Introduction to Hybrid Ensemble-Variational Data Assimilation Methods and Recent Research, Development and Application for Global to Storm Scale NWP

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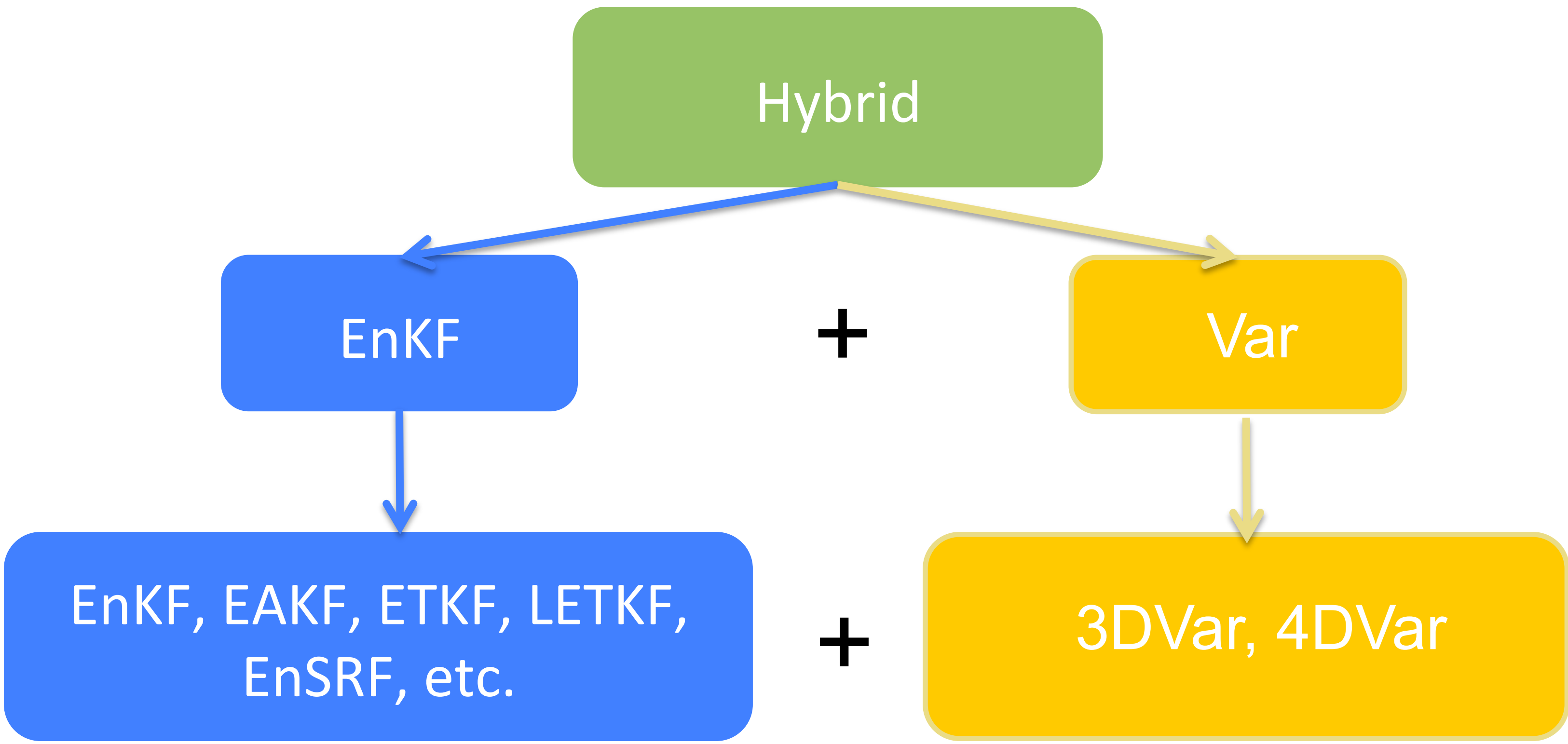
Outline

- ❑ Introduction to hybrid DA methods
- ❑ Examples of recent research, development and application of hybrid DA
  - ➢ Recent R&D to improve global forecasts
  - ➢ Recent R&D to improve high resolution hurricane forecasts
  - ➢ Recent R&D for convective scale weather forecasts over CONUS
- ❑ Future work and challenges

What is Hybrid?



Background

- History of hybrid DA

2000

Development of theory

Research with simple model and simulated data

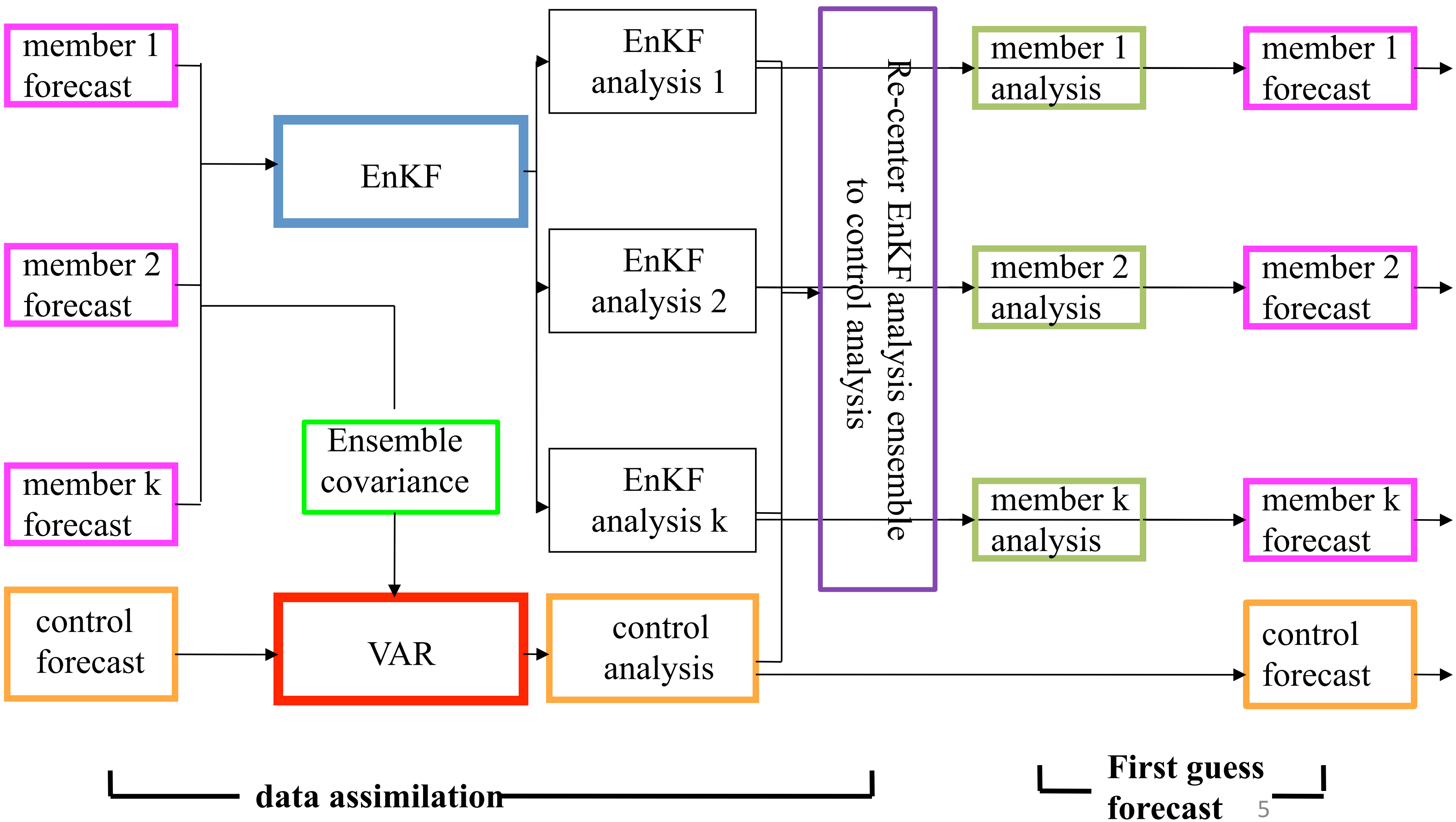
System development for real NWP model and test real data

2010

Operational implementation at NWP centers for global NWP, US NWS, Env. Canada, US Navy, ECMWF, UK Met, etc.

Active R&D for regional, convective scale NWP

Hybrid DA system



Wang et al. 2013, MWR

How to incorporate ensemble in VAR? Method 1

- Direct combination of static and ensemble covariances (Hamill and Snyder 2000; Wang et al. 2007a, 2008a; Kuhl et al. 2013):

$$J\left(\mathbf{x}'\right) = J_b + J_o$$

$$= \frac{1}{2}\mathbf{x}'^T\left(\underline{\frac{1}{\beta_1}\mathbf{B} + \frac{1}{\beta_2}\mathbf{P}^e \circ \mathbf{S}}\right)^{-1}\mathbf{x}' + \frac{1}{2}\left(\mathbf{y}^{o'} - \mathbf{H}\mathbf{x}'\right)^T \mathbf{R}^{-1}\left(\mathbf{y}^{o'} - \mathbf{H}\mathbf{x}'\right)$$

Blended static and ensemble covariances

$$\mathbf{P}^e = \sum_{k=1}^{K}\left(\mathbf{x}_k^e\right)\left(\mathbf{x}_k^e\right)^T, \quad \frac{1}{\beta_1} + \frac{1}{\beta_2} = 1$$

$\mathbf{B}$ 3DVAR static covariance; $\mathbf{R}$ observation error covariance; K ensemble size;
$\mathbf{S}$ correlation matrix for ensemble covariance localization; $\mathbf{x}_k^e$ kth ensemble perturbation; $\mathbf{x}'$ total (hybrid) increment; $\mathbf{y}^{o'}$ innovation vector;
$\mathbf{H}$ linearized observation operator; β_1 weighting coefficient for static covariance;
β_2 weighting coefficient for ensemble covariance

How to incorporate ensemble in VAR? Method 2

- Extended control variable (ECV) method (Lorenz 2003; Buehner 2005; Wang et al. 2007b, 2008a; Wang 2010):

$$J\left(\mathbf{x}_s^{'},\boldsymbol{\alpha}\right)=\beta_1 J_s+\beta_2 J_e+J_o$$

$$=\beta_1\frac{1}{2}\mathbf{x}_s^{'T}\mathbf{B}^{-1}\mathbf{x}_s^{'}+\beta_2\underline{\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{C}^{-1}\boldsymbol{\alpha}}+\frac{1}{2}\left(\mathbf{y}^{o'}-\mathbf{H}\mathbf{x}^{'}\right)^T\mathbf{R}^{-1}\left(\mathbf{y}^{o'}-\mathbf{H}\mathbf{x}^{'}\right)$$

Extra term associated with extended control variable

$$\mathbf{x}^{'}=\mathbf{x}_s^{'}+\underline{\sum_{k=1}^{K}\left(\boldsymbol{\alpha}_k\circ\mathbf{x}_k^e\right)}$$

Extra increment associated with ensemble

$\mathbf{B}$ 3DVAR static covariance; $\mathbf{R}$ observation error covariance; K ensemble size;
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$\mathbf{x}_s^{'}$ 3DVAR increment; $\mathbf{x}^{'}$ total (hybrid) increment; $\mathbf{y}^{o'}$ innovation vector;
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β_2 weighting coefficient for ensemble covariance; $\boldsymbol{\alpha}$ extended control variable.

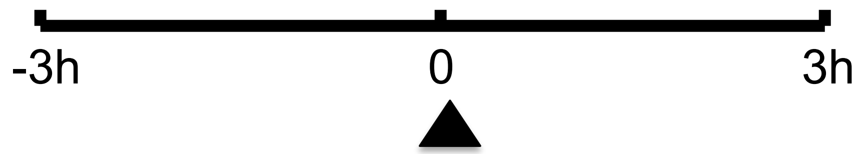
Theoretical Equivalence

- The solutions from method 1 and method 2 are proved to be equivalent (Wang et al. 2007b).
- Most operational NWP centers use method 2
- For method 2, specific implementation can be different depending on the preconditioning used in variational minimization (e.g., Lorenc 2003 (UK Met); Wang et al. 2008a (WRFVAR hybrid); Wang 2010 (GSI hybrid)).

3DEnVar vs. 4DEnVar Hybrid

- **3DEnVar**

e.g. Wang et al. 2008a



$$J\left(\mathbf{x}_s^{'},\boldsymbol{\alpha}\right)=\beta_1 J_s+\beta_2 J_e+J_o$$

$$=\beta_1\frac{1}{2}\mathbf{x}_s^{'T}\mathbf{B}^{-1}\mathbf{x}_s^{'}+\beta_2\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{C}^{-1}\boldsymbol{\alpha}+\frac{1}{2}\left(\mathbf{y}^{o'}-\mathbf{H}\mathbf{x}^{'}\right)^T\mathbf{R}^{-1}\left(\mathbf{y}^{o'}-\mathbf{H}\mathbf{x}^{'}\right)$$

Extra term associated with extended control variable

$$\mathbf{x}^{'}=\mathbf{x}_s^{'}+\sum_{k=1}^{K}\left(\boldsymbol{\alpha}_k\circ\mathbf{x}_k^e\right)$$

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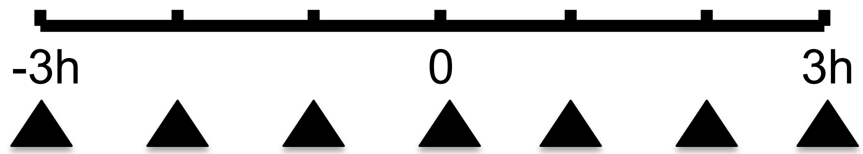
3DEnVar vs. 4DEnVar Hybrid

- **4DEnVar**

Wang and Lei 2014



Add time dimension in 4DEnVar

$$J\left(\mathbf{x}_s^{'},\boldsymbol{\alpha}\right)=\beta_1 J_s+\beta_2 J_e+J_o$$

$$=\beta_1\frac{1}{2}\mathbf{x}_s^{'T}\mathbf{B}^{-1}\mathbf{x}_s^{'}+\beta_2\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{C}^{-1}\boldsymbol{\alpha}+\frac{1}{2}\boxed{\sum_{t=0}^{T}(y_t^{o'}\text{-}\mathrm{H}_t\mathbf{x}_t^{'})^{\mathrm{T}}\mathrm{R}_t^{-1}(y_t^{o'}\text{-}\mathrm{H}_t\mathbf{x}_t^{'})}$$

$$\mathbf{x}_t^{'}=\mathbf{x}_s^{'}+\boxed{\sum_{k=1}^{K}\left(\alpha_k\circ(\mathbf{x}_k^e)_t\right)}$$

Use ensemble temporal covariance to propagate increments over time. No tangent linear adjoint.

