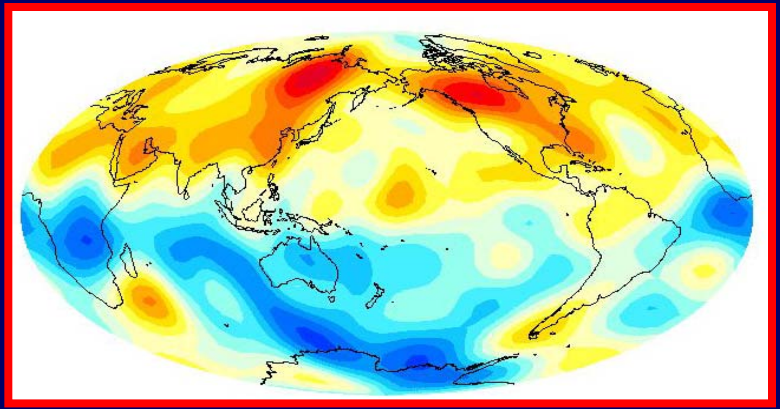
  - parameter values

  - boundary conditions

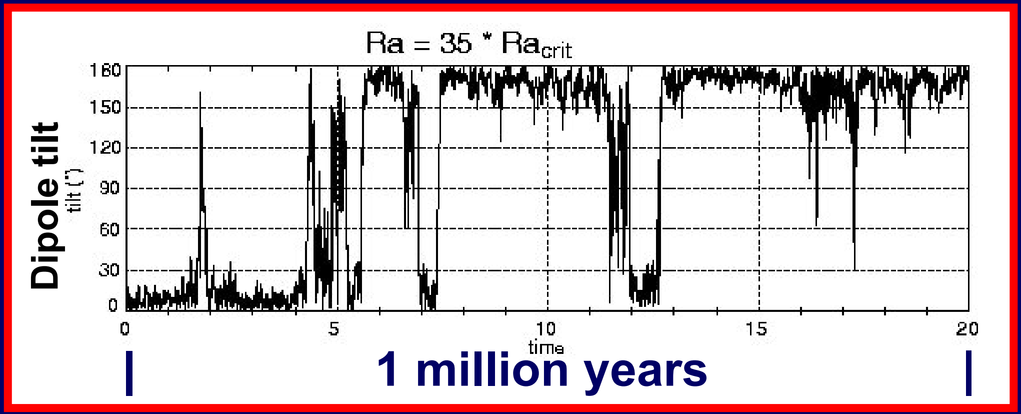
  - mode of driving convection



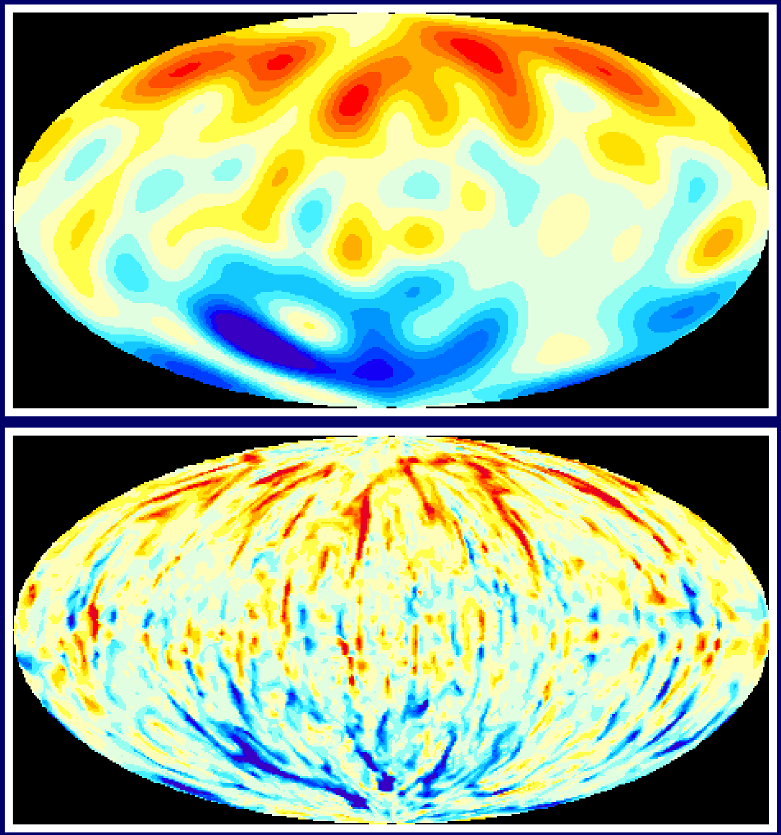
Successes of geodynamo modeling



Earth's radial magnetic field at CMB



Tilt of dipole axis vs. time in dynamo model



Dynamo model field at full resolution and filtered to $\ell < 13$

Control parameters

	Name	Force balance	Earth value	Model values
Ra*	Rayleigh number	Buoyancy Retarding forces	5000 x critical	< 50 x critical
E	Ekman number	Viscosity Coriolis force	10^{-14}	$\geq 3 \times 10^{-6}$
Pr	Prandtl number	Viscosity Thermal diffusion	0.1 - 1	0.1 - 10
Pm	Magnetic Prandtl #	Viscosity Magnetic diffusion	10^{-6}	0.06 - 10

Models all wrong ?

- Geodynamo models successfully reproduced the properties of the geomagnetic field
- This is surprising, because several control parameters are far from Earth values (viscosity and thermal diffusivity too large)
- Pessimistic view: Models give right answer for wrong reasons
- Optimistic assumption: Models are already in regime where diffusion plays no first-order role

Scaling laws

- Use many dynamo models, covering decent range in control parameters.
- Try to find laws that relate the characteristic velocity, magnetic field strength and heat flow to the control parameters
- If diffusion plays no role, these properties must be characterised by non-dimensional numbers unrelated to any diffusivity

Control parameters

Three diffusivities enter into the dynamo problem:

ν: viscous κ: thermal λ: magnetic

Parameters describing diffusive effects:

$E = \nu/\Omega D^2$ Ekman number

$Pr = \nu/\kappa$ Prandtl number

$Pm = \nu/\lambda$ magnetic Prandtl number

Control parameter independent of any diffusivity:

$Ra^* = \alpha g \Delta T/(\Omega^2 D)$ modified Rayleigh number

Choose basic scales for time (Ω^{-1}) and magnetic field ($[\rho\mu]^{1/2}\Omega D$) that do not depend on diffusivities

Boussinesq equations

$$\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) + 2\vec{e}_z \times \vec{u} + \vec{\nabla}P = \boxed{E}\,\nabla^2\vec{u} + \boxed{Ra^*}\,\frac{\vec{r}}{r_o}T + (\vec{\nabla}\times\vec{B})\times\vec{B}$$

Inertia **Coriolis** **Viscosity** **Buoyancy** **Lorentz**

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\vec{\nabla}T = \boxed{E_\kappa}\,\nabla^2 T \qquad \left(E_\kappa = \frac{E}{Pr}\right)$$

Advection **Diffusion**

$$\frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{B} = \vec{B}\cdot\vec{\nabla}\vec{u} + \boxed{E_\lambda}\,\nabla^2\vec{B} \qquad \left(E_\lambda = \frac{E}{Pm}\right)$$

Advection **Induction** **Diffusion**

Output parameters

- Ro $= U/(D\Omega)$ Rossby number non-dim velocity
- Lo $= B/[\rho\mu]^{1/2}\Omega D$ Lorentz number non-dim field strength
- Nu* $= \gamma Q_{adv}/(\rho c_p \Delta T \Omega D^3)$ Modified Nusselt number

Relations to conventional parameters

Rayleigh number	Ra	$Ra^* = Ra\, E^2/Pr$
Nusselt number	Nu	$Nu^* = (Nu-1)\, E/Pr$
Elsasser number	Λ	$Lo = (\Lambda\, E/Pm)^{1/2}$
magn. Reynolds #	Rm	$Ro = Rm\, E/Pm$

Data basis and case selection

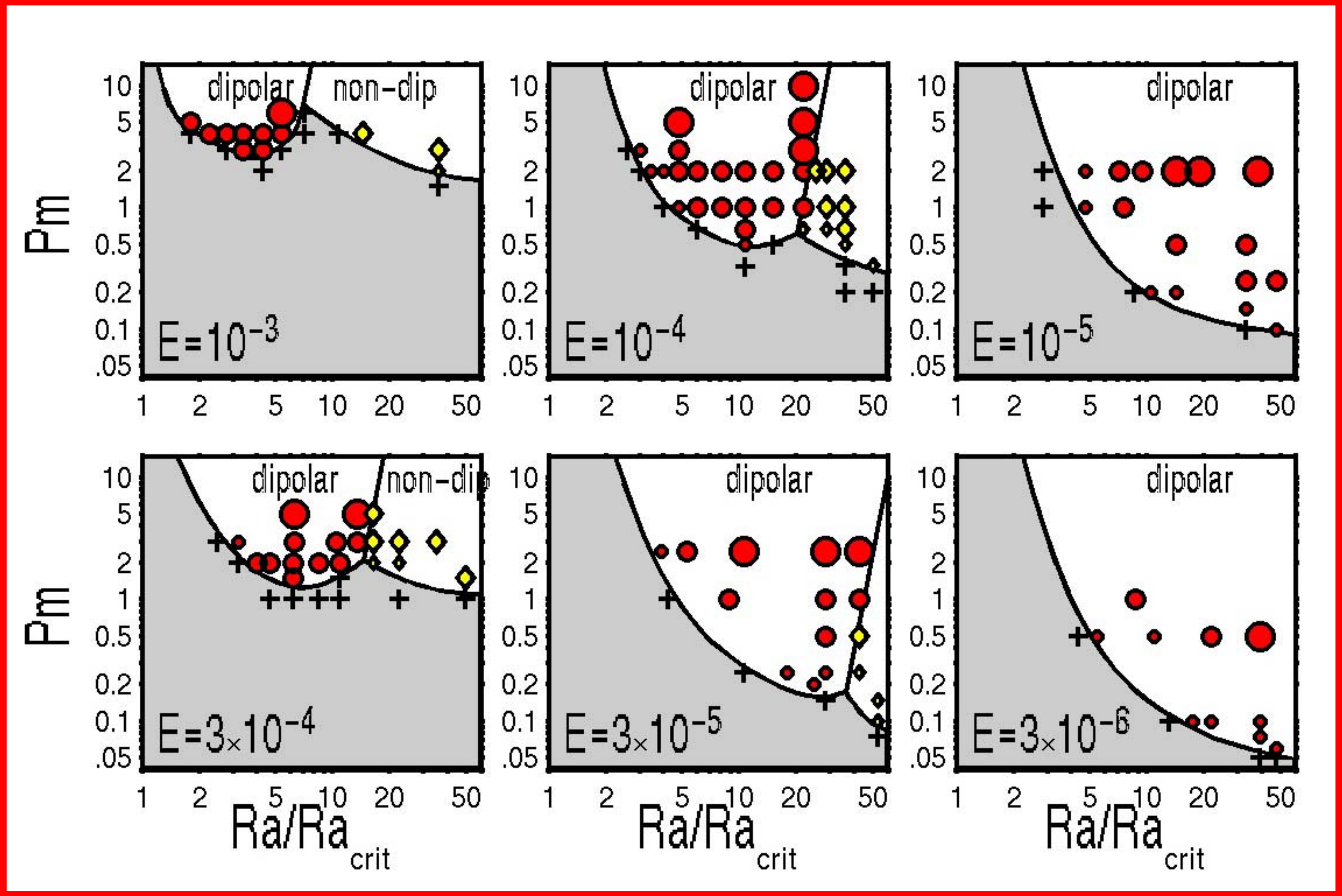
- Total of 162 model cases (127 dynamos, 35 failed dynamos)
- Driven by fixed ΔT, no-slip boundaries
- All parameters varied by at least two orders of magnitude
- Each run for at least 50 advection times (some much longer)
- Symmetry in longitude assumed for $E \leq 10^{-5}$