$\mathbf{B}$ stat 3DVAR static covariance; $\mathbf{R}$ observation error covariance; K ensemble size; $\mathbf{C}$ Concatenated correlation matrix for ensemble covariance localization; $\mathbf{x}_k^e$ kth ensemble perturbation;
$\mathbf{x}_s^{'}$ 3DVAR increment; $\mathbf{x}^{'}$ total (hybrid) increment; $\mathbf{y}^{o'}$ innovation vector;
$\mathbf{H}$ linearized observation operator; β_1 weighting coefficient for static covariance;
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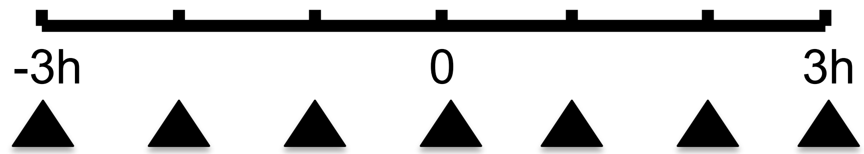
4DEnVar vs. En4DVar Hybrid

- **En4DVar**

Lorenc et al. 2015



$$J\left(\mathbf{x}_s^{'},\boldsymbol{\alpha}\right)=\beta_1 J_s+\beta_2 J_e+J_o$$

$$=\beta_1\frac{1}{2}\mathbf{x}_s^{'T}\mathbf{B}^{-1}\mathbf{x}_s^{'}+\beta_2\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{C}^{-1}\boldsymbol{\alpha}+\frac{1}{2}\sum_{t=0}^{T}(y_t^{o\prime}\text{-}\mathrm{H}_t\mathbf{x}_t^{'})^{\mathrm{T}}\mathrm{R}_t^{-1}(y_t^{o\prime}\text{-}\mathrm{H}_t\mathbf{x}_t^{'})$$

$$\mathbf{x}_t^{'}=\mathbf{M}_t\left[\mathbf{x}_s^{'}+\sum_{k=1}^{K}\left(\boldsymbol{\alpha}_k\circ(\mathbf{x}_k^e)_0\right)\right]$$

Use tangent linear and adjoint to propagate increments over time

$\mathbf{B}$ stat 3DVAR static covariance; $\mathbf{R}$ observation error covariance; K ensemble size; $\mathbf{C}$ Concatenated correlation matrix for ensemble covariance localization; $\mathbf{x}_k^e$ kth ensemble perturbation; $\mathbf{M}_t$ tangent linear model

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Summary of Hybrid flavors

	Number of time levels of ensemble perturbations incorporated in the DA window during the variational minimization	Tangent linear and adjoint of the forecast model
3DEnVar hybrid	One, usually valid at the center of the DA window	Not needed
4DEnVar hybrid	Multiple	Not needed, same static covariance is used for multiple time levels, equivalent to assuming a numerical model of identity matrix
En4DVar hybrid	One, usually valid at the beginning of the DA window	Needed

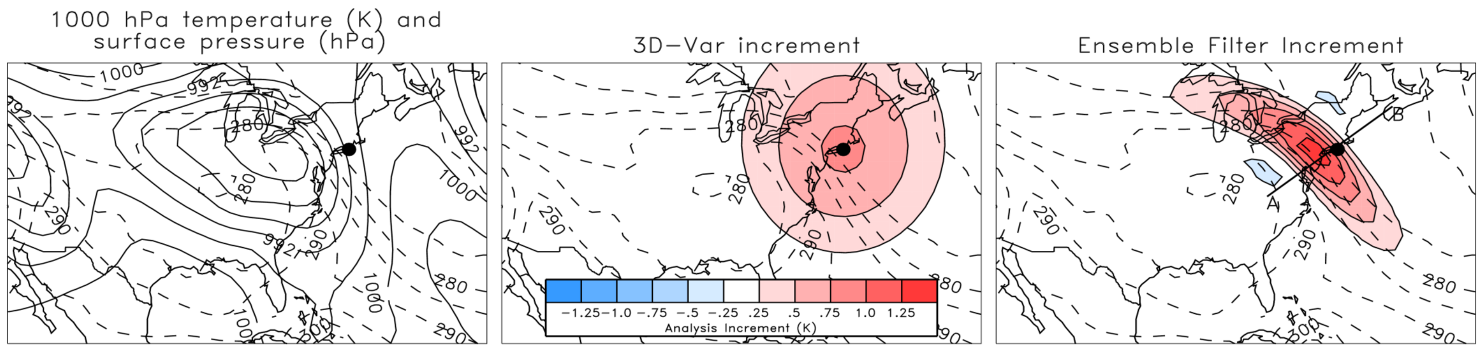
Adapted from Wang and Lei 2014

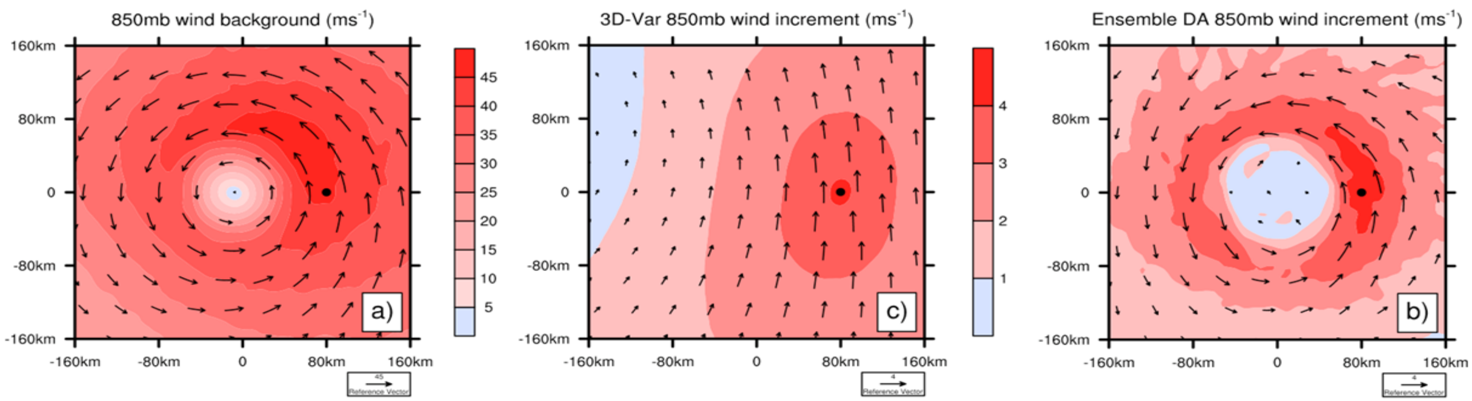
Why Hybrid?

	VAR (3D, 4D)	EnKF	hybrid	References (e.g.)
Benefit from use of flow dependent ensemble covariance instead of static B		yes	yes	Hamill and Snyder 2000; Wang et al. 2007a,2008ab, 2009, 2013; Buehner et al. 2010ab; Wang 2011; etc.
Robust for small ensemble			yes	Wang et al. 2007b, 2009;
Model space covariance localization	yes		yes	Campbell et al. 2010
Flexible to add various dynamical/physical constraints	yes		yes	Wang et al. 2013
Built in outer loops for nonlinearity treatment	yes		yes	
Use of various existing capabilities in VAR, e.g. dual resolution hybrid	yes		yes	

Summarized in Wang 2010, MWR

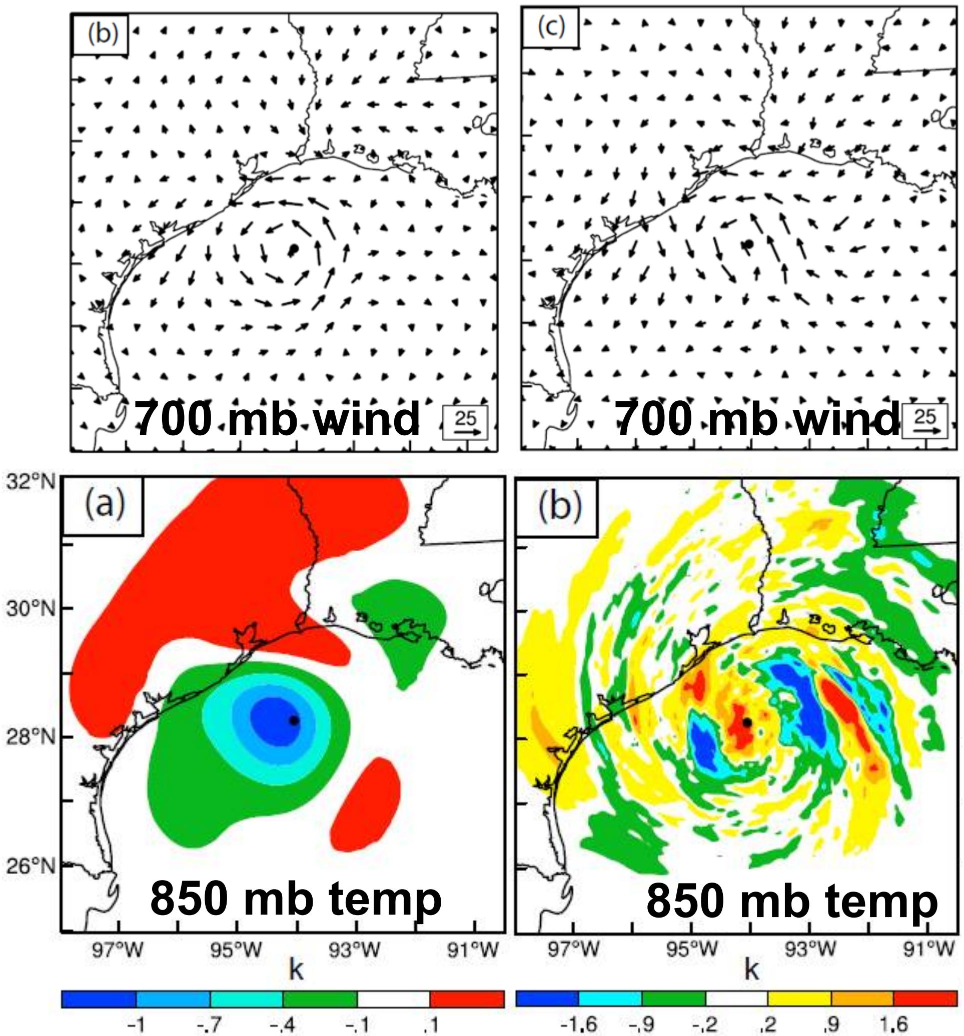
Static vs. flow dependent covariance





Cross-variable increment

3DVAR hybrid

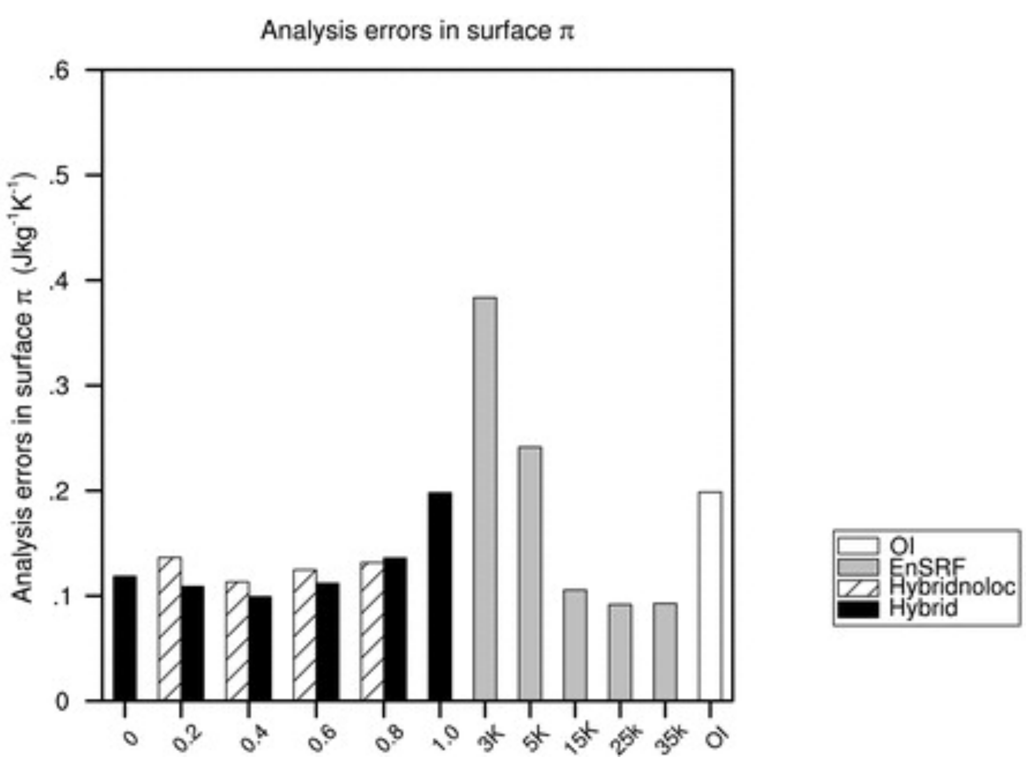


- Hurricane IKE 2008
- WRF ARW: Δx=5km
- Observations: radial velocity from two WSR88D radars (KHGX, KLCH)
- WRFVAR hybrid DA system (Wang et al. 2008ab)

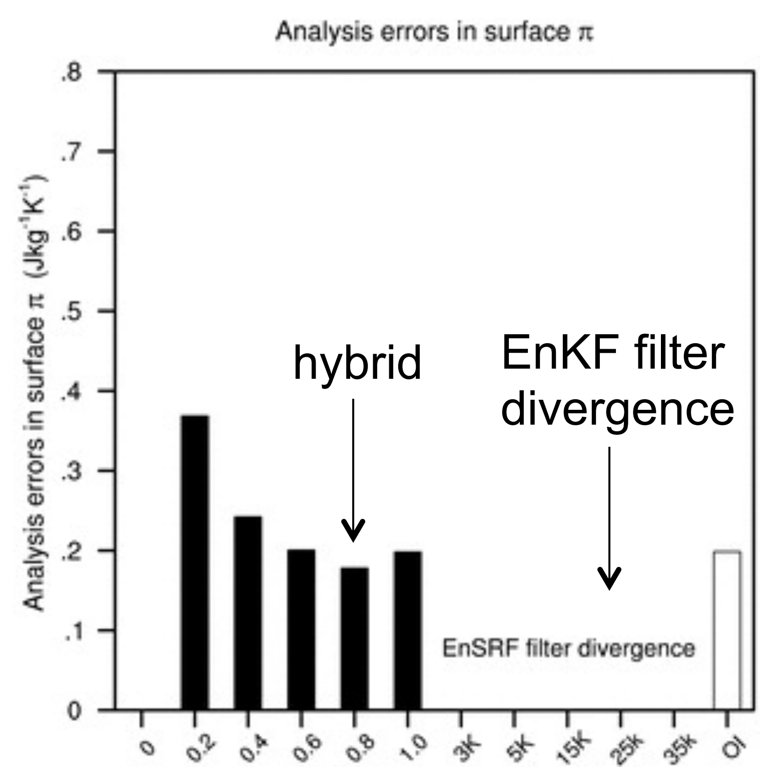
Li et al., 2012

Hybrid robust for small ensemble

50 member

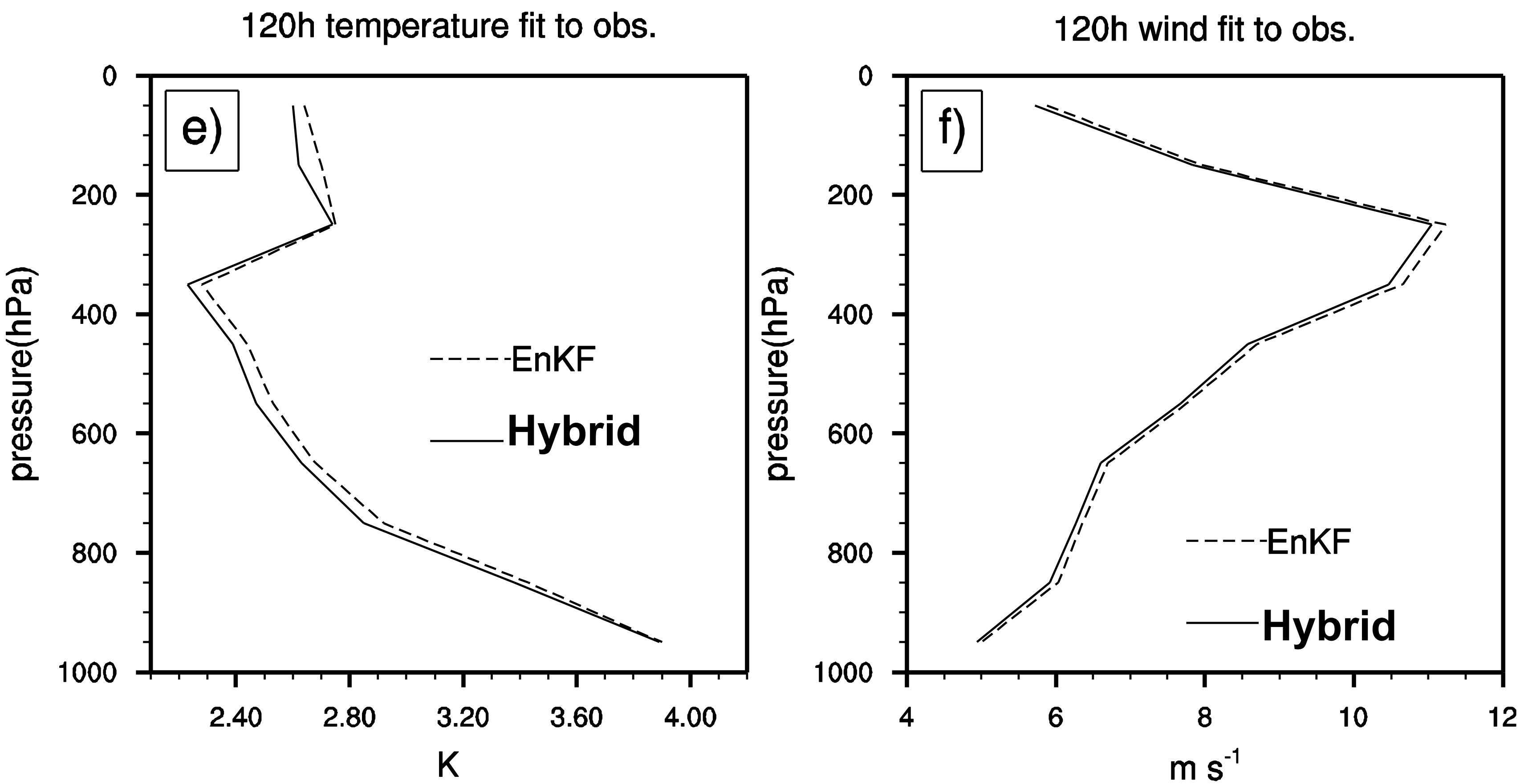


5 member



Wang et al. 2007b, 2009

Hybrid benefits from variational constraints: example from GSI hybrid for GFS



- Hybrid with full ensemble covariance was better than EnKF due to the use of tangent linear normal mode balance constraint