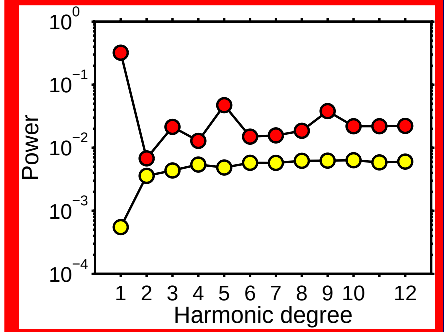
Selection criteria for scaling analysis:

- Self-sustained dynamo
- Dipole-dominated magnetic field ($f_{dip} > 0.35$)
- $E \leq 3 \times 10^{-4}$
- Fully developed convection (Nusselt-# > 2.0)

→ 70 model cases pass these criteria

Dynamo regimes (at Pr=1)

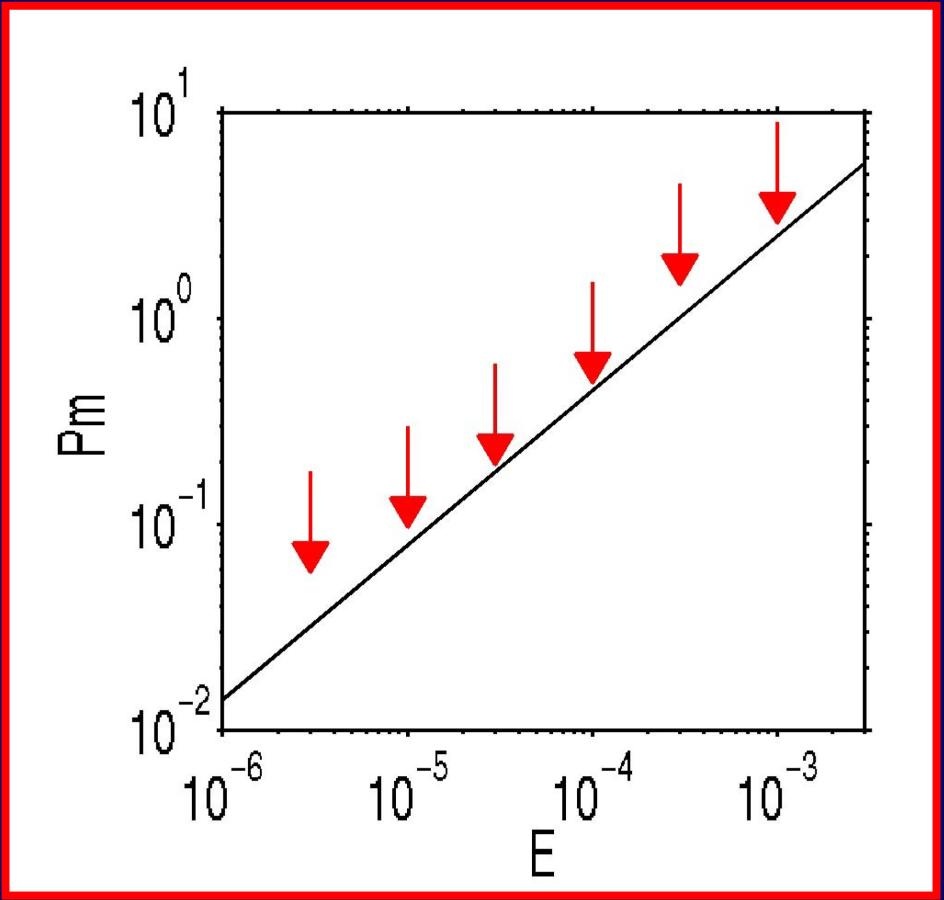




Typical magnetic spectra on outer boundary

As the Ekman number is lowered, dipolar dynamos occupy a broader region and are found at lower magnetic Prandtl #

Minimum magnetic Prandtl number

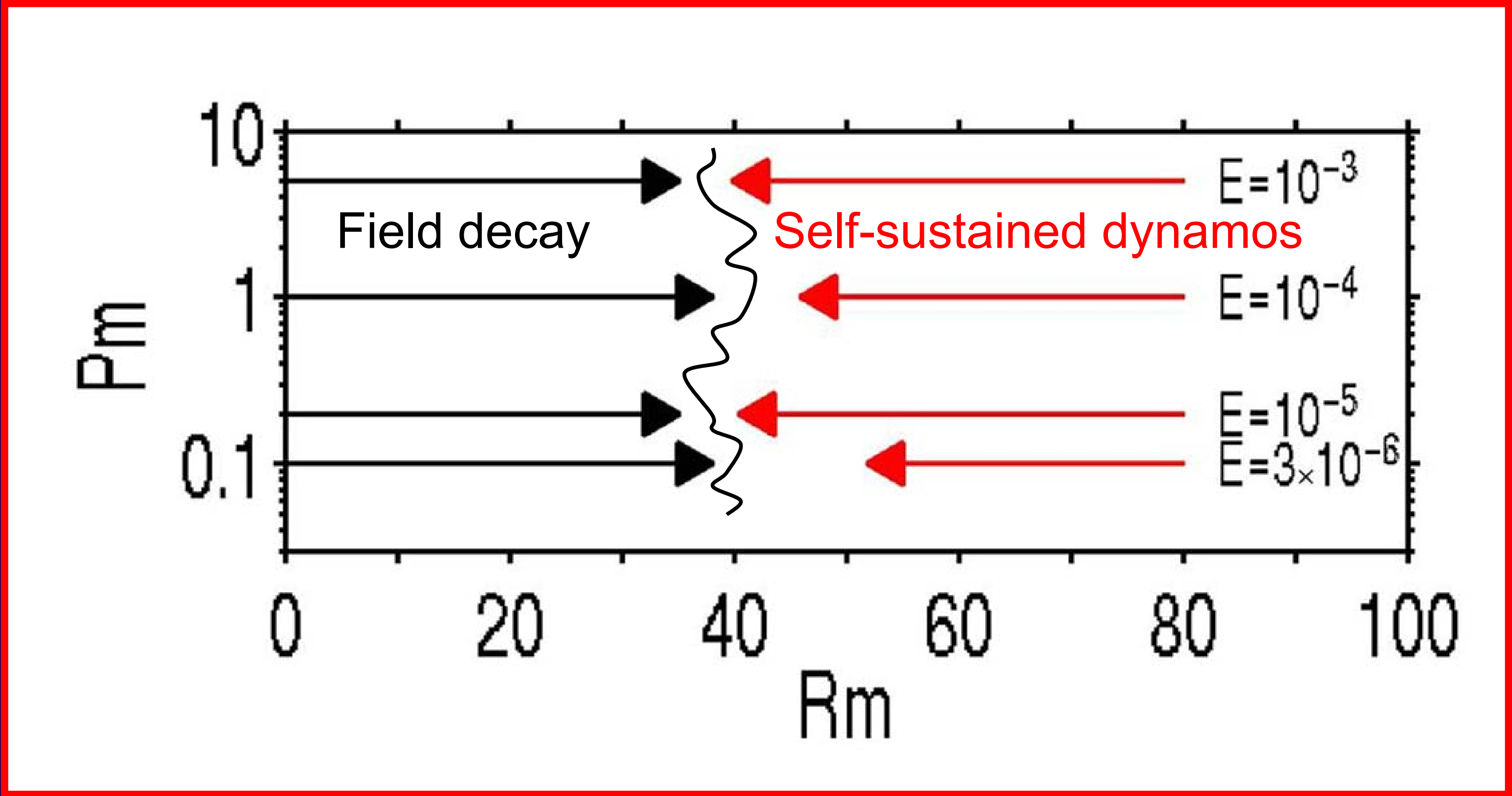


$$Pm_{min} \approx 450\ E^{3/4}$$

Earth values:

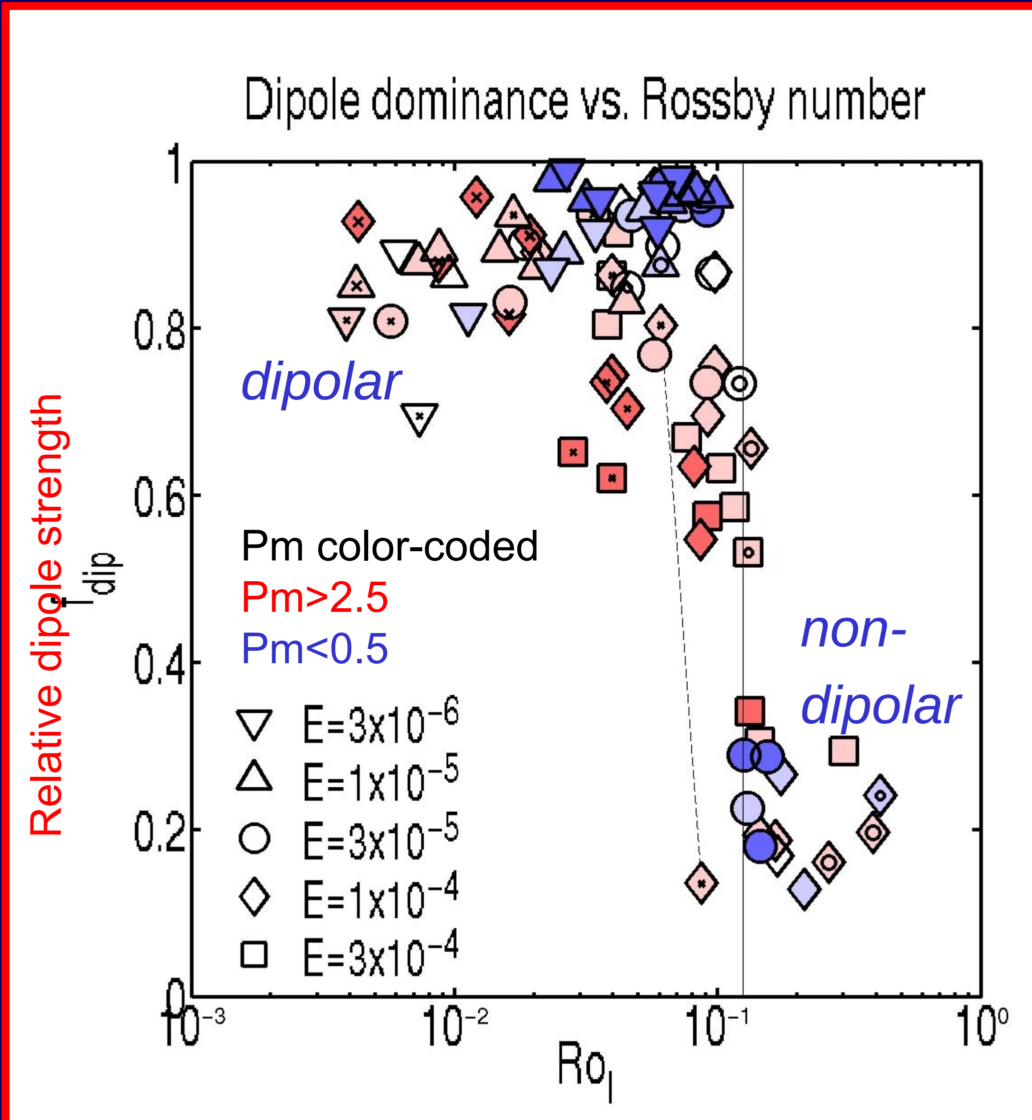
$$E \approx 10^{-14} \longrightarrow Pm_m \approx 2x10^{-8}$$

Critical magnetic Reynolds number



The mimimum magnetic Reynolds number for self-sustained dynamos is ≈ 40, irrespective of the value of the magnetic Prandtl number.

Dynamo regimes (0.1 ≤ Pr ≤ 10)



Inertial vs. Coriolis force:

Local Rossby number Ro_L calculated with mean length scale ℓ in the kinetic energy spectrum

$$Ro_L = U/(\ell\Omega)$$

When inertia dominates the dipolar regime breaks down

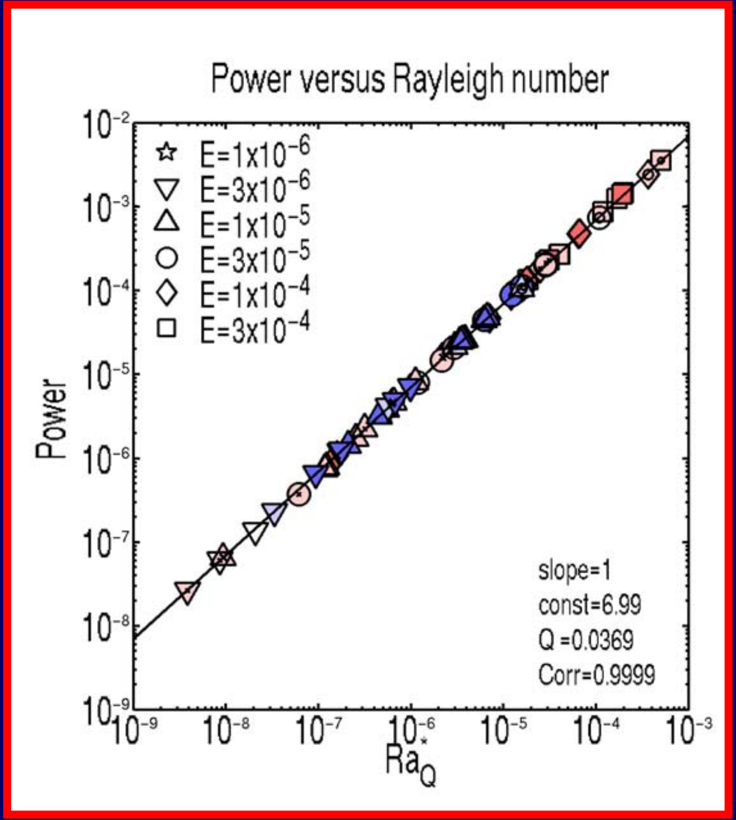
Flux-based Rayleigh number

Modified Rayleigh # based on buoyancy flux:

$$Ra_Q^* \quad = \quad \gamma g_o Q_{buoy} / (\rho c_p \Omega^3 D^2)$$

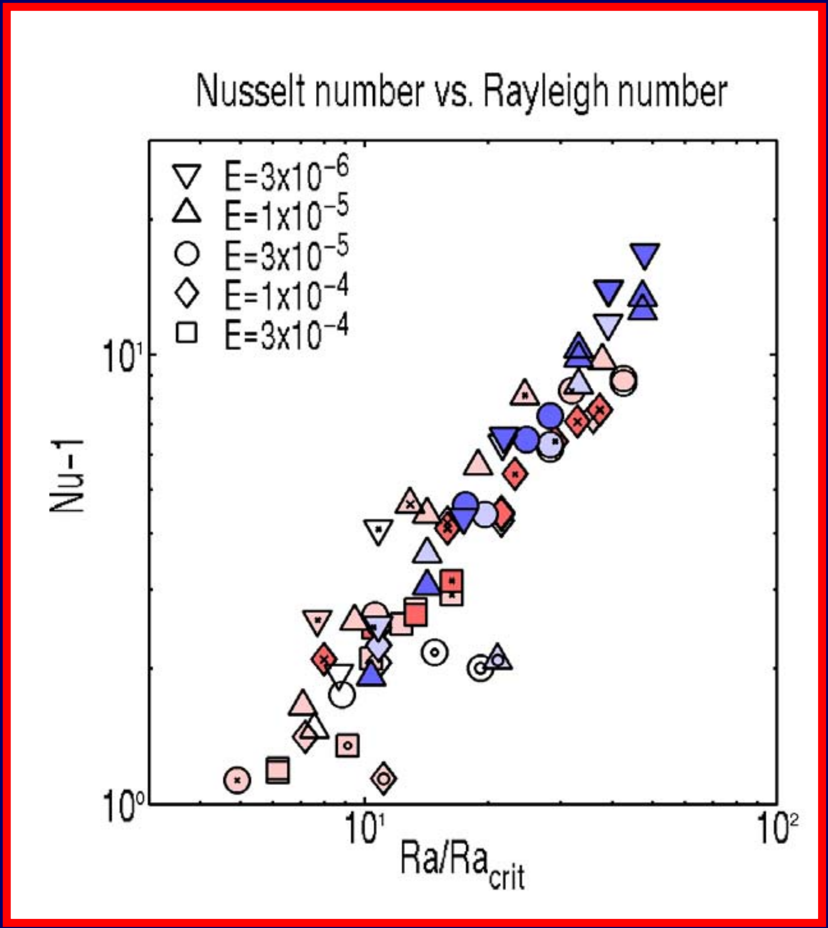
$$Ra_Q^* = \quad Ra^* Nu^* \; = \; Ra\,(Nu-1) E^3 Pr^{-2}$$

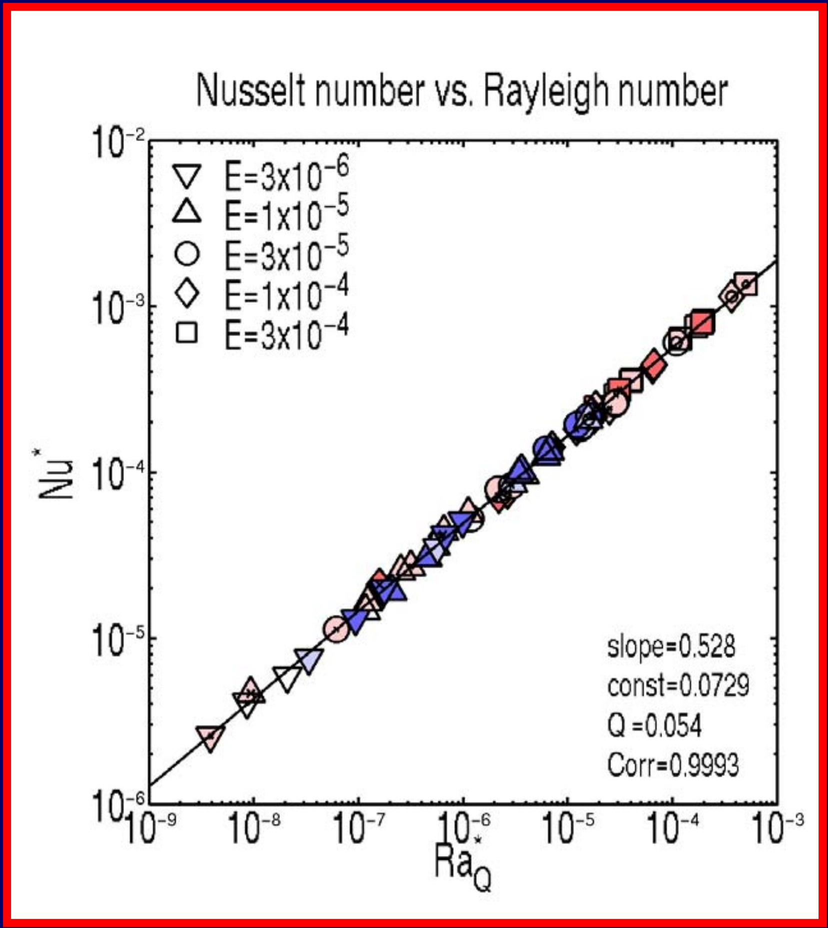
To very good approximation Ra_Q^* is equivalent to the power generated by buoyancy forces



$\gamma = (1-\eta^2)/4\pi\eta$: geometry factor

Scaling of Nusselt number





Use of modified „diffusionless“ parameters allows to collapse the data and express the dependence by a single power-law.

Compared to non-rotating convection, the exponent is very large (≈ 0.5).

Dimensional heat flow

For an exponent of 0.5:

$$Q_{adv} \sim \gamma \alpha g \rho c_p D^2 \Delta T^2 / \Omega$$

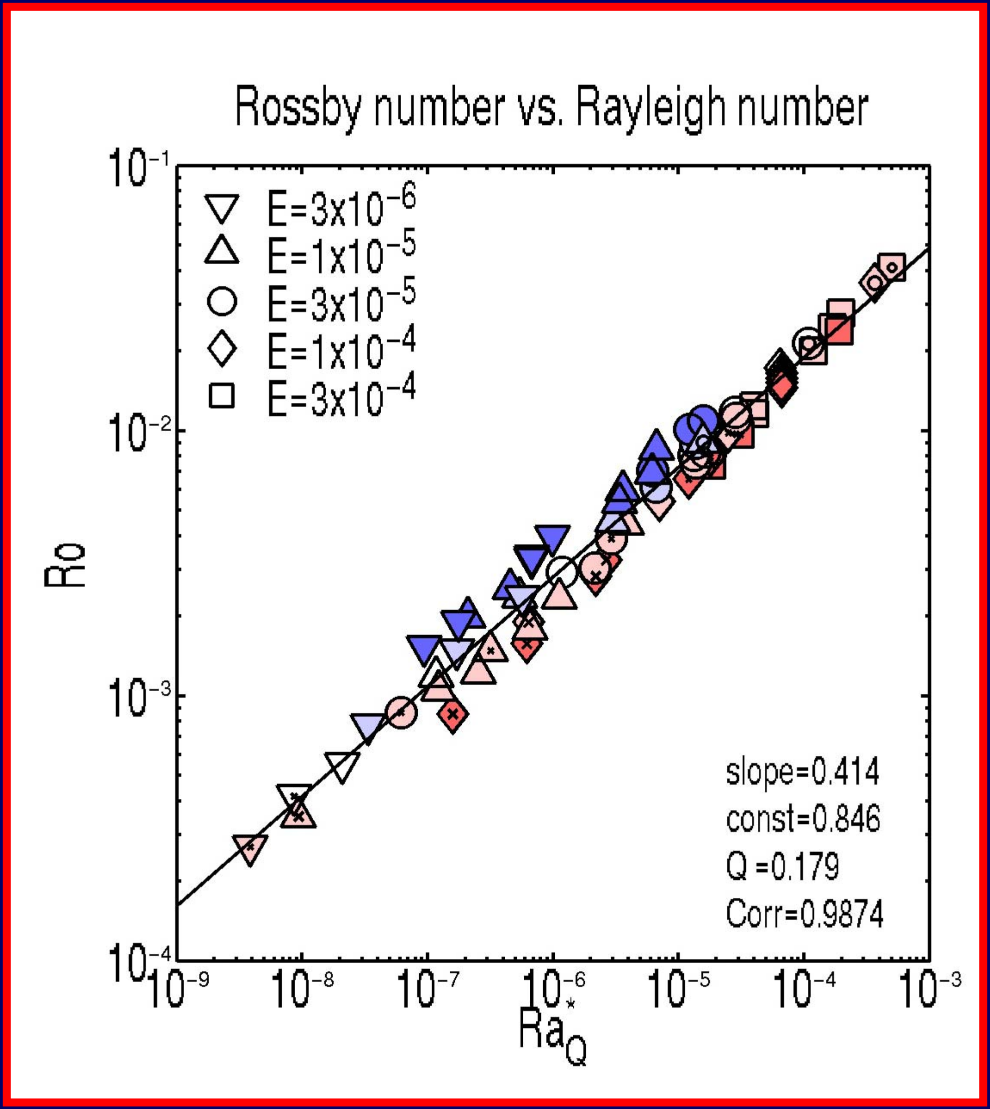
Advected heat flow is independent of thermal conductivity.

Earth‘s core: $Q_{adv} = 2$ TW $\rightarrow$ $\Delta T_{superadiab} \approx 1$ mK

Requirement for validity is probably that the thermal boundary layer thickness is larger than the Ekman layer thickness, which in terms of non-dimensional parameters requires $Ra^*_Q < 400\, E\, Pr^{-2}$. Satisfied in the numerical models and perhaps for thermal convection in the Earth’s core.

Velocity: Rossby number

Velocity ⟶

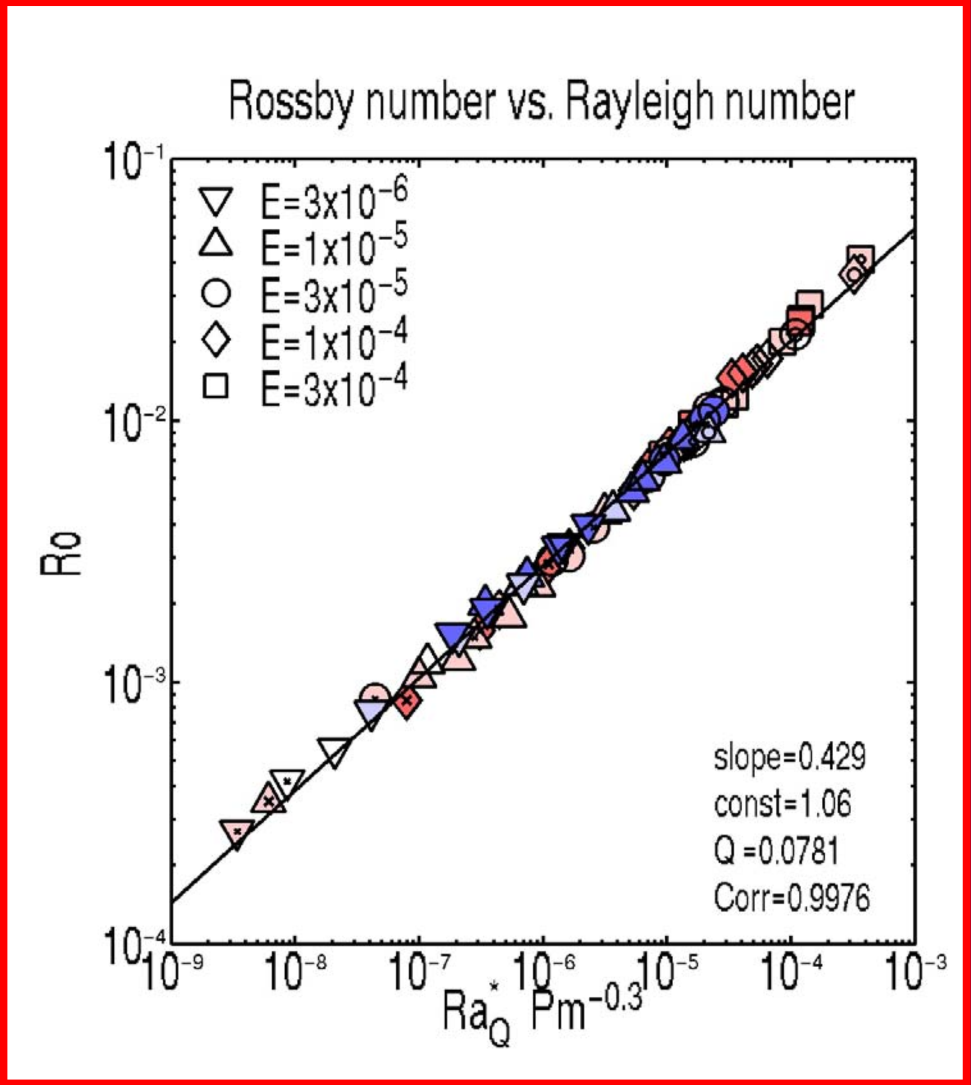


Power ⟶

$$Ro \sim Ra_Q^{*\ 2/5}$$

Dependence on magnetic Prandtl number ?

Velocity scaling II



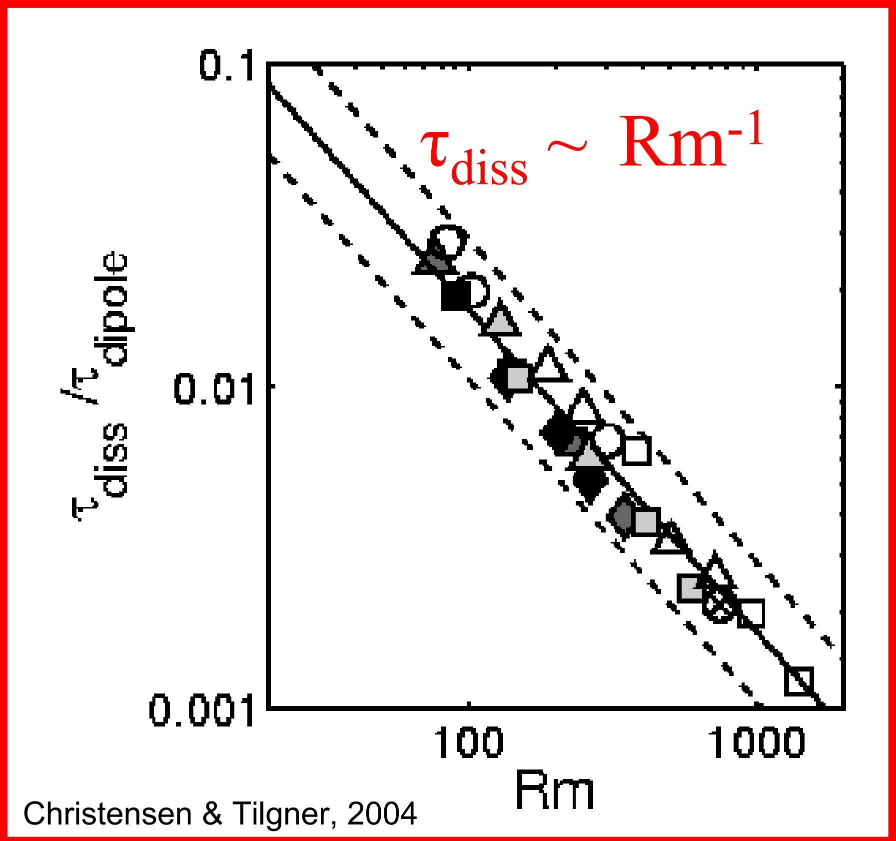
$$Ro \sim Ra^{*}_{Q}{}^{0.43}\, Pm^{-0.13}$$

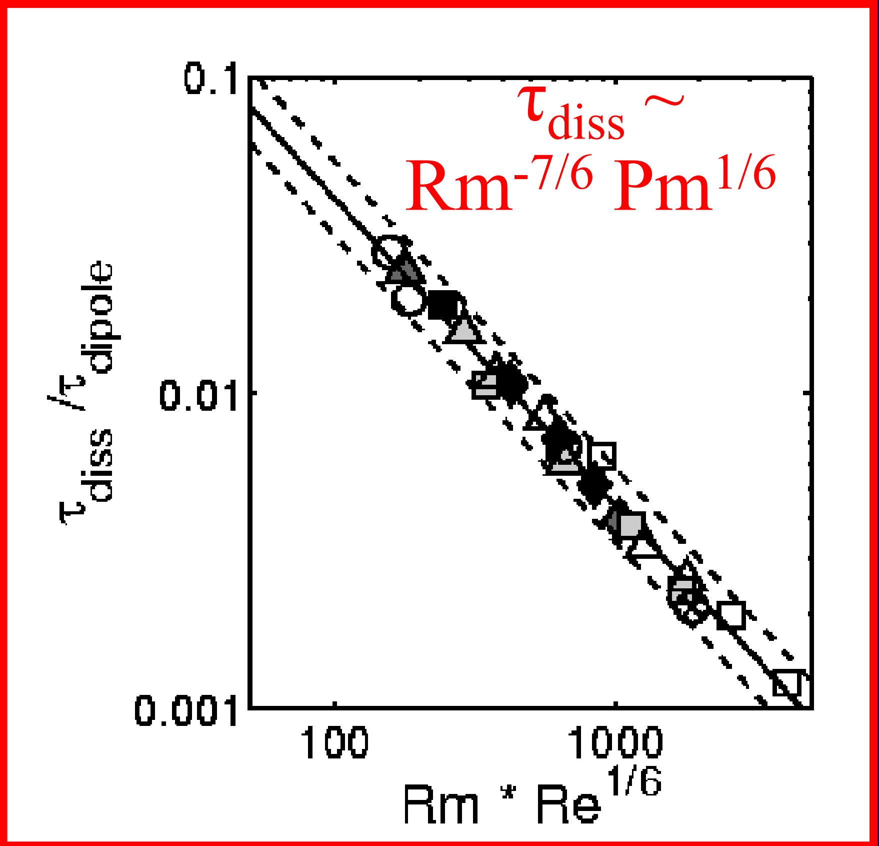
Two-parameter regression reduces misfit by factor 2.5.

But is dependence on the magnetic Prandtl number definitely required ?