Wang et al. 2013

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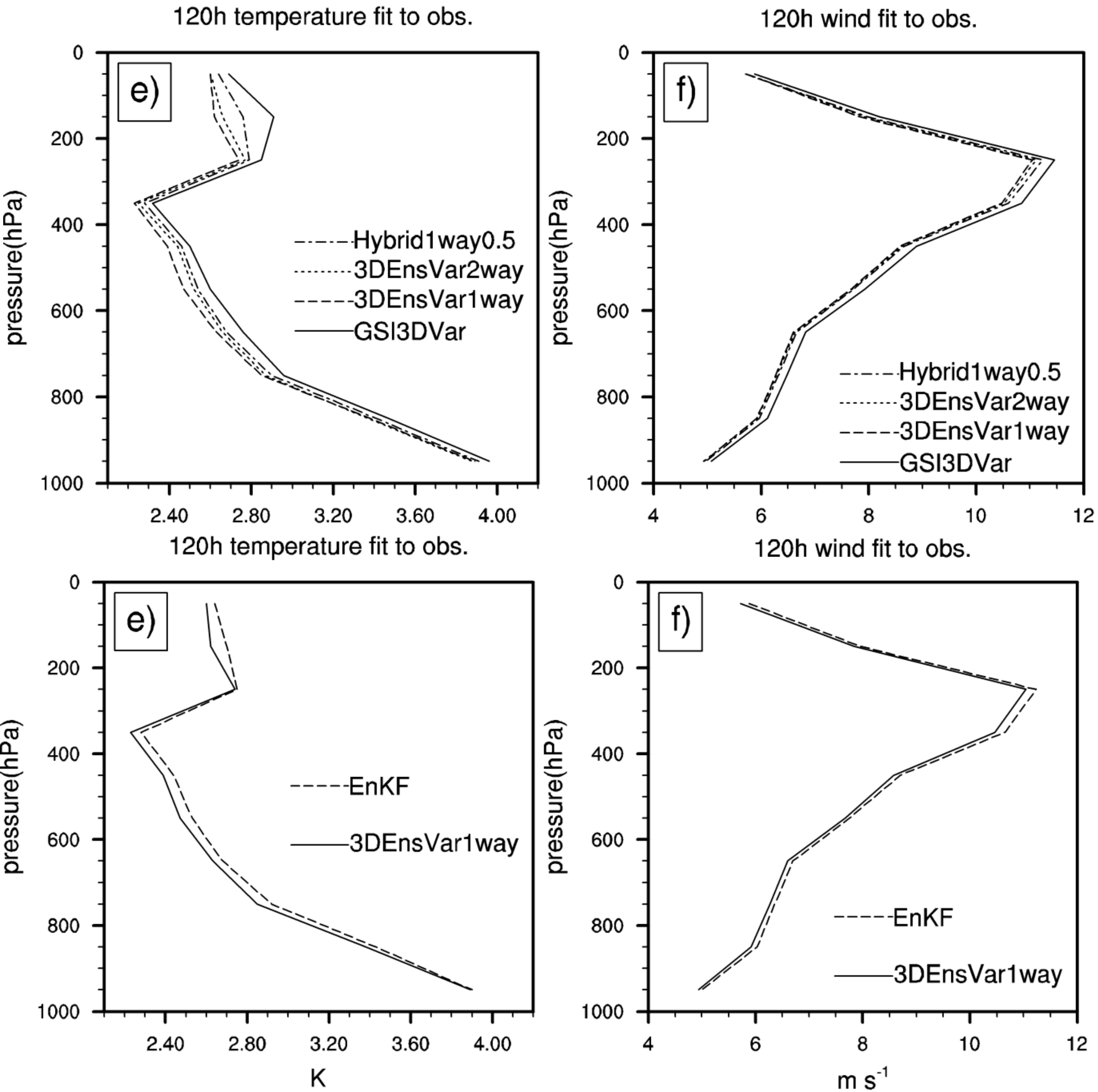
- ❑ Future work and challenges

GSI hybrid for GFS: GSI 3DVar vs. 3DEnVar Hybrid vs. EnKF

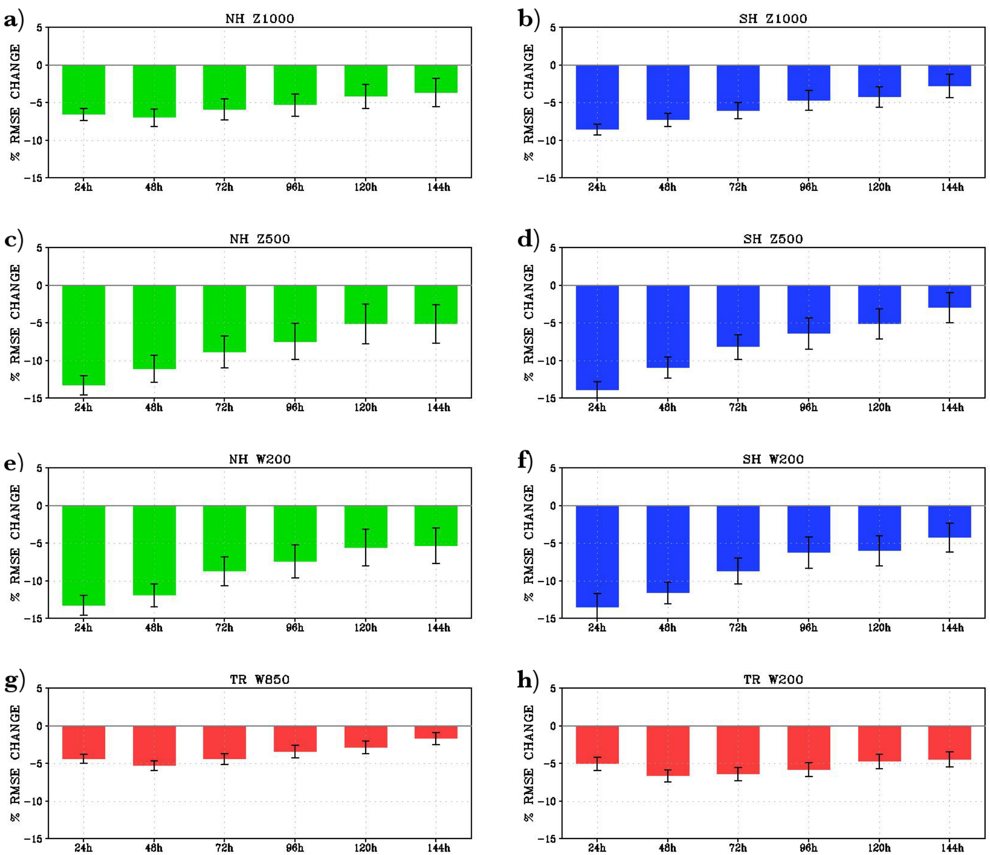


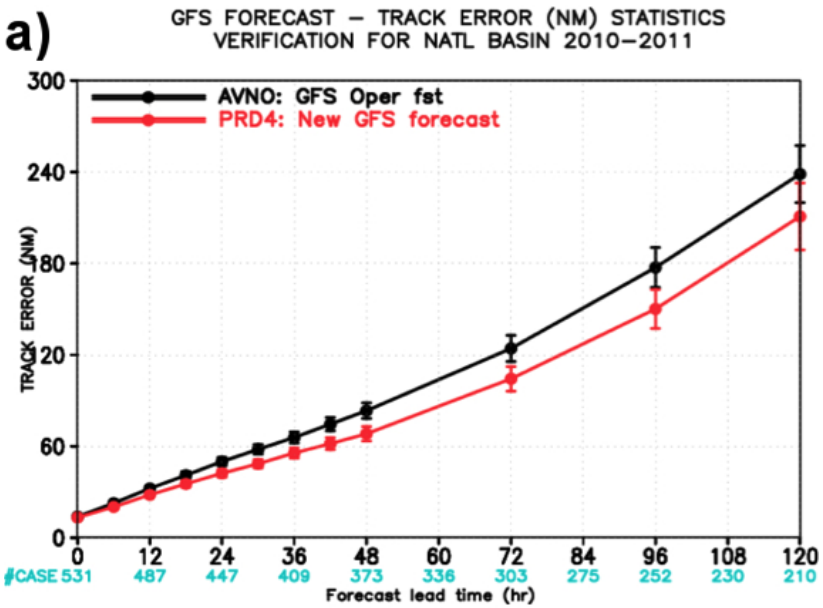
- 3DEnVar Hybrid was better than 3DVar due to use of flow-dependent ensemble covariance

- 3DEnVar was better than EnKF due to the use of tangent linear normal mode balance constraint (TLNMC)

Wang et al., MWR, 2013, 141, 4098-4117

GSI hybrid for GFS: NCEP pre-implementation test



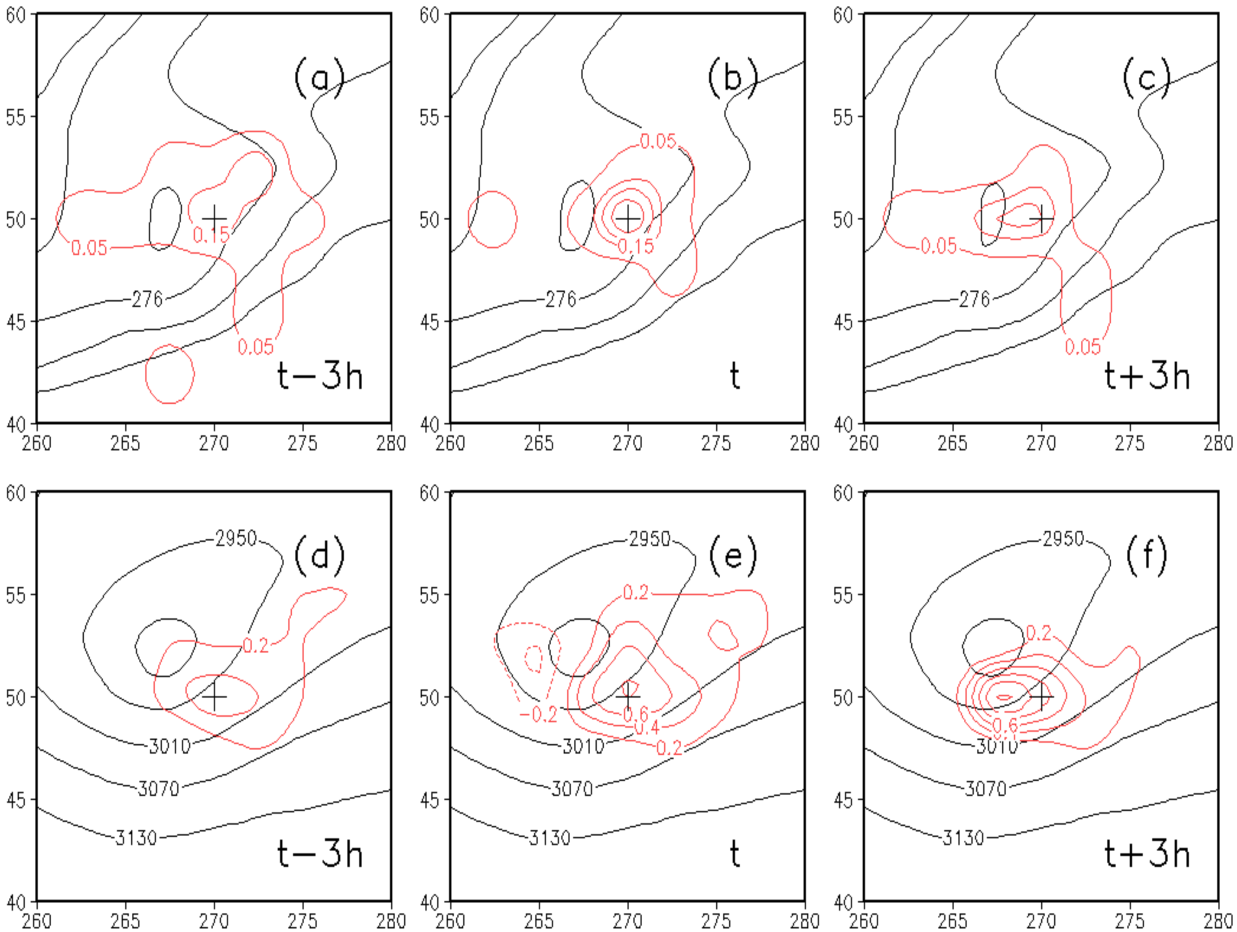


Courtesy: Daryl Kleist

Temporal evolution of error covariance by GSI 4DEnVar

Temp.

Height



Downstream impact

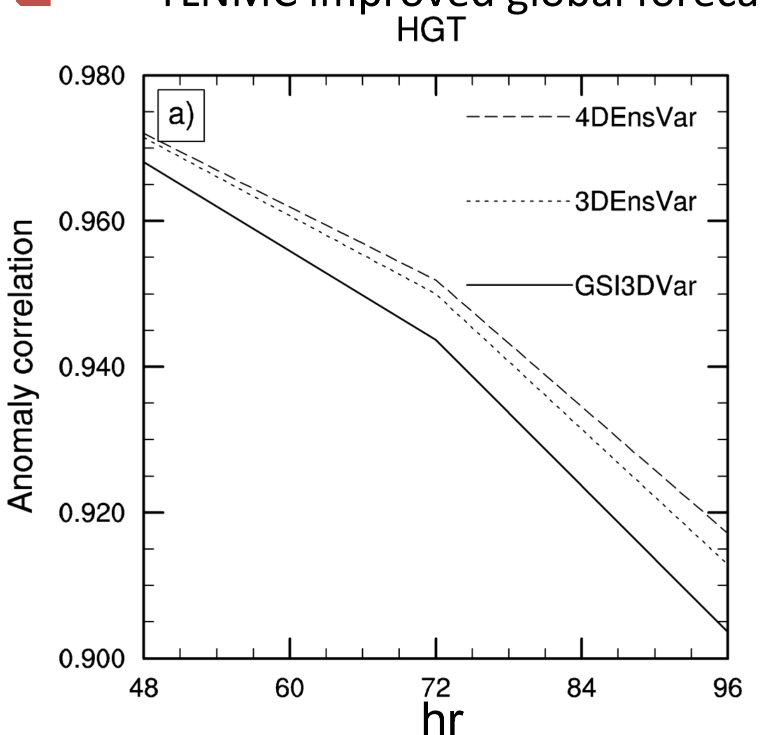
Upstream impact

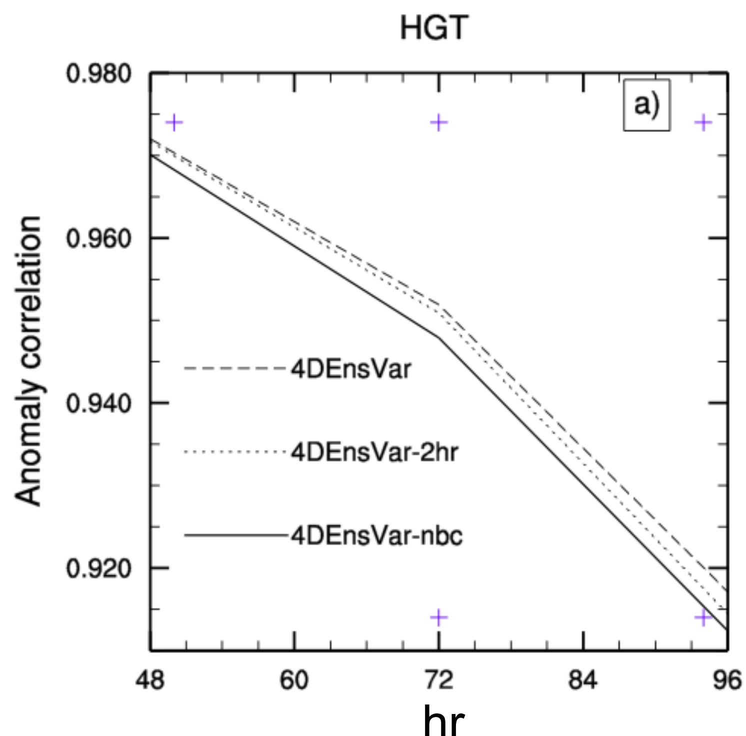
Wang and Lei, 2014, MWR

GSI hybrid for GFS: 3DEnVar vs. 4DEnVar

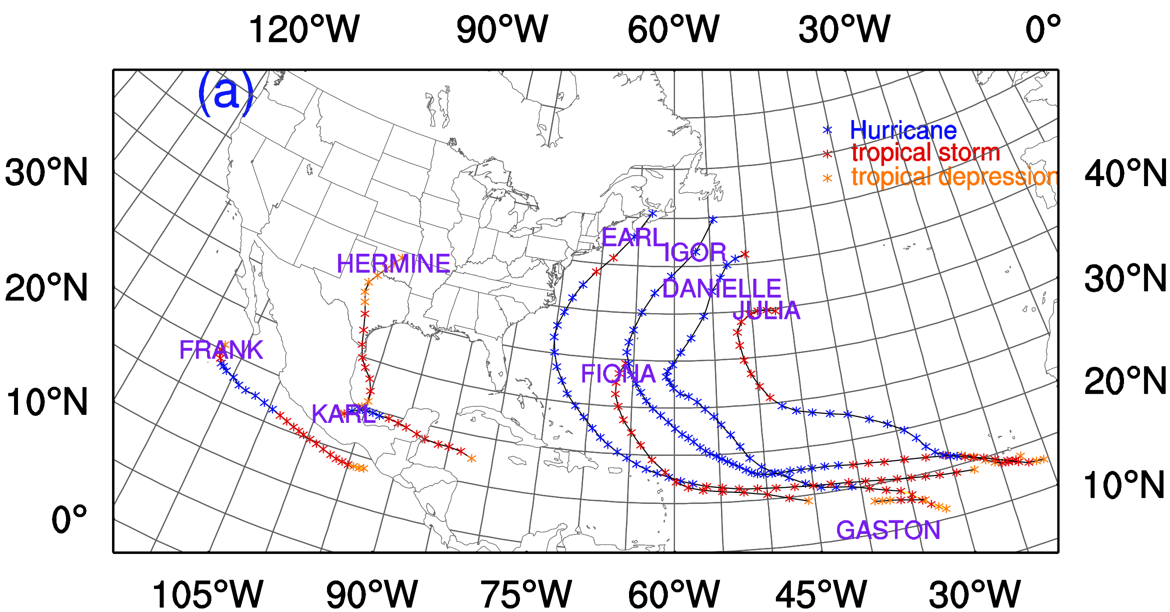
Results from Single Reso. Experiments (Wang and Lei 2014, MWR)