Scaling ohmic dissipation time

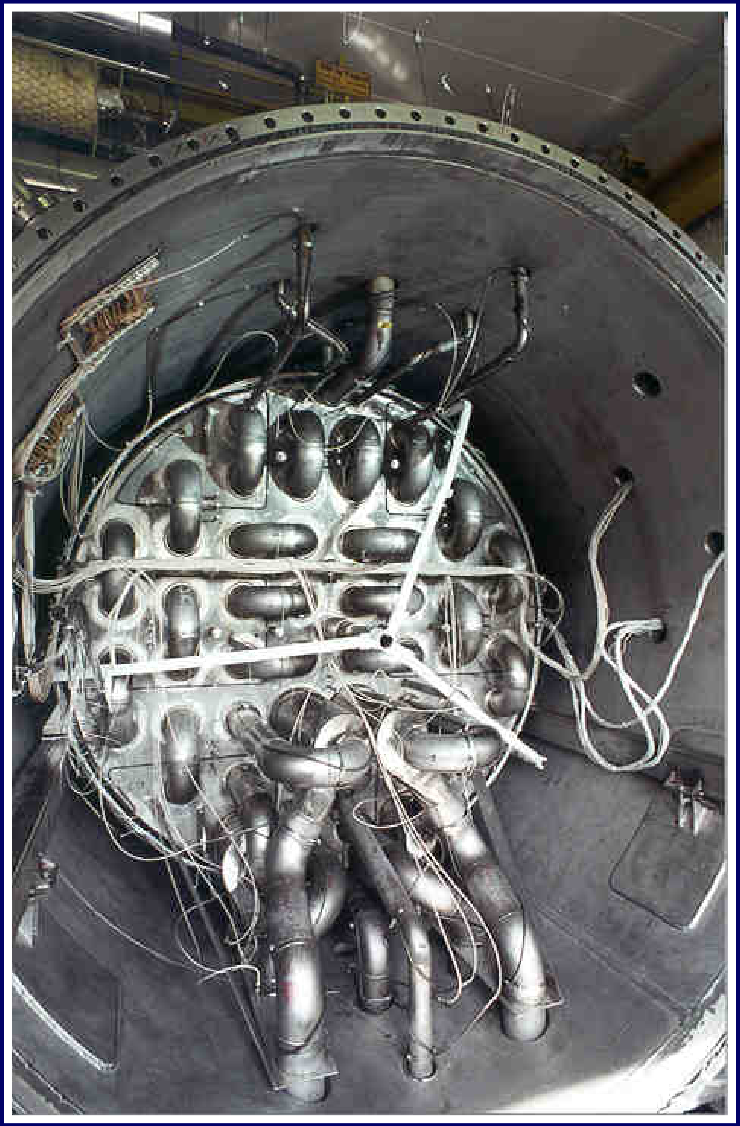
τ_{diss} = Magnetic energy / Ohmic dissipation

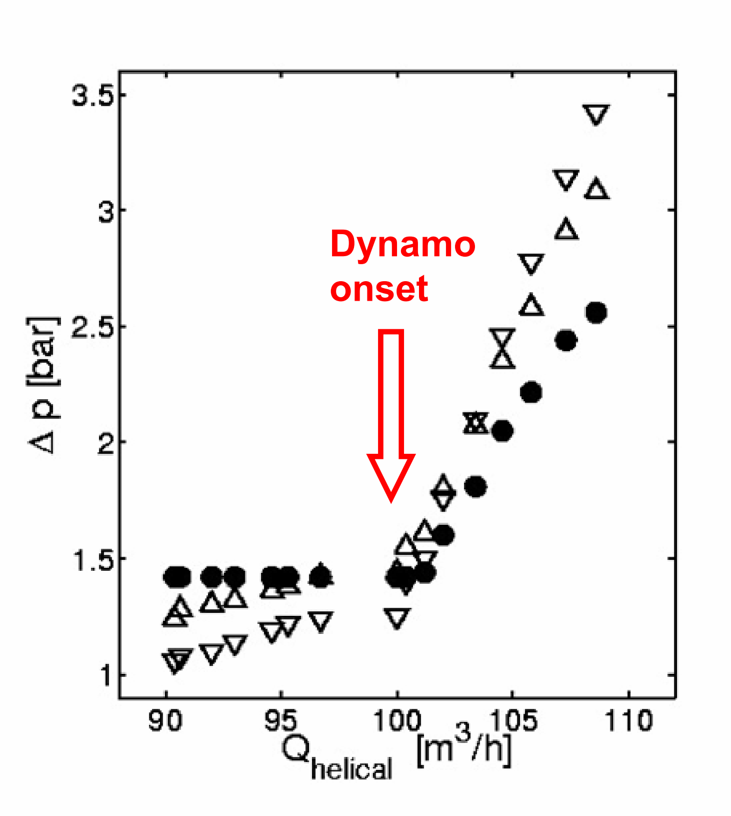




Which law is right ?

Karlsruhe laboratory dynamo



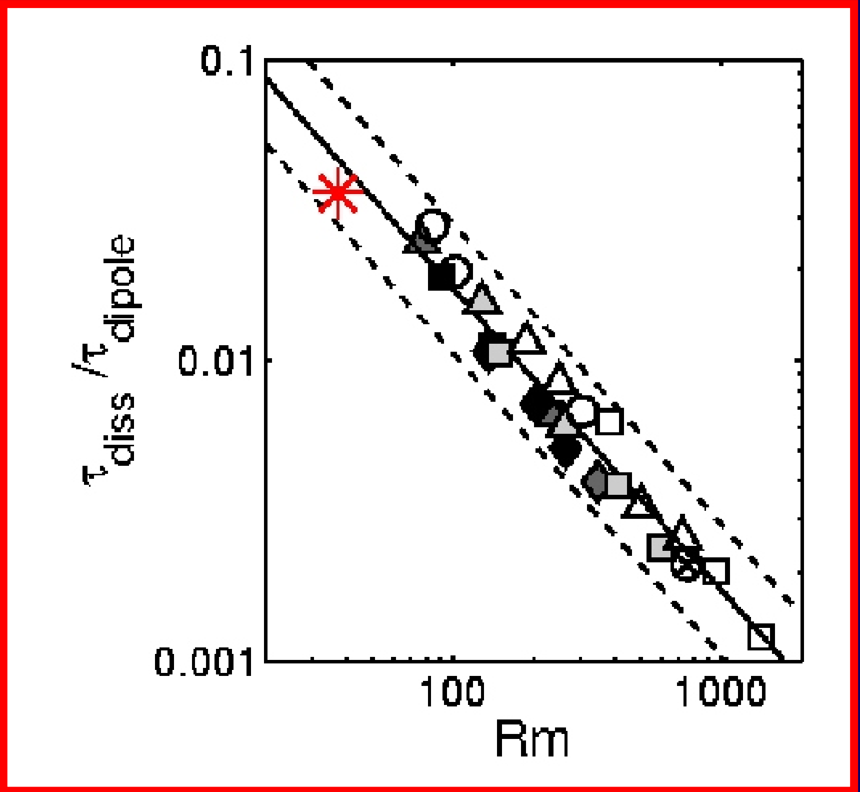


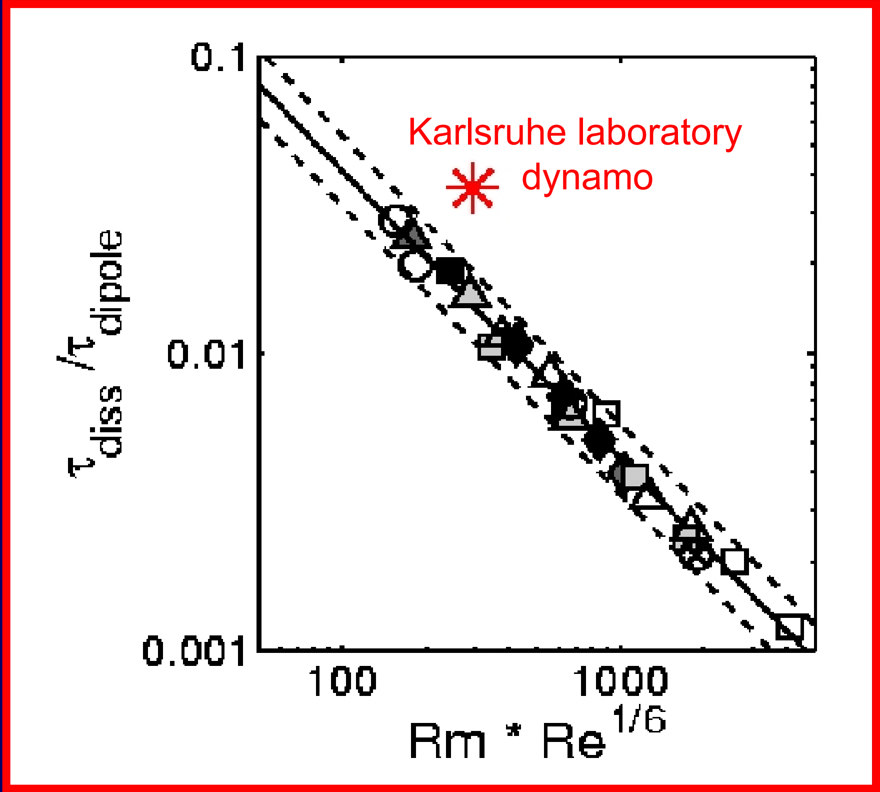
Magnetic Prandtl number

≈ 1 in numerical models

≈ 10^{-5} for liquid sodium

Scaling ohmic dissipation time



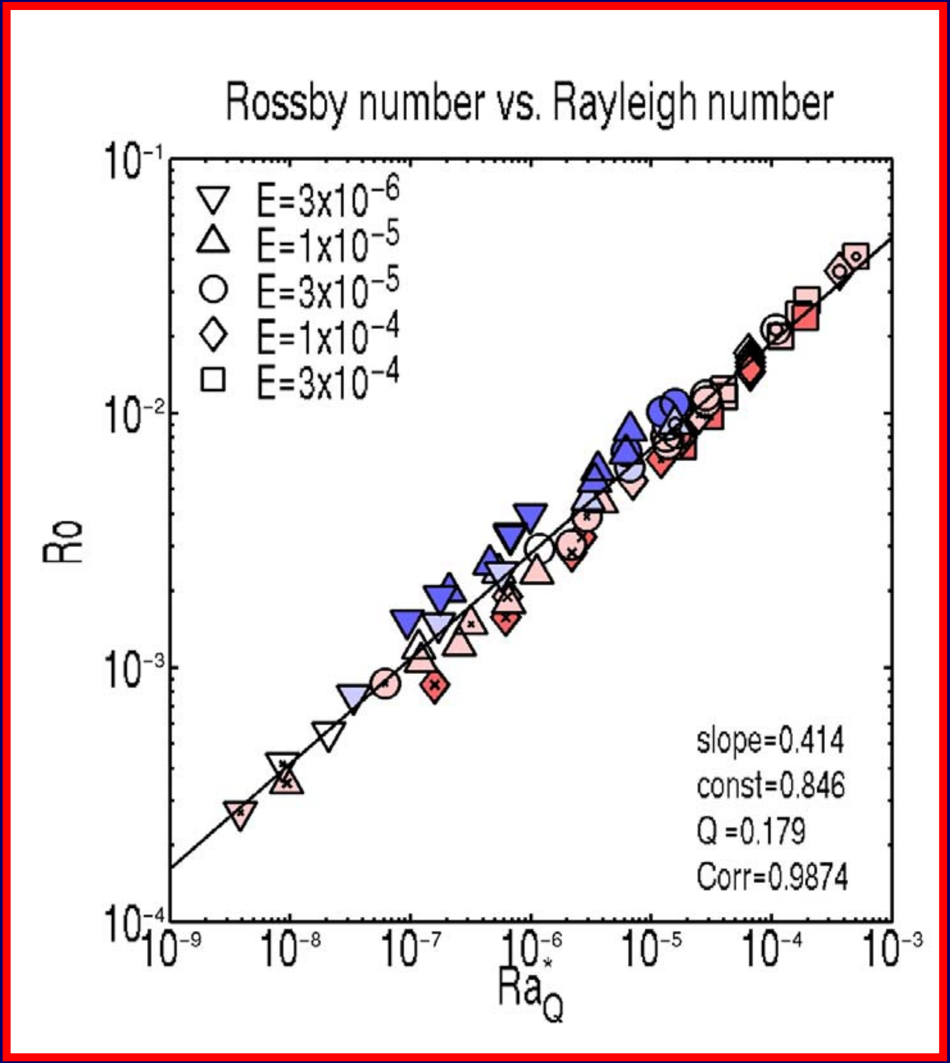


$\tau_{diss} \sim Rm^{-1}$

$\tau_{diss} \sim Rm^{-7/6} \; Pm^{1/6}$

Better agreement with the simple scaling law

Velocity Scaling: stay simple



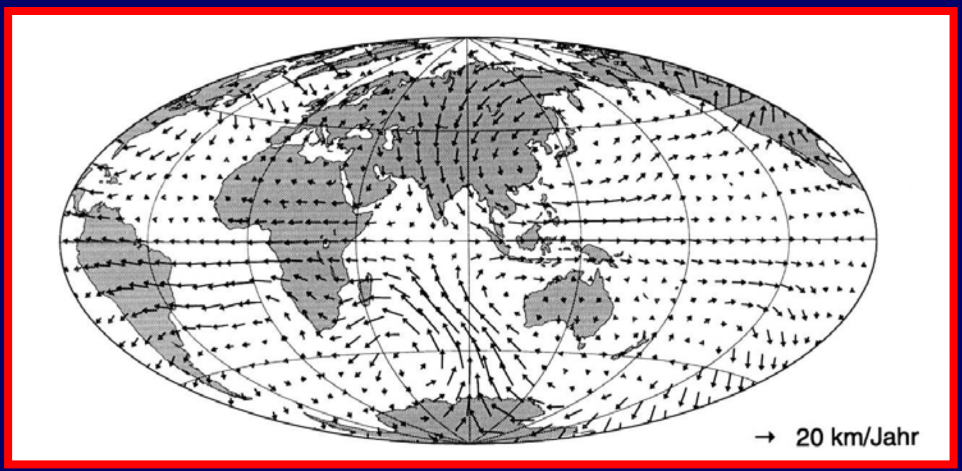
Let us assume that in general the magnetic Prandtl number has no influence (at least in the limit Pm << 1)

$$\boxed{Ro = 0.85 \; Ra^{*}_{Q}{}^{2/5}}$$

Rayleigh number of the core

Core velocity estimated from secular variation:

Large-scale flow ~ 0.5 mm/sec   Total rms-flow ~ 1 mm/sec



$Ro \sim 6 \times 10^{-6}$

Core Rayleigh number → $Ra^*_Q \sim 3 \times 10^{-13}$

The conventional Rayleigh number is 10^{23} ($5000 \times Ra_c$)

Light element flux and inner core growth rate

$Ra^*_Q \sim 3 \times 10^{-13} \rightarrow Q_{buoy} \sim 3 \times 10^{4}$ kg s^{-1}

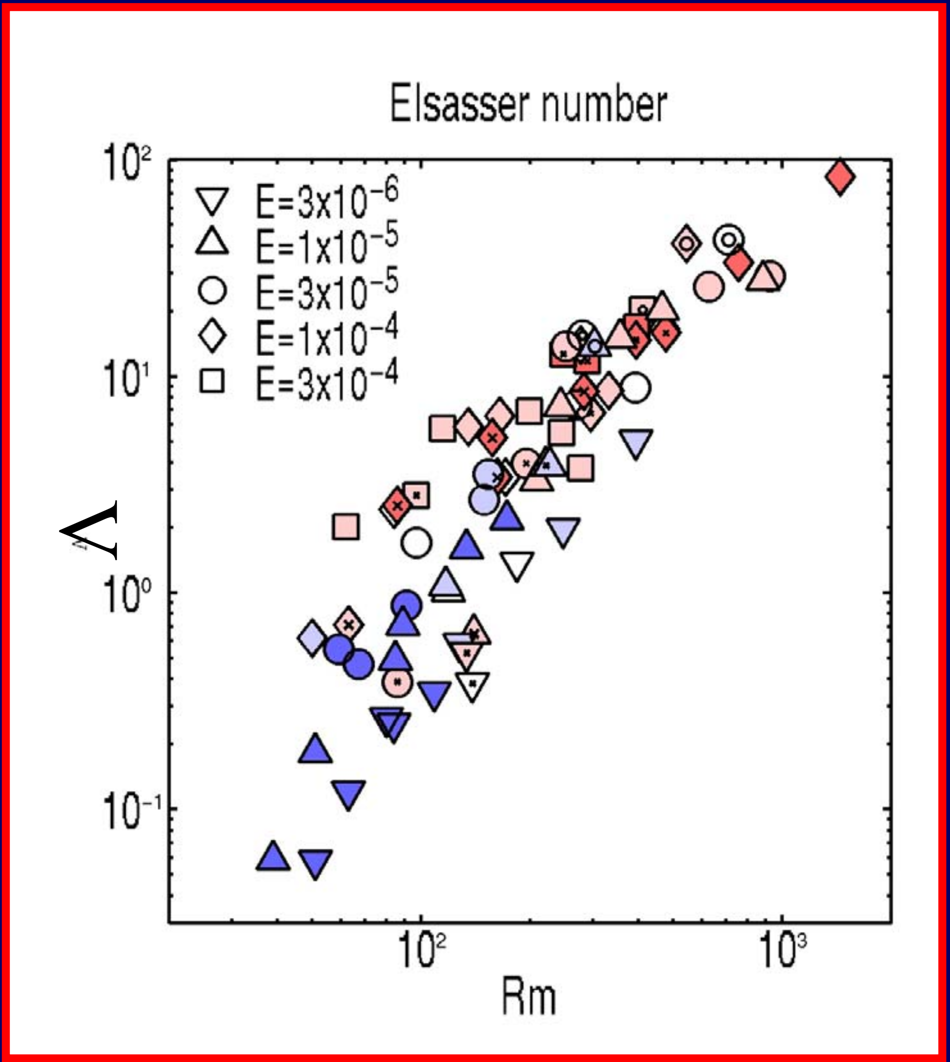
If convection is driven entirely by compositional flux associated with inner core growth, the growth rate is obtained as $dR_{ic}/dt = Q_{buoy} / (4\pi r_i^2 \Delta\rho_{ic})$.

$\rightarrow \quad dR_{ic}/dt \quad \sim 0.1$ mm/yr

Implication: Inner core is old:

3.5 ± 1.5 Gyr

What controls the strength of the magnetic field?



Magnetostrophic balance often associated with an Elsasser number $\Lambda = B^2/\mu\eta\rho\Omega$ ~ O(1)

In the numerical models, the Elsasser number varies widely.

→ Force balance not magneto-strophic, or Λ not good measure.

Alternative scaling: Magnetic field strength based on available power ?

Power-controlled field strength

Power driving convection ~ Ra_Q^*

Magnetic energy density ~ $1/2 \; Lo^2$

Ratio magn. energy / dissipation $= \tau_{diss}$

Prediction:

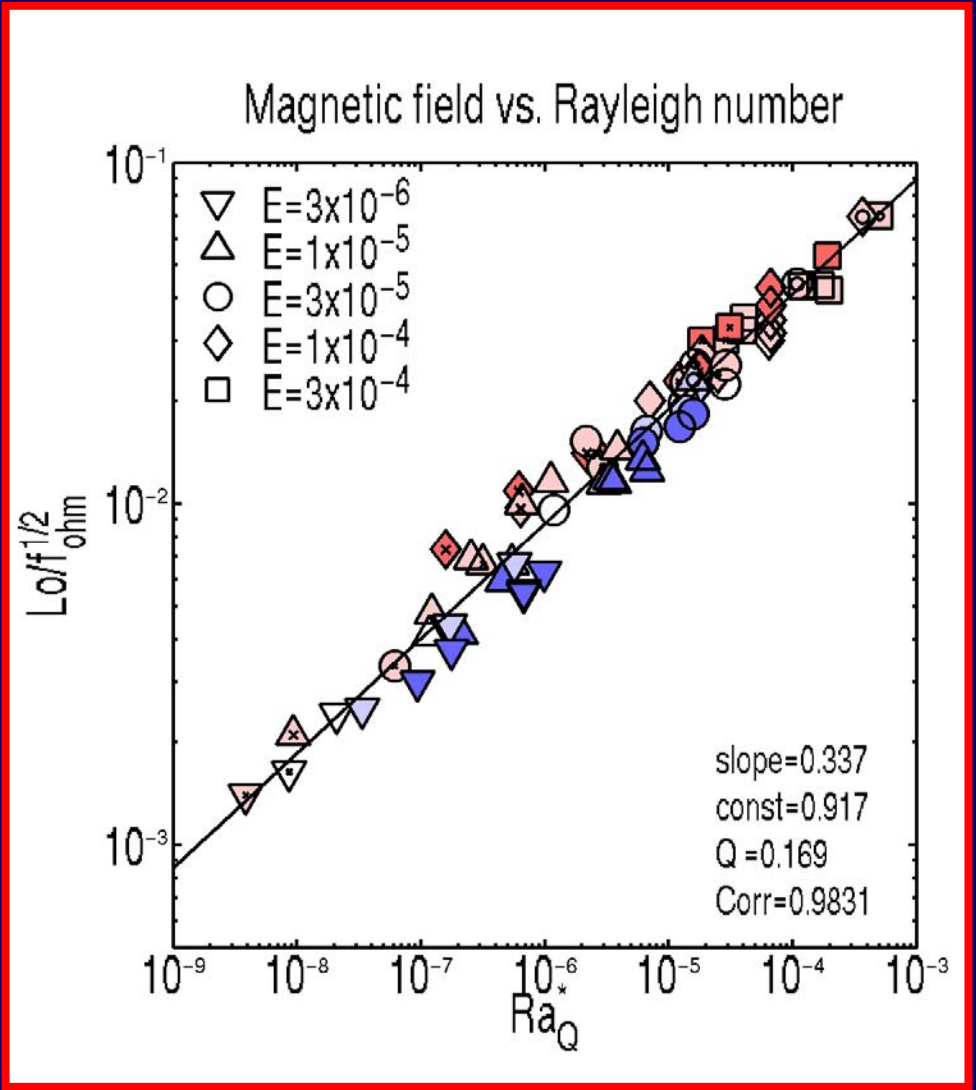
If $Ro \sim Ra_Q^{*2/5}$ and $\tau_{diss} \sim Rm^{-1} \sim Pm\,(Ro\,E)^{-1}$ then

$$\boxed{Lo/\sqrt{f_{ohm}} \sim Ra^*_Q{}^{0.3}}$$

where f_{ohm} is the relative fraction of energy dissipated by ohmic losses (30 – 85% in the models)

Magnetic Field Scaling

Magnetic field strength →



Power →

$$Lo \sim Ra^{*\,0.34}_{Q}$$

Empirical fit is close to predicted dependence

Magnetic Field Scaling II

Assume $f_{ohm} \approx 1$ in the core

In dimensional form the magnetic field strength is

$$\boxed{B \sim \mu^{1/2} \rho^{1/6} (g Q_{buoy}/D)^{1/3}}$$

B independent of conductivity and rotation rate

Strength of core magnetic field

With the estimated buoyancy flux of 3.5×10^4 kg/s the predicted magnetic field strength in the core is

$$B \approx 1.2 \text{ mT}$$

Compare to:

- „Observed“ field at CMB: 0.39 mT ($\ell < 13$)
- $B_s \approx 0.4$ mT inside core from torsional oscillations (Zatman & Bloxham, 1997)

How robust are the scaling laws?