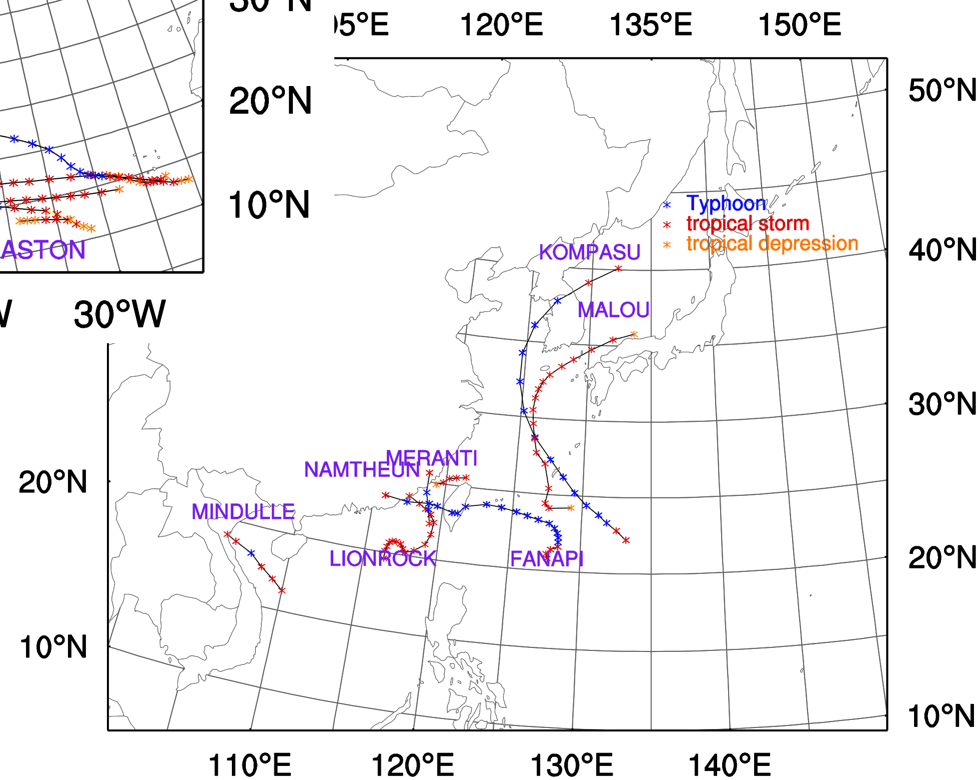
- 4DEnVar improved general global forecasts
- 4DEnVar improved the balance of the analysis
- Performance of 4DEnVar improved if more frequent ensemble perturbations used
- 4DEnVar approximates nonlinear propagation better with more frequent ensemble perturbations
- TLNMC improved global forecasts





GSI hybrid for GFS: 3DEnVar vs. 4DEnVar





16 named storms in Atlantic and Pacific basins during 2010

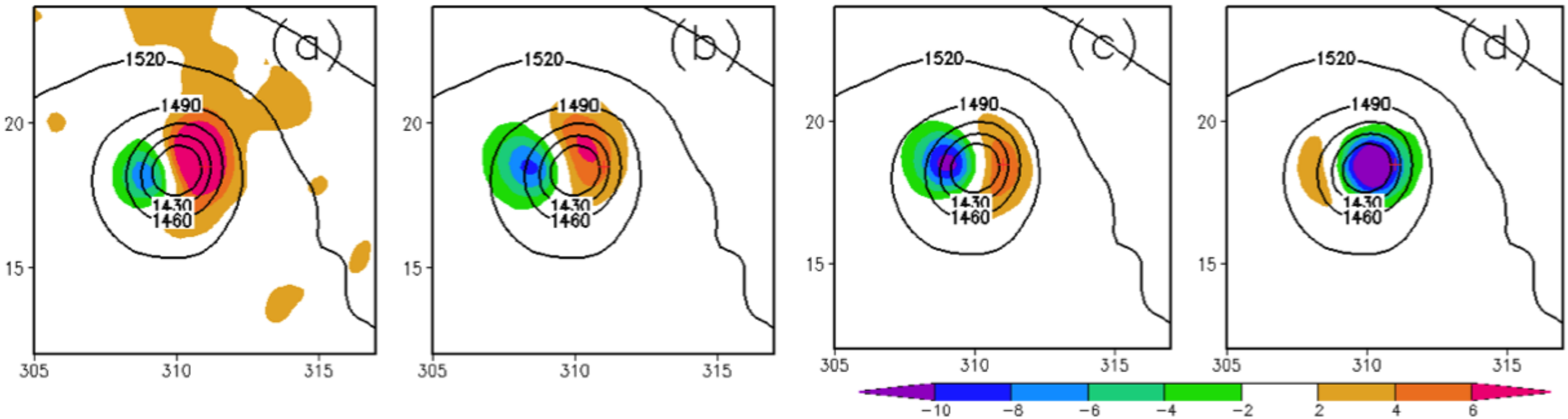
Approximation to nonlinear propagation

–3h increment propagated by model integration

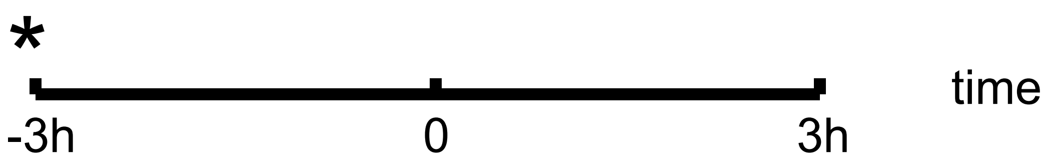
4DEnVar (hrly pert.)

4DEnVar (2hrly pert.)

3DEnVar

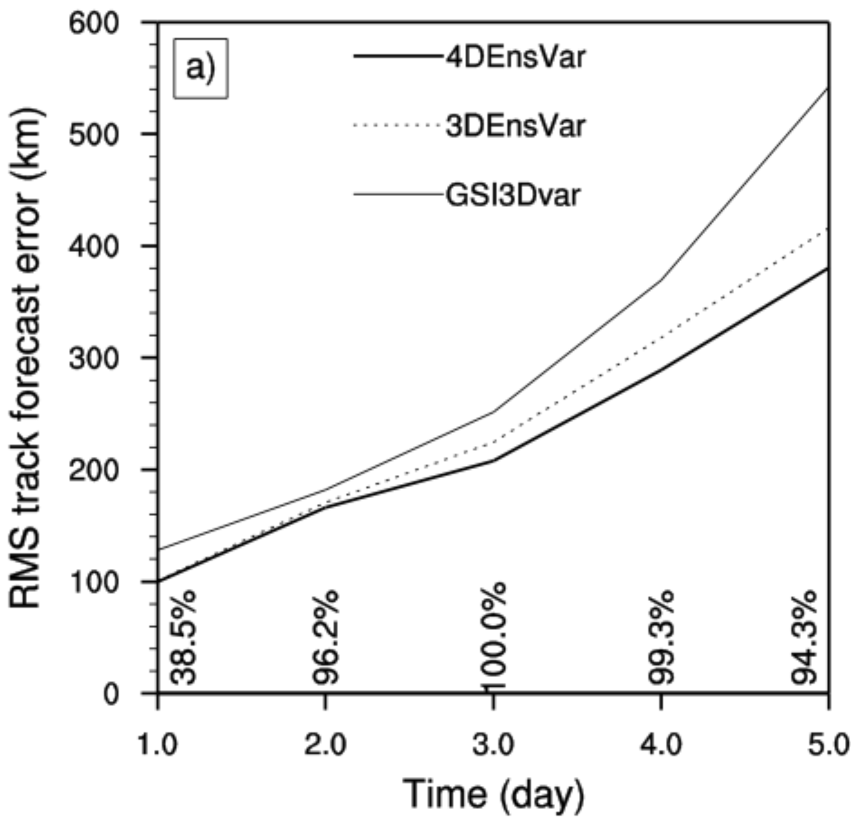


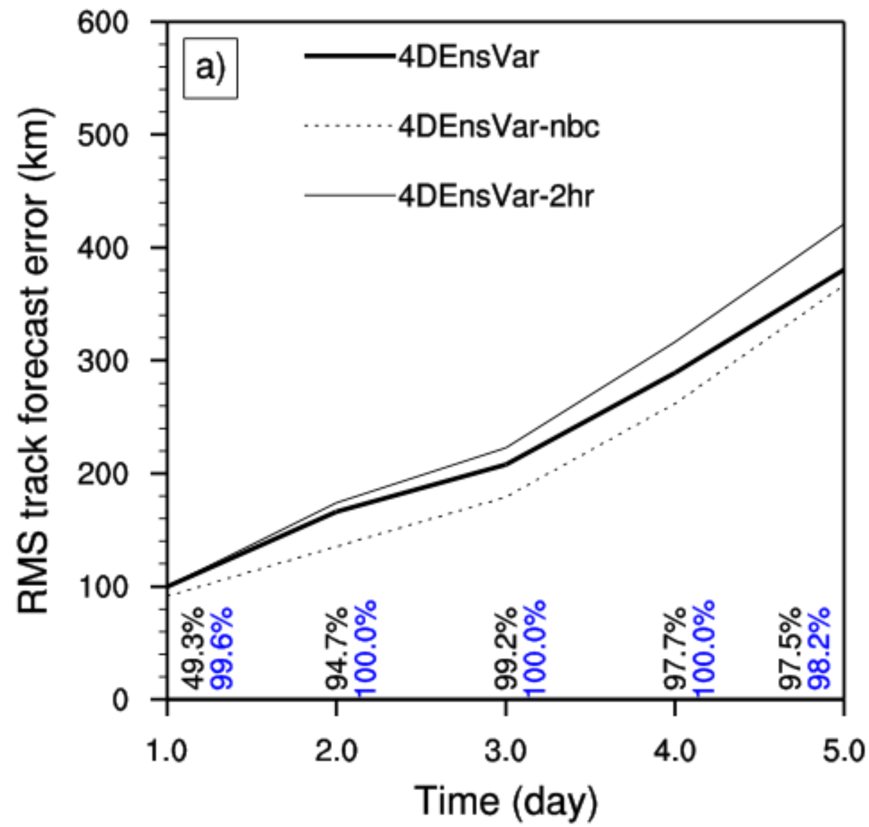
Hurricane Daniel 2010



Wang and Lei, 2014, MWR

Verification of hurricane track forecasts





- 3DEnVar outperforms GSI3DVar.
- 4DEnVar is more accurate than 3DEnVar after the 1-day forecast lead time.
- Negative impact if using less number of time levels of ensemble perturbations.
- Negative impact of TLNMC on TC track forecasts.

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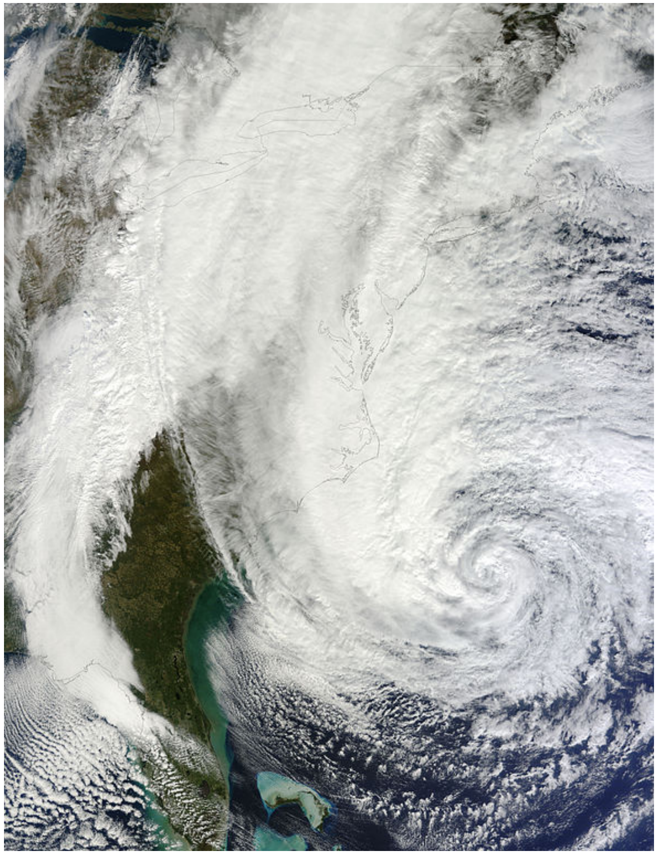
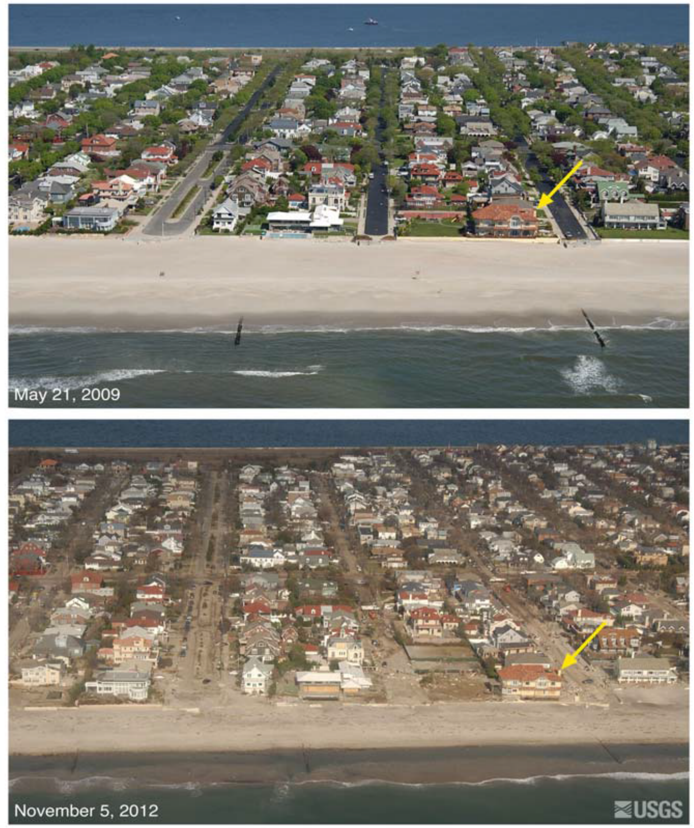
GSI hybrid for HWRF
Hurricane Sandy, Oct. 2012



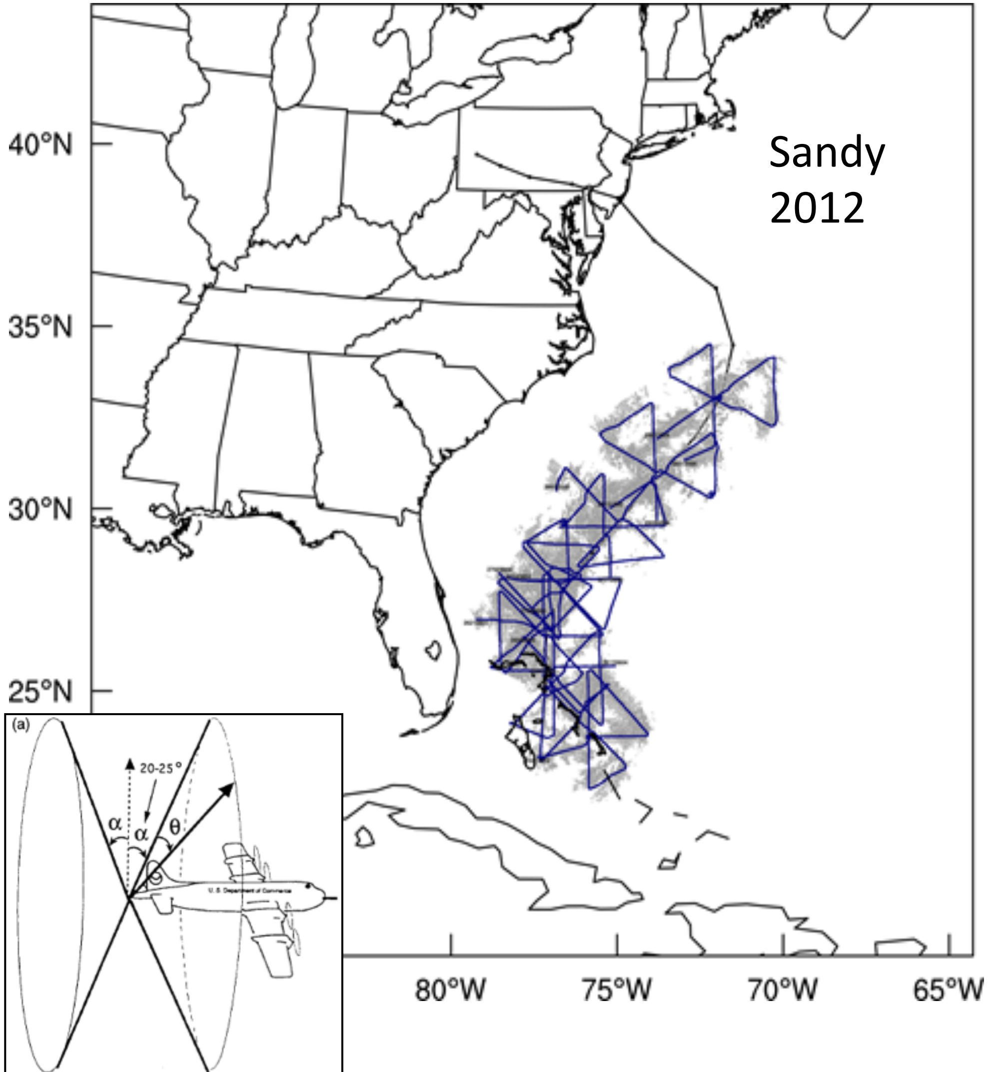


- Complicated evolution
- Tremendous size
- 147 direct deaths across Atlantic Basin
- US damage $50 billion

New York State before and after

nhc.noaa.gov

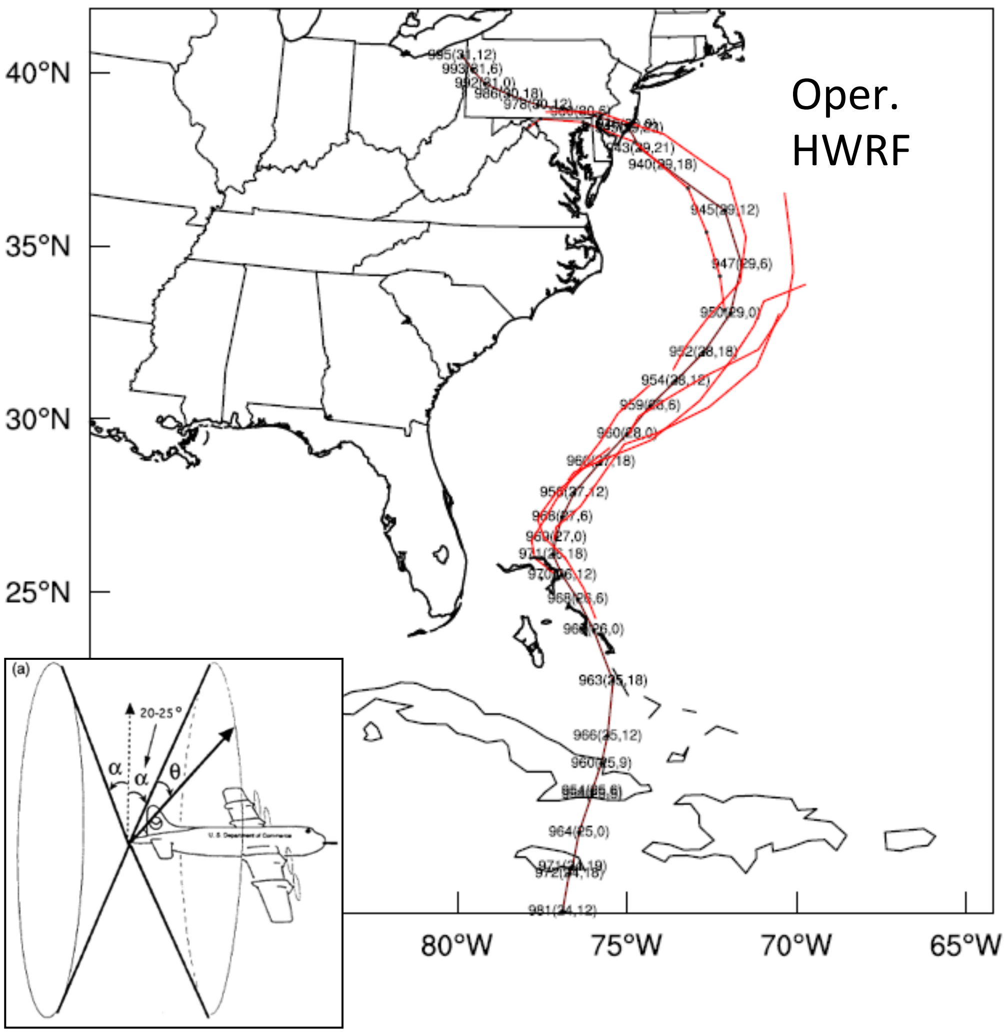
Experiment Design



- Model: HWRF
- Observations: radial velocity from Tail Doppler Radar (TDR) onboard NOAA P3 aircraft
- Initial and LBC ensemble: GFS global hybrid DA system
- Ensemble size: 40

Lu et al. 2015a

Experiment Design



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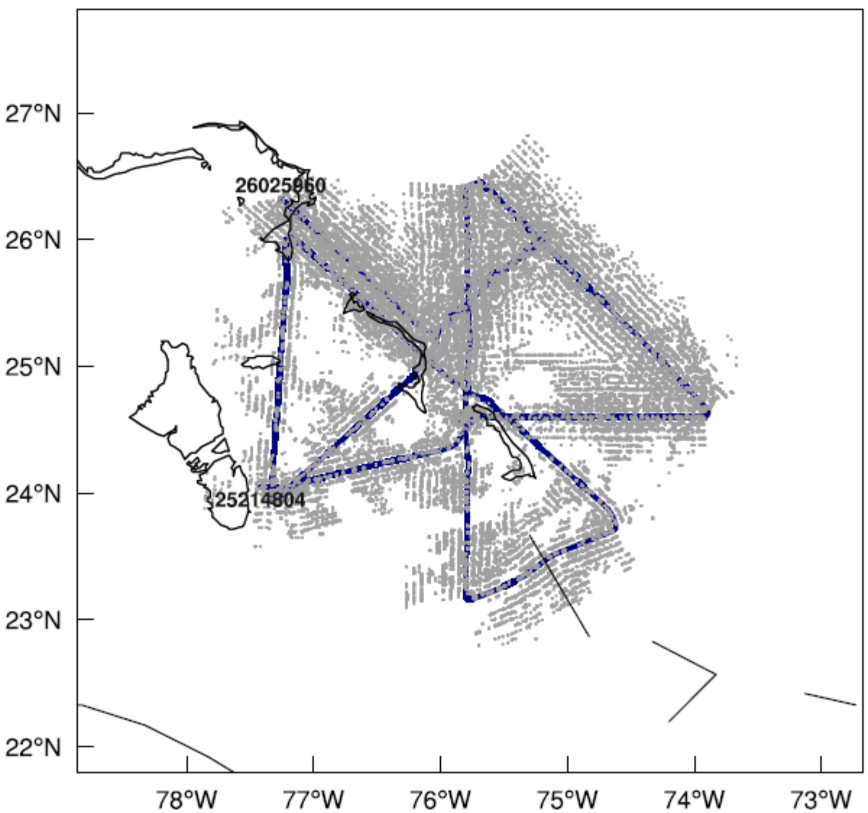
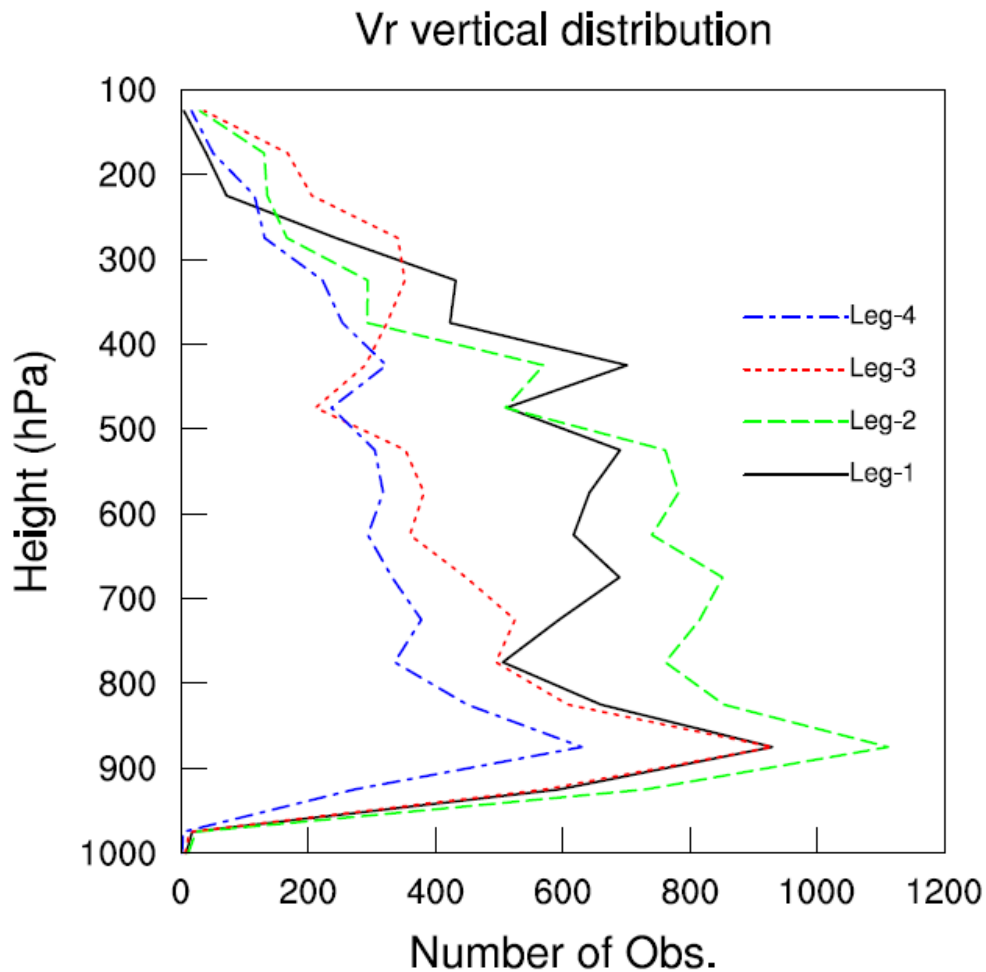
TDR data distribution (mission 1)