Tests:

- Exclude cases with $E \geq 10^{-4}$ from fit

  $\longrightarrow$ Results virtually unchanged

- Exclude cases with $Ro_\ell > 0.05$ (high inertia) from fit $\longrightarrow$ Results virtually unchanged

- Attempt general least-squares fit of the form:

$$Y = A \; Ra_Q{}^{*\alpha} \; Pm^{\beta} \; E^{\gamma} \; Pr^{\delta}$$

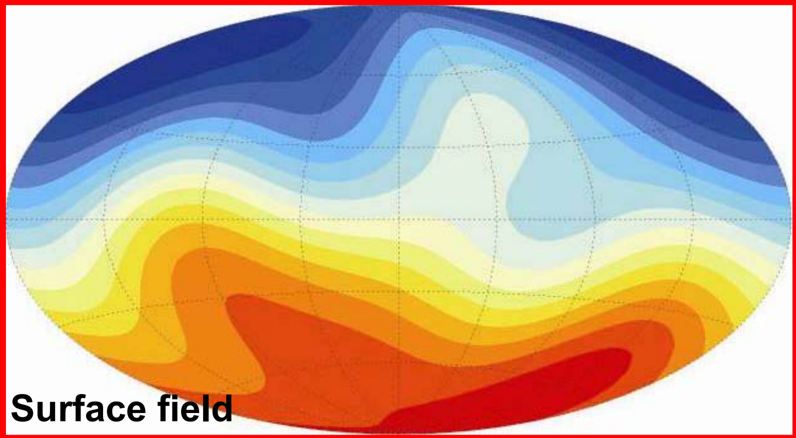
where Y stands for Nu*, Ro, or $Lo/\sqrt{f_{ohm}}$

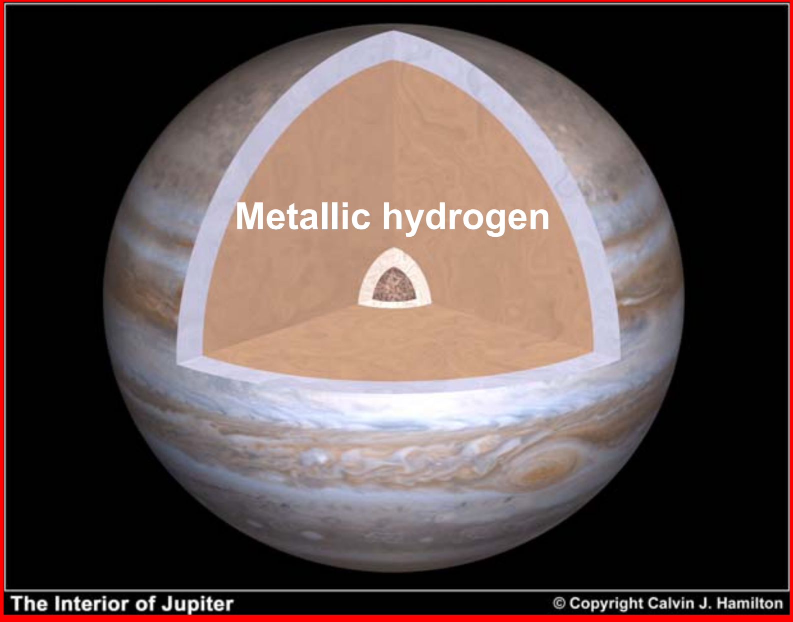
$\rightarrow$ Exponents for E and Pr very small (<0.03)

Other planets: Jupiter

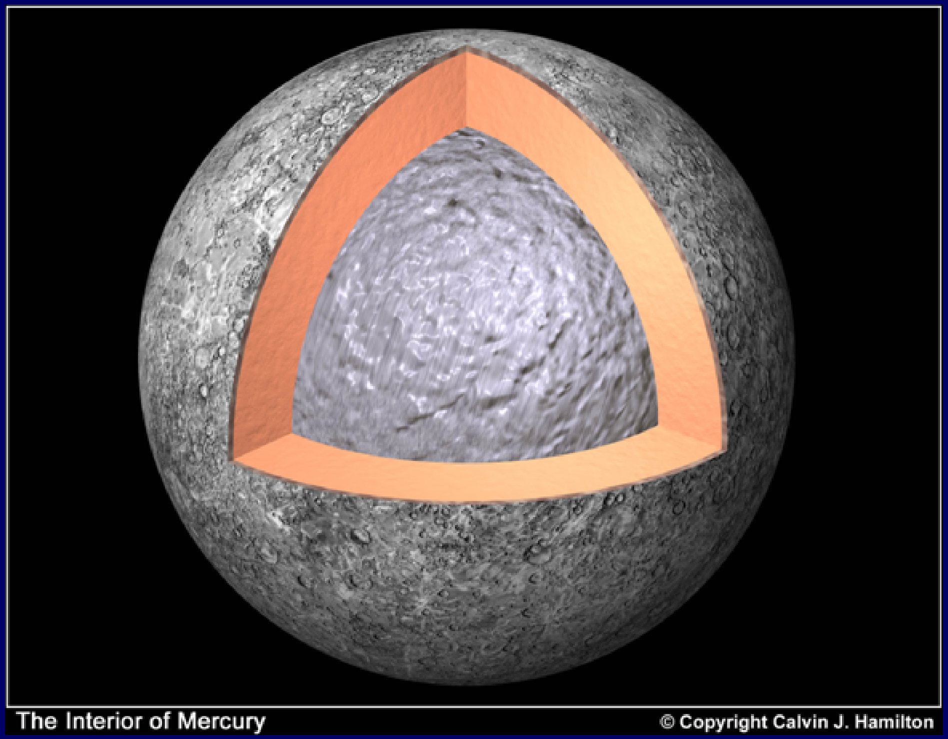
For the observed excess heat flow (5.4 W/m^2) and reasonable estimates for other relevant parameters, the magnetic field in the dynamo region is $B \approx 8$ mT.

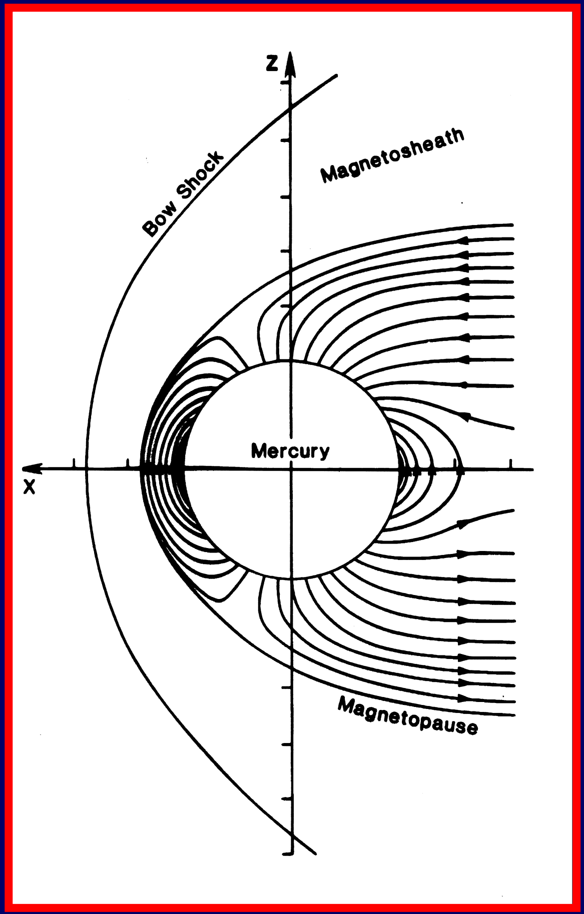
This is in agreement with Jupiter's surface field strength roughly 10 times the Earth value.





Mercury





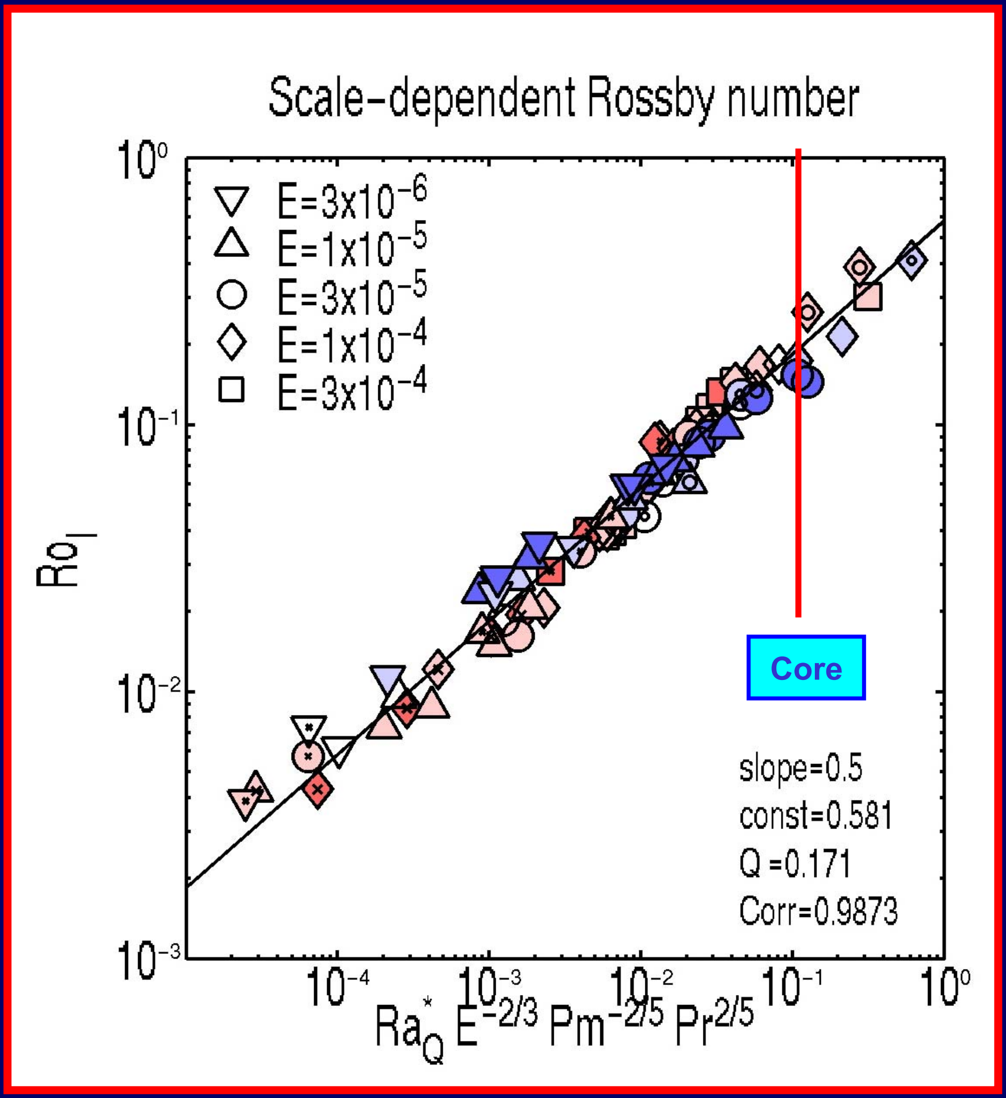
Surface field strength ≈ 1/100 Earth value.