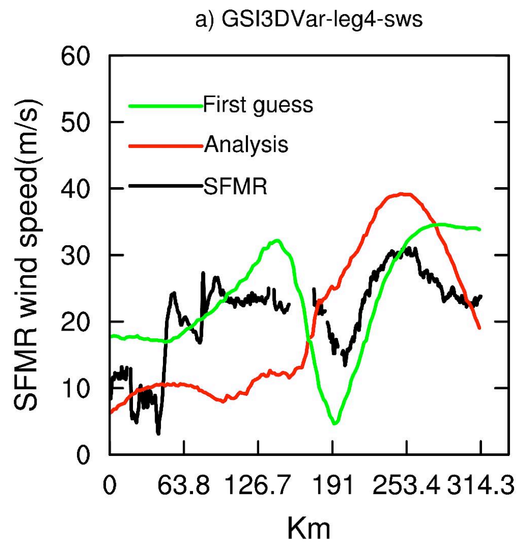
P3 Mission 1

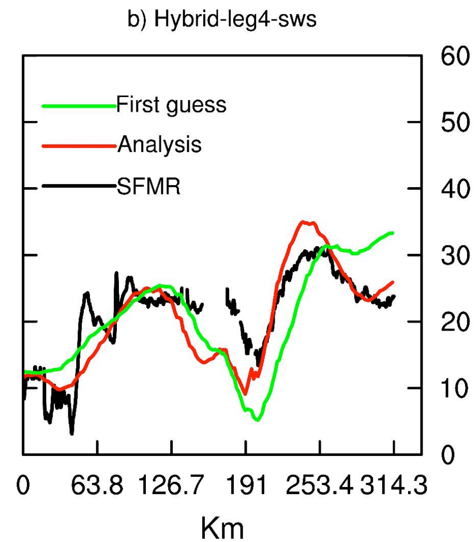


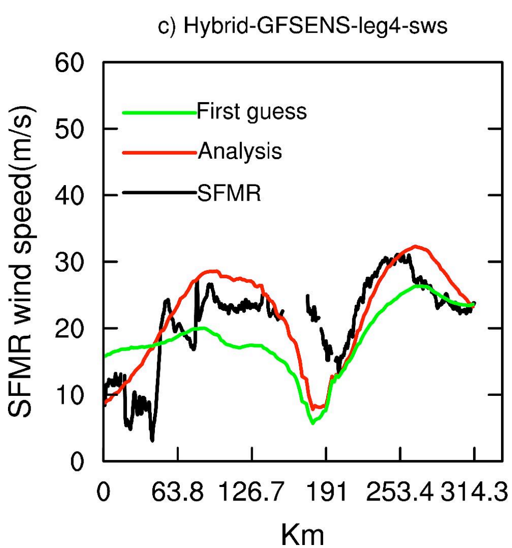


Verification against SFMR wind speed

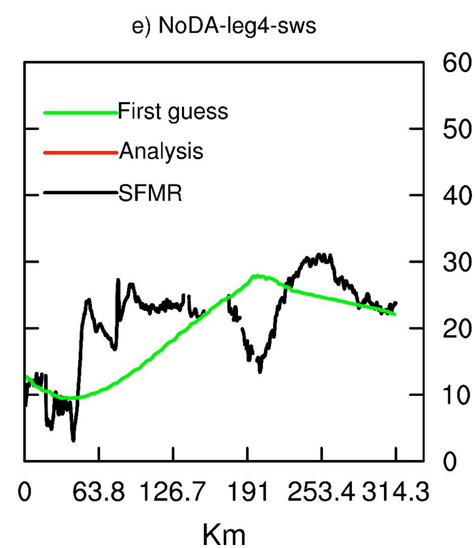
Last Leg



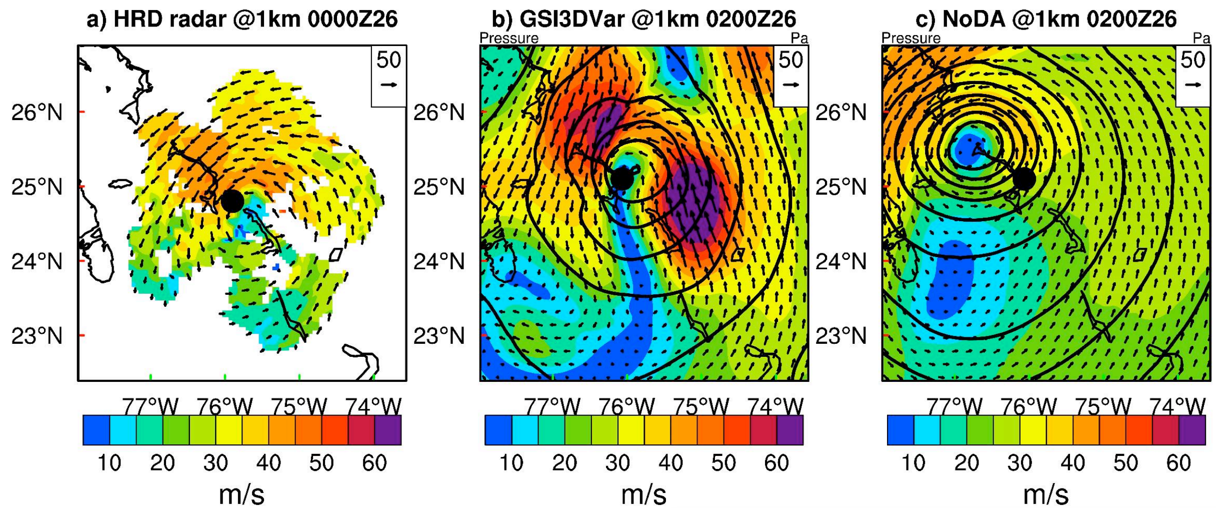


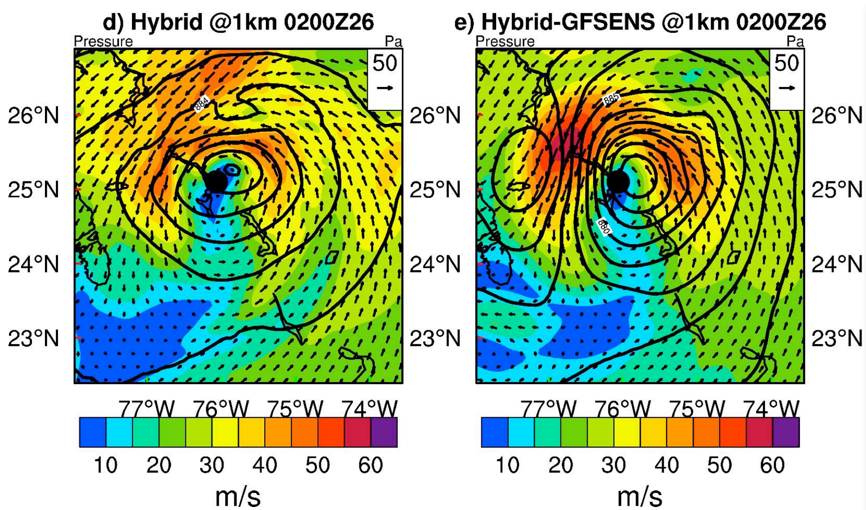


- Ingesting the GFS, the posterior was much improved compared to using 3DVar.
- For both the prior and the posterior, the hybrid using HWRF's own ensemble was better than ingesting coarse resolution GFS ensemble



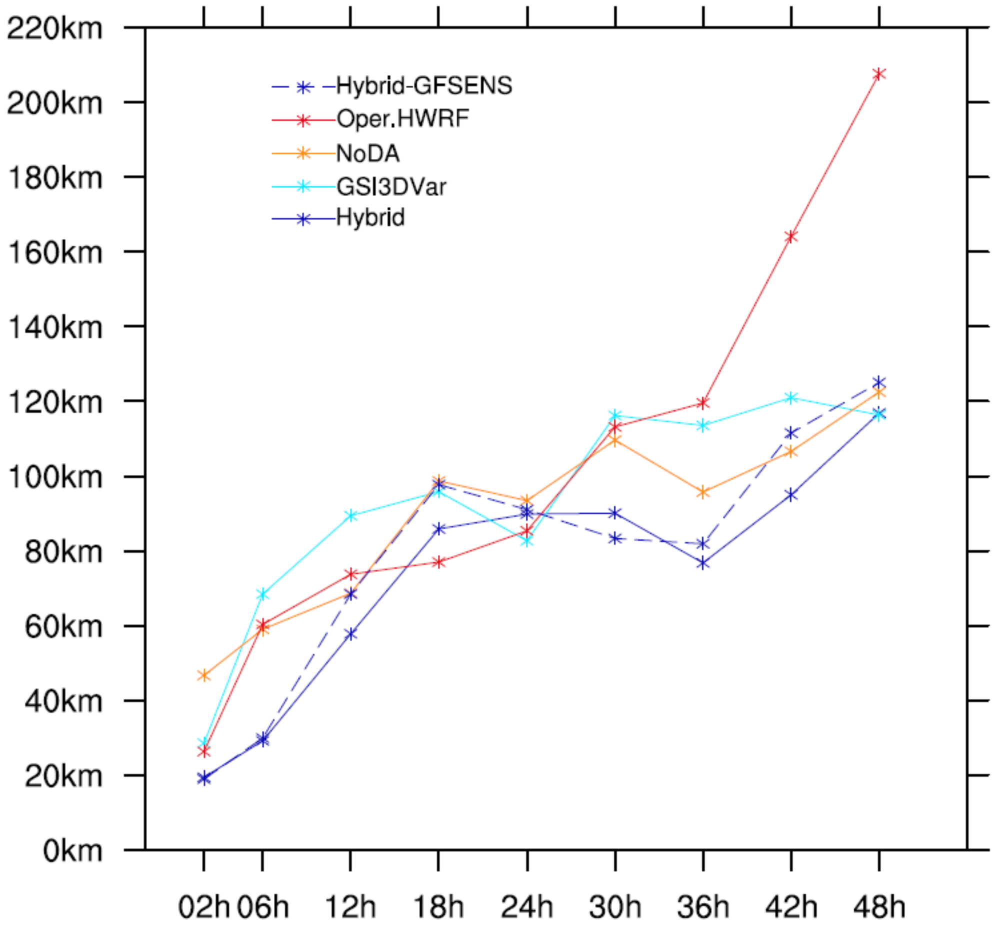
Comparison with HRD radar wind analysis

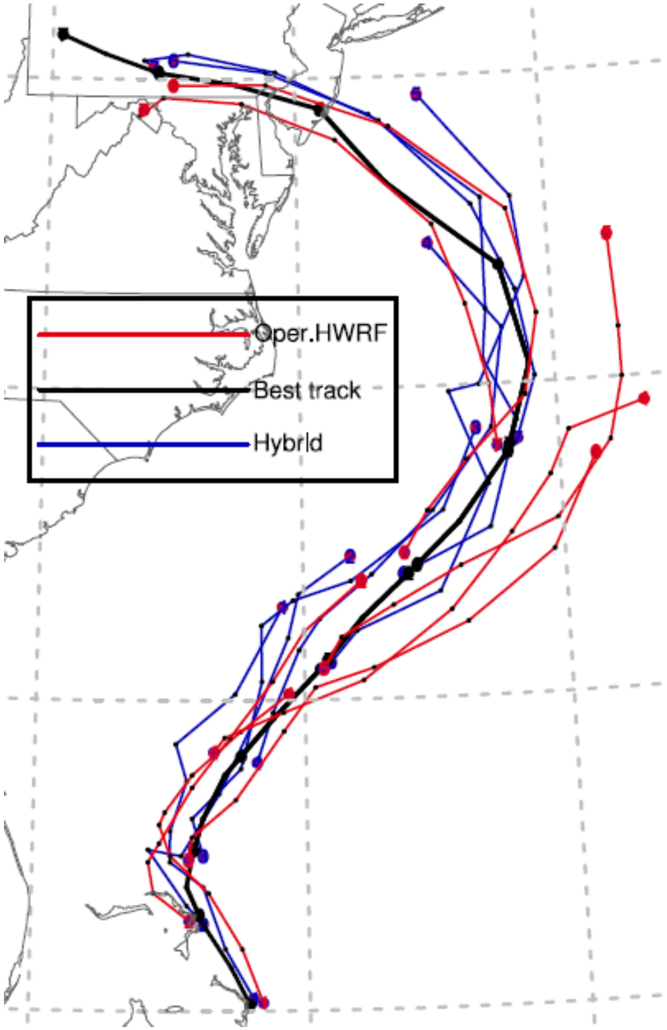




- Using hybrid, the wind pattern fits the HRD wind analysis the best.
- Compared to 3DVar, HWRF hybrid ingesting GFS ensemble produces the wind pattern which looks a lot like the HRD wind composite, but it is not as good as hybrid ingesting HWRF ensemble.

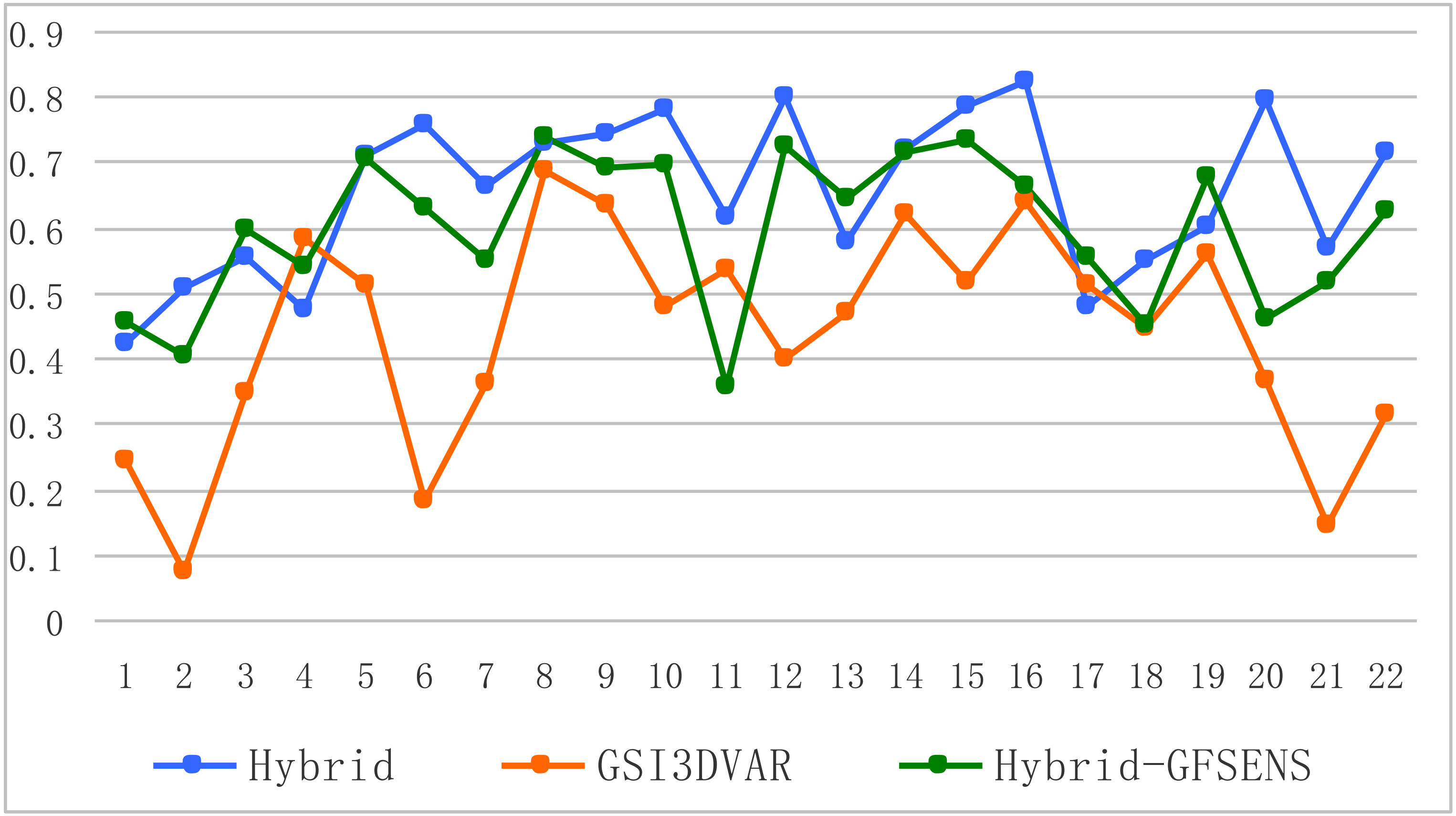
Track forecast (RMSE for 7 missions)





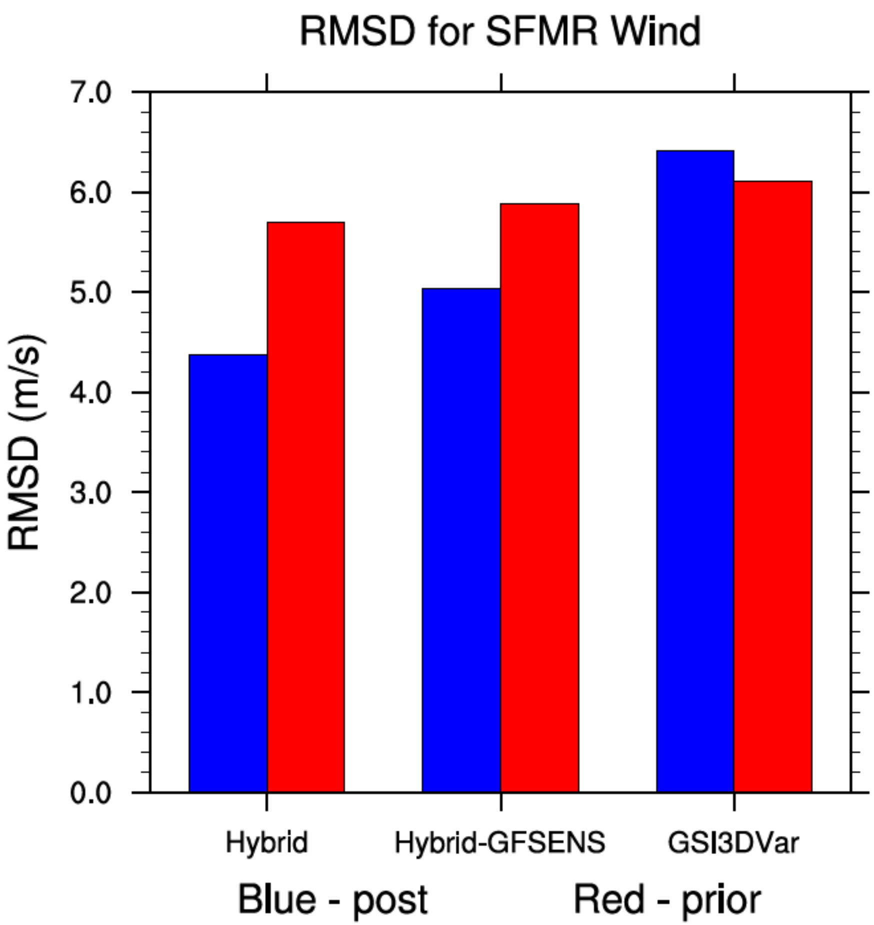
Experiments for 2012-2013 seasons

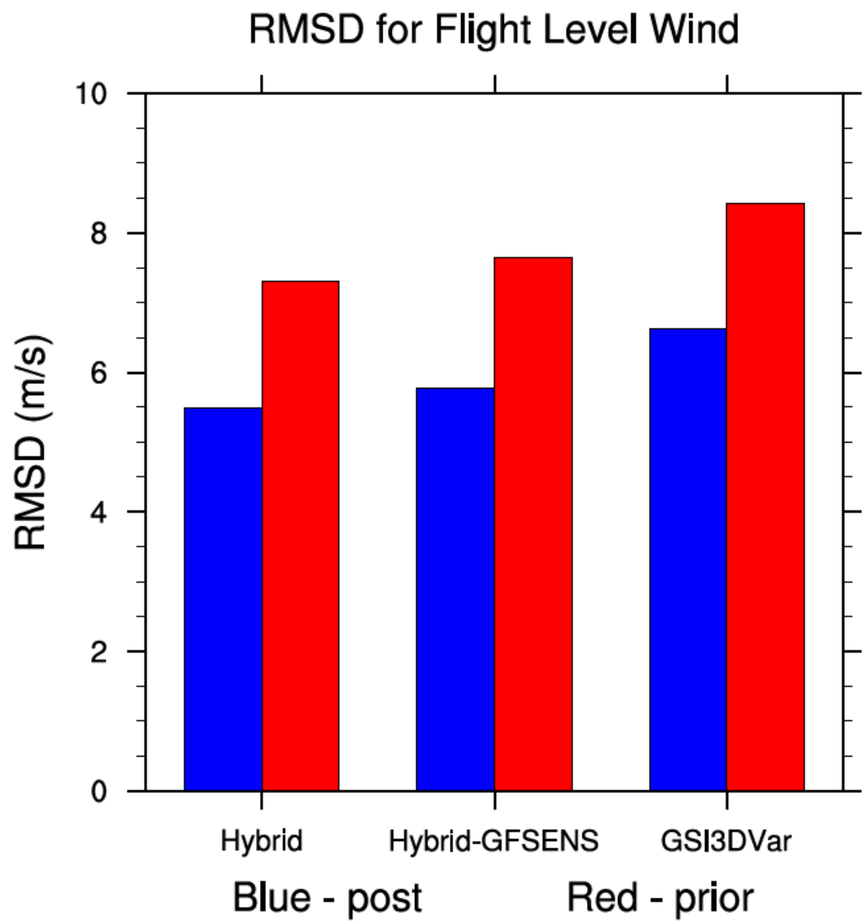
Correlation between HRD radar wind analysis and analyses from various DA methods



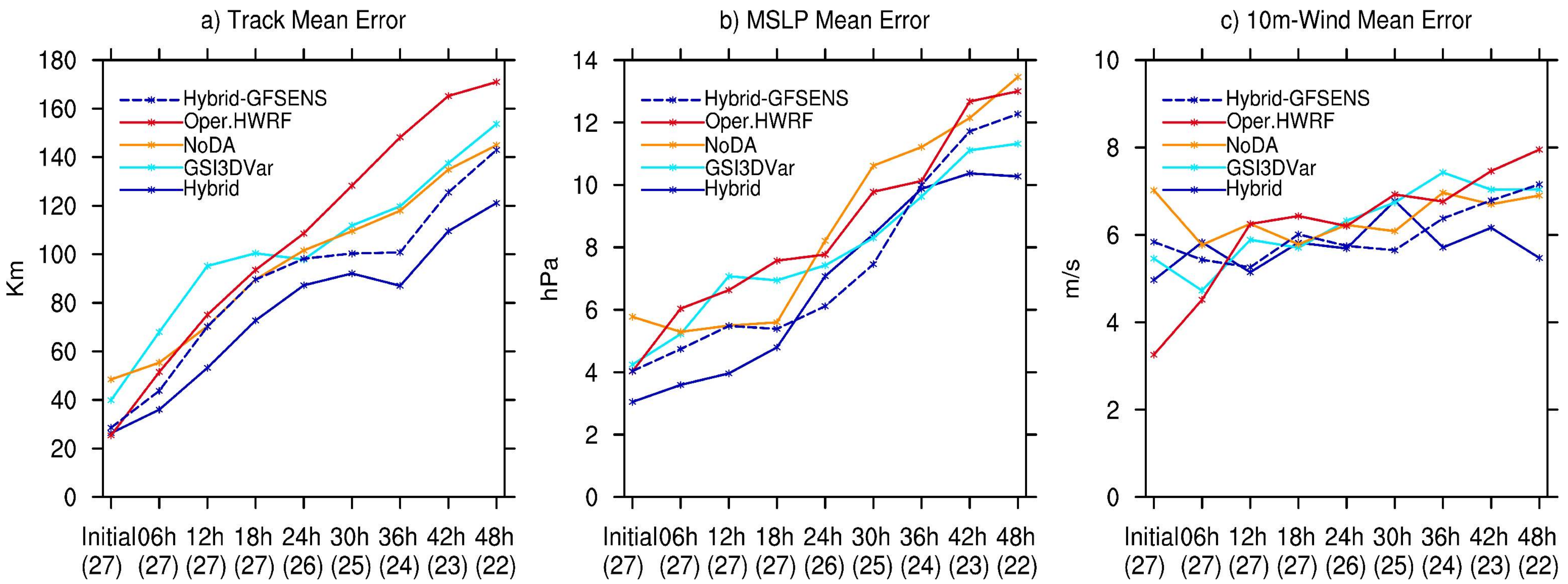
Case#

Verification against SFMR and flight level data

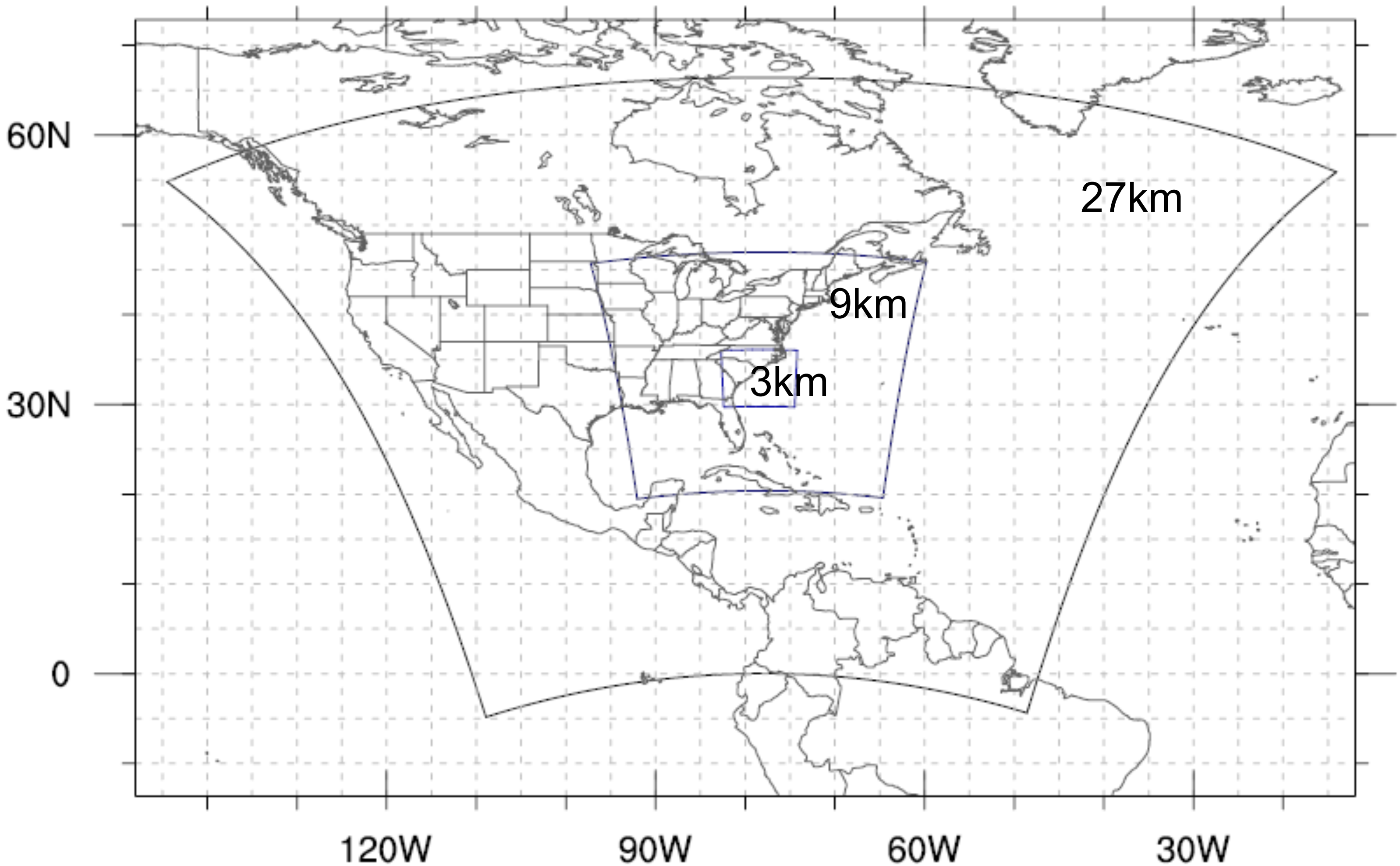




Experiments for 2012-2013 season



HWRF Dual Resolution Hybrid DA



Lu et al. 2015b

Experiments for Edouard 2014

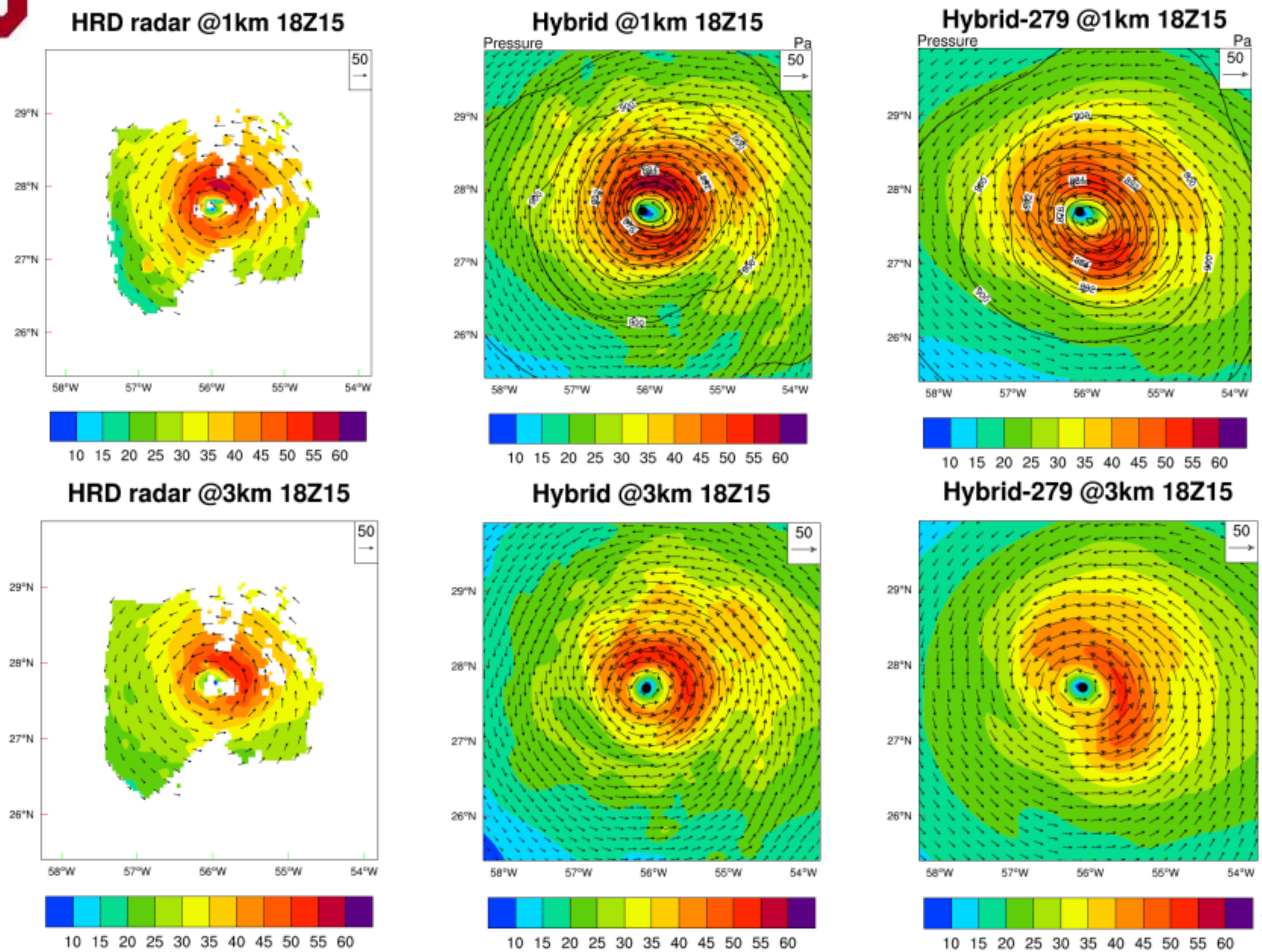
Observations assimilated:

3km domain:
Conventional in-situ data in prepbufr, satellite wind, TDR and tcvital

9km domain:
Conventional in-situ data in prepbufr, satellite wind, TDR and tcvital
Satellite radiances

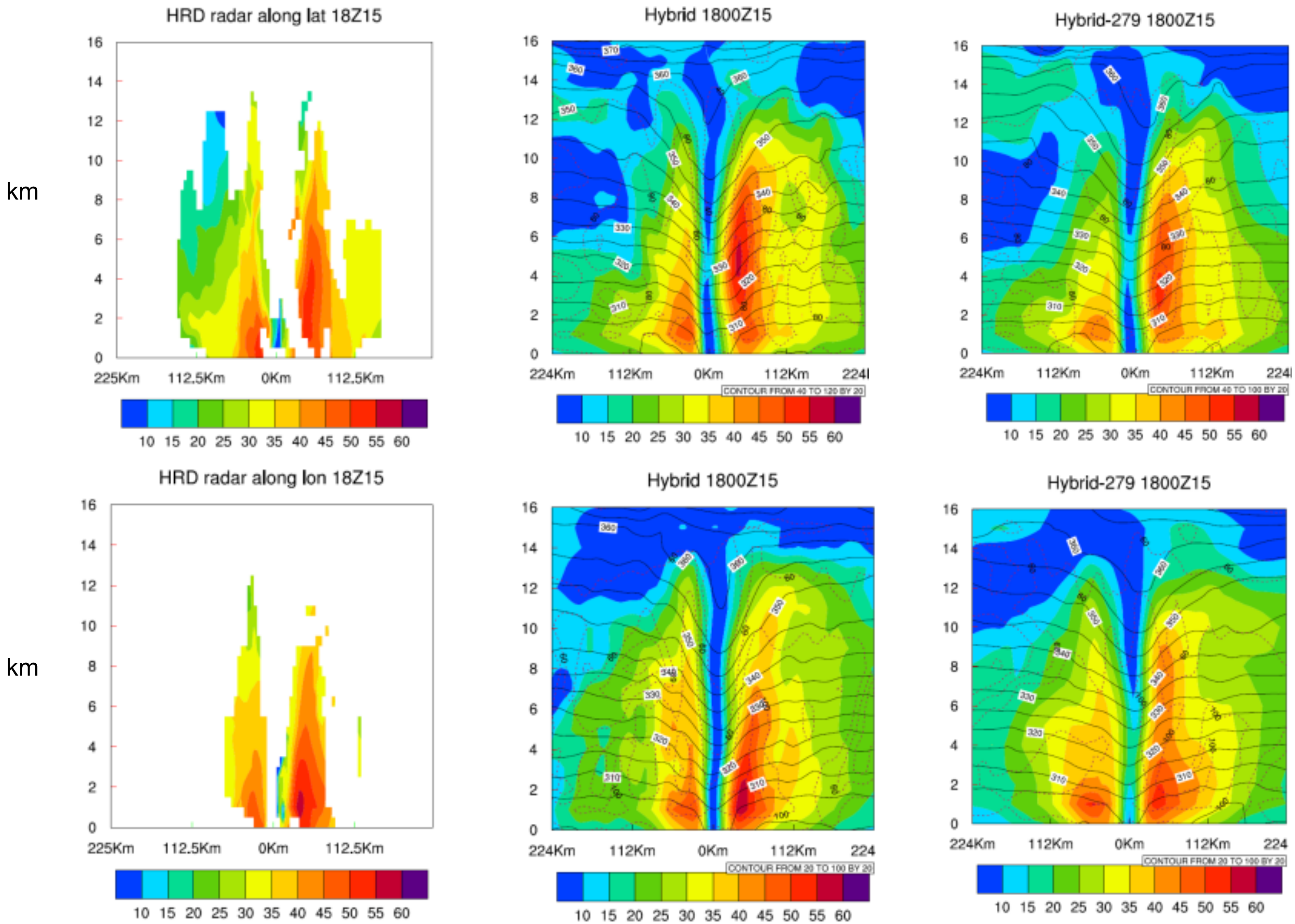
Hybrid vs Hybrid-279

analyzed Edouard structure @2014091518



Hybrid vs Hybrid-279

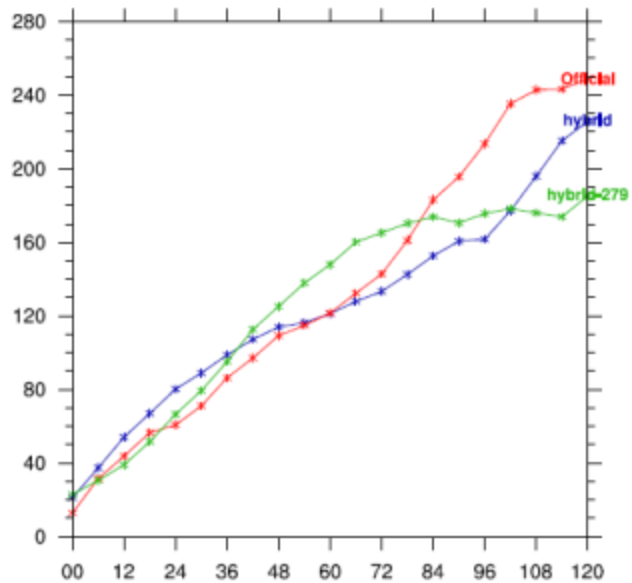
analyzed Edouard structure @2014091518



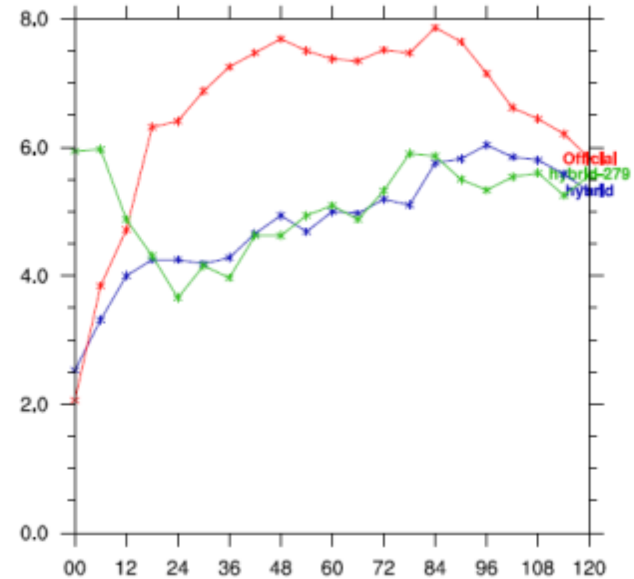
Hybrid vs Hybrid-279 vs operational HWRF