Cannot be explained by low buoyancy flux, because it would imply magnetic Reynold number below critical.

Conclusions

- Numerical dynamo models reproduce observed properties of the geomagnetic field, even though viscous and thermal diffusivity are far too large.
- Scaling laws that are independent of diffusivities fit the model results well and lead to reasonable predictions for the Earth.
- This supports the view that the models are already in a regime similar to that of the Earth‘s core.
- The range of validity of laws must be explored. Is there a dependence on the magnetic Prandtl number? Laboratory dynamo experiments will help to decide.

Scaling of local Rossby number



Decent fit possible, but involves all four control parameters

$Ro_L \sim Ra_Q^{*1/2} E^{-1/3} Pr^{1/5} Pm^{-1/5}$

Predicted Earth value is $Ro_L \approx 0.1 - 0.2$, very close to the transition point between dipolar and non-dipolar dynamos.

Enstrophy balance

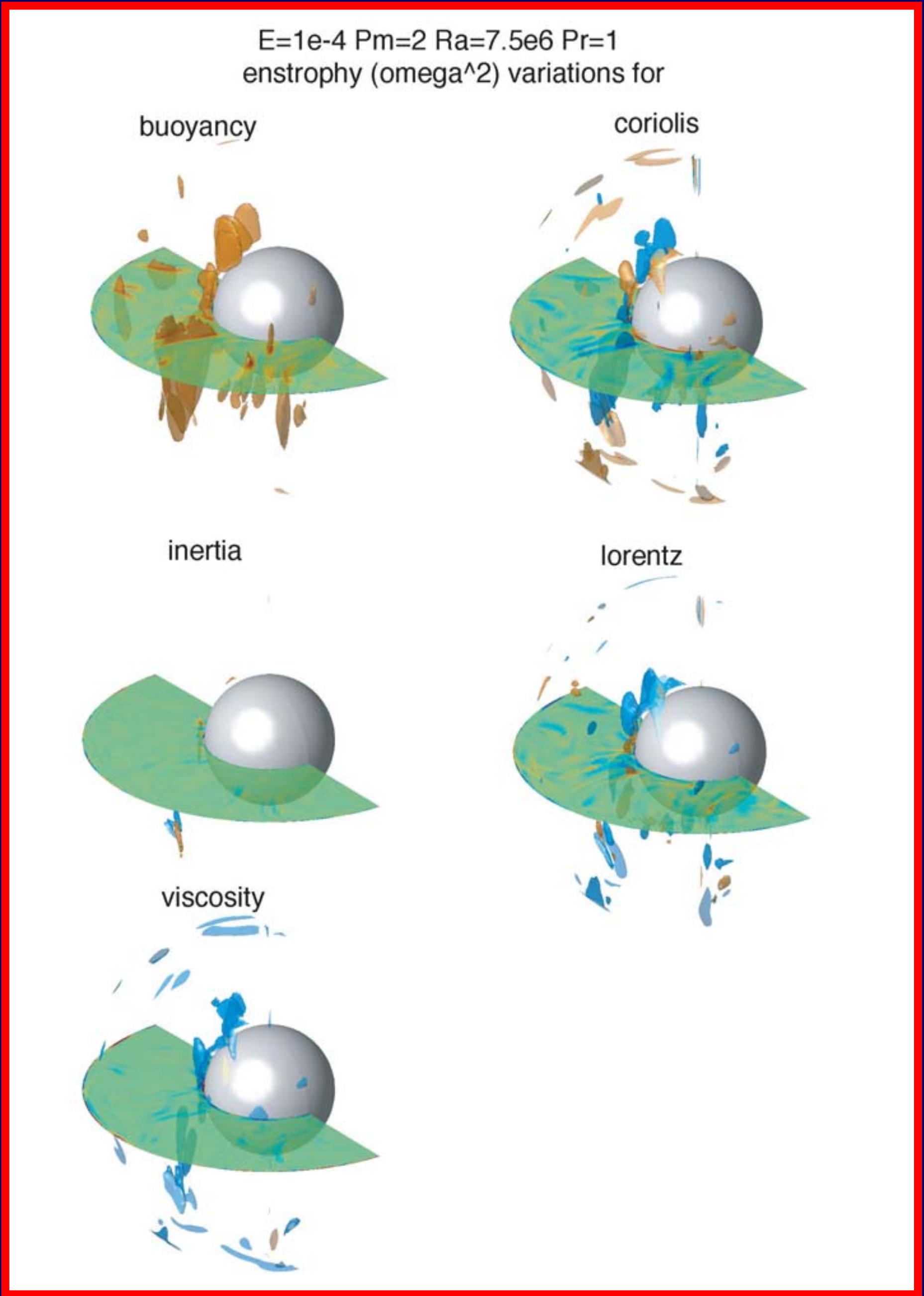
Vorticity ω

Enstrophy = $\omega \cdot \omega$ „Energy of vorticity“

$$\frac{\partial \vec{\omega}^2}{\partial t} =$$

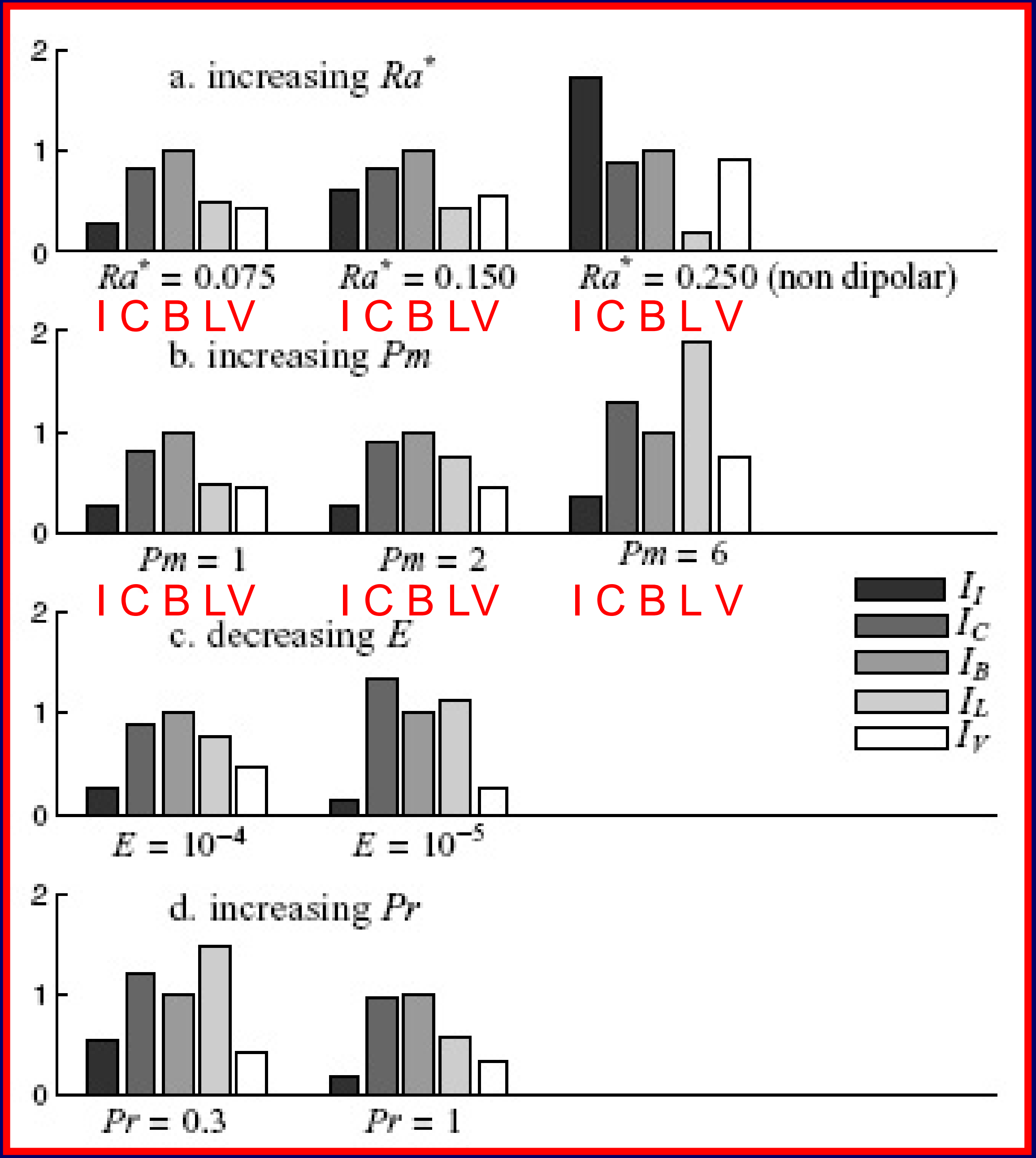
$$-\vec{\nabla} \times [\vec{\omega} \times \vec{u}] \cdot \vec{\omega} - 2\frac{\partial \vec{u}}{\partial z} \cdot \vec{\omega} + E\,\nabla^2 \vec{\omega} \cdot \vec{\omega} + \frac{Ra^*}{r_o} \vec{\nabla} \times T\vec{r} \cdot \vec{\omega} + \vec{\nabla} \times [(\vec{\nabla} \times \vec{B}) \times \vec{B}] \cdot \vec{\omega}$$

Inertia *Coriolis* *Viscosity* *Buoyancy* *Lorentz*



Sources & sinks of enstrophy

Buoyancy is main source and Lorentz force and viscosity is main sink



Enstrophy budget

Coriolis ~ buoyancy

Lorentz variable

Inertia large in non-dipolar case