RMSE for all cycles

Track (km)



MSLP (hPa)



- Hybrid improved especially MSLP and Vmax forecasts compared to operational HWRF
- Dual resolution hybrid improved MSLP and Vmax forecast for the first 12/18 hours than single 9km resolution hybrid

Outline

- ❑ Introduction to hybrid DA methods
- ❑ Examples of recent research, development and application of hybrid DA
  - ➢ Recent R&D to improve global forecasts
  - ➢ Recent R&D to improve high resolution hurricane forecasts
  - ➢ Recent R&D for convective scale weather forecasts over CONUS
- ❑ Future work and challenges

Motivation for Hybrid DA Development and Research for Convective Scales

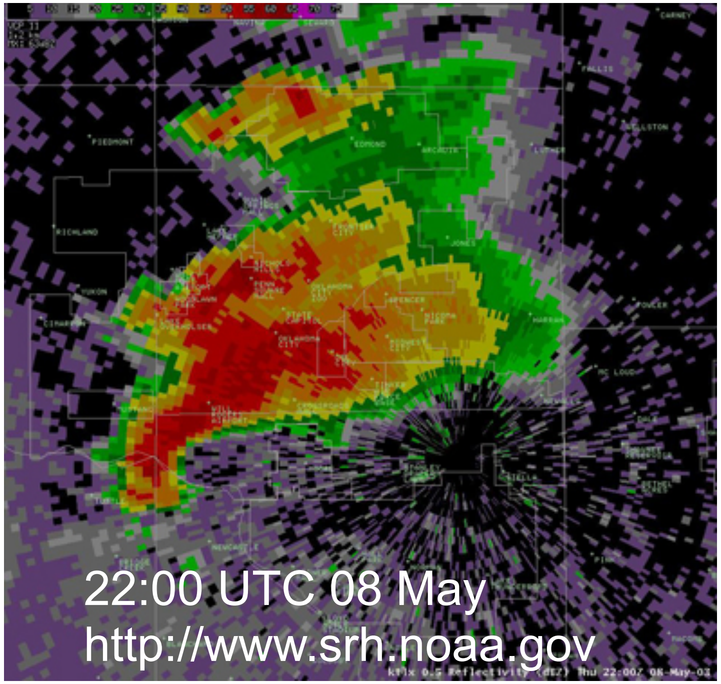
- Convective scale analysis and forecasting is a multi-scale problem, requiring an accurate estimate of both the synoptic/mesoscale environment and the convective scale details.

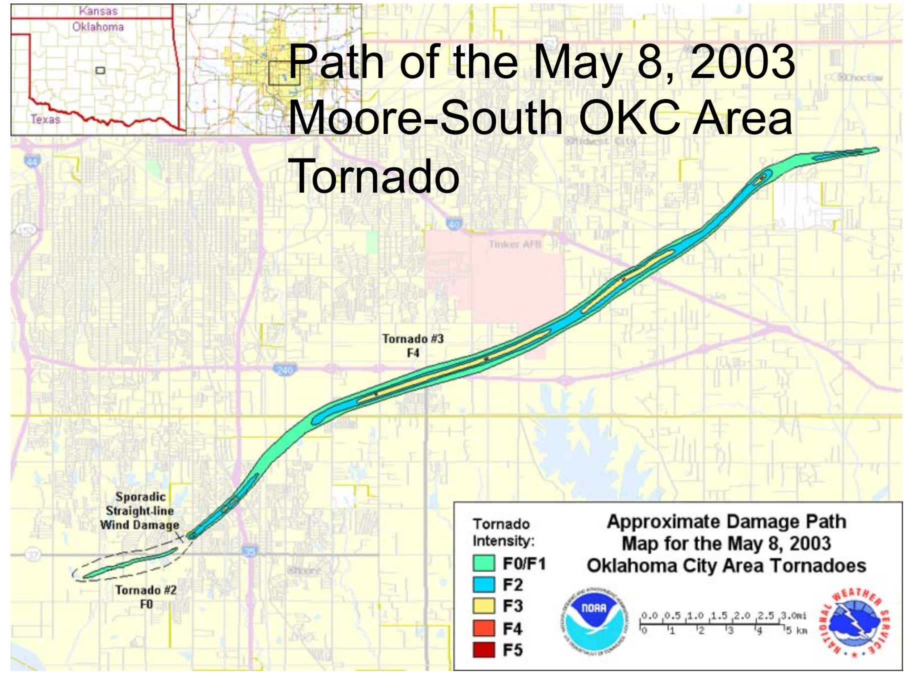
- Convective scale observations (i.e., radar, satellite radiances) require unique observation operators and inclusion of additional state variables (e.g., hydrometeors).

- Accurate cross-variable covariance is especially important.

- Comparison study among Var, EnKF or 3DEnVar or 4DEnVar hybrid for convective scales including both complex cases with multiple storm modes and interactions, and a largely heterogeneous environment, and tornadic storms is still limited.

May 8th 2003 OKC Tornadic Supercell



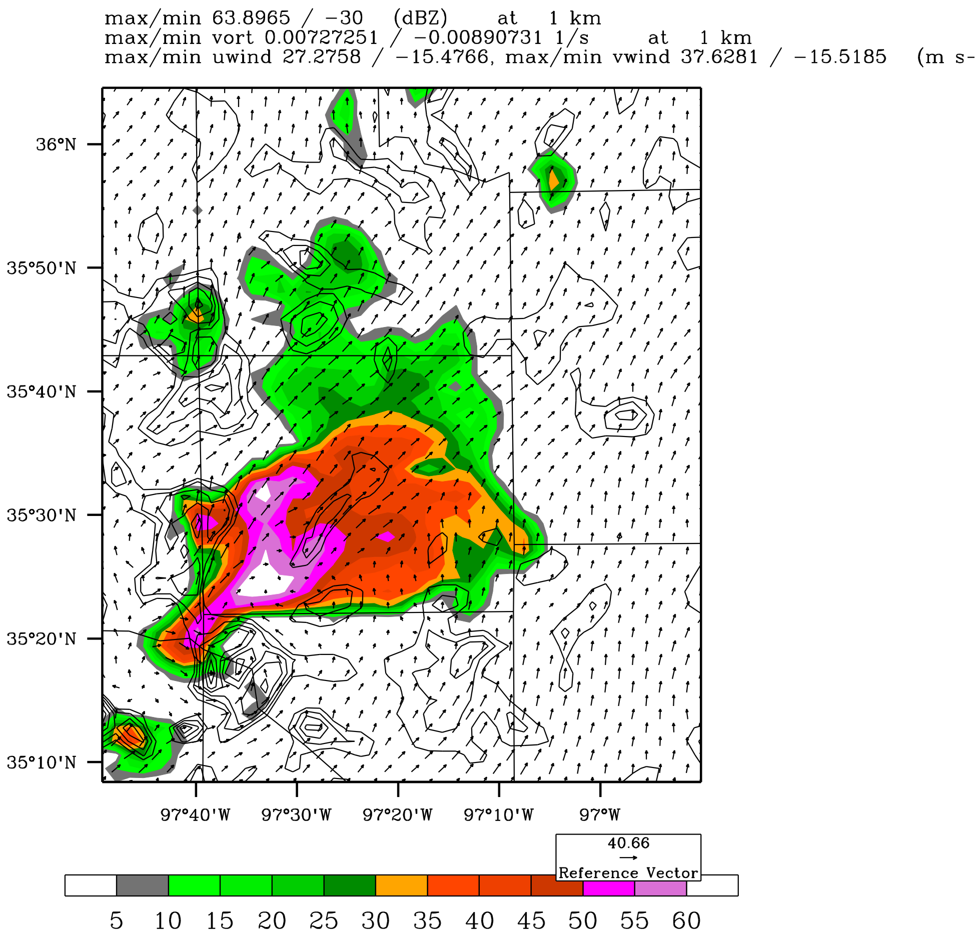


- An isolated supercell case that produced F-4 intensity tornadoes in Moore and Oklahoma City (OKC) during about 2210—2240 UTC.
- Supercell maintained well beyond 2300 until about 2400 UTC.

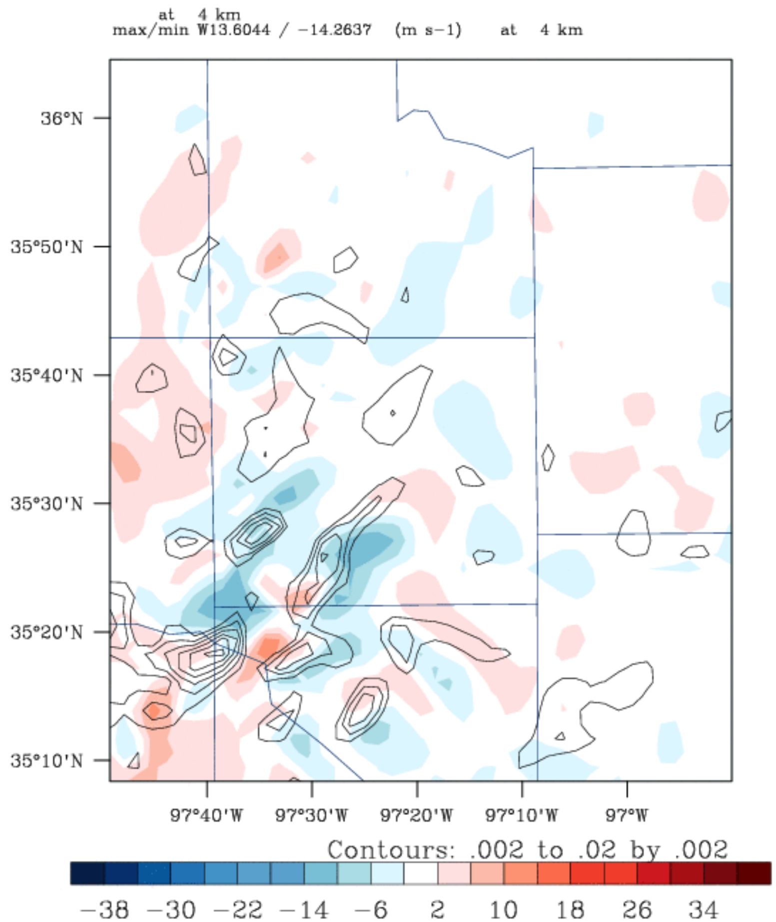
Analysis at 2200 UTC: GSI-3DVar

Wang Y. et al. 2015

22:00:00



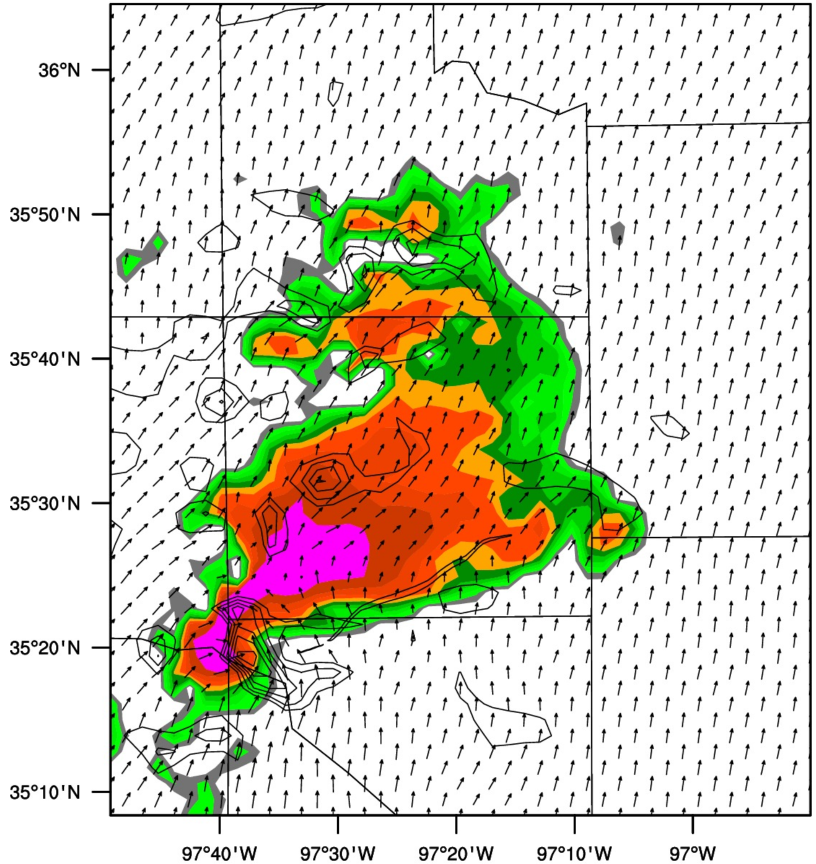
Ref and vorticity at 1 km

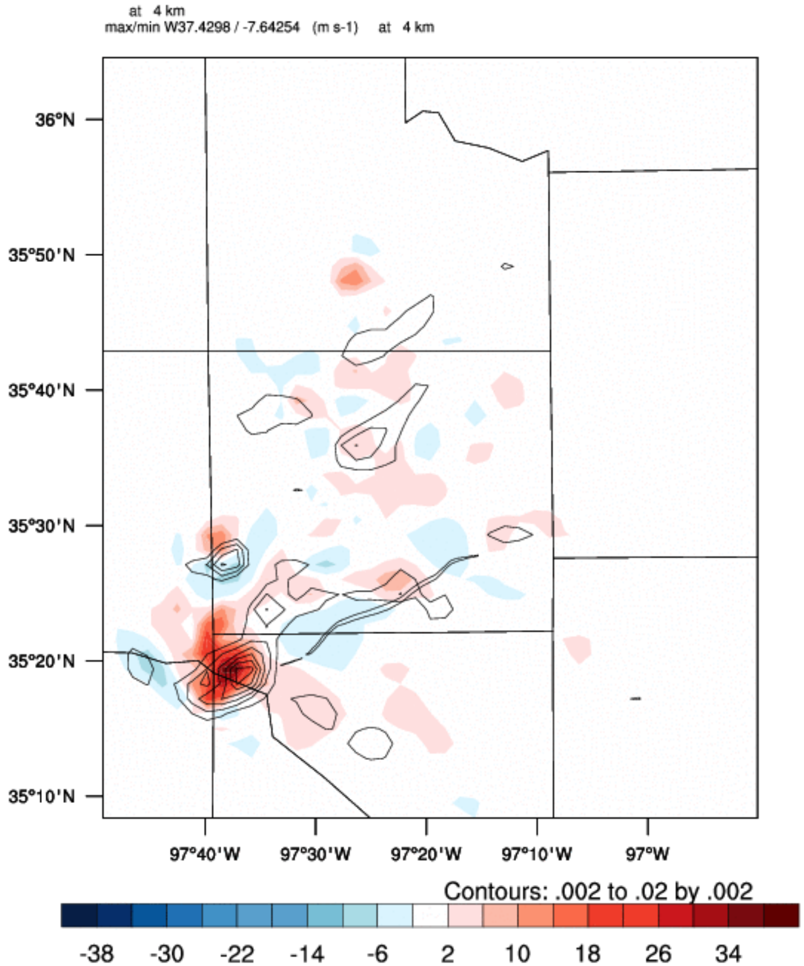


W at 4 km

Analysis at 2200 UTC: GSI-Hybrid

Wang Y. et al. 2015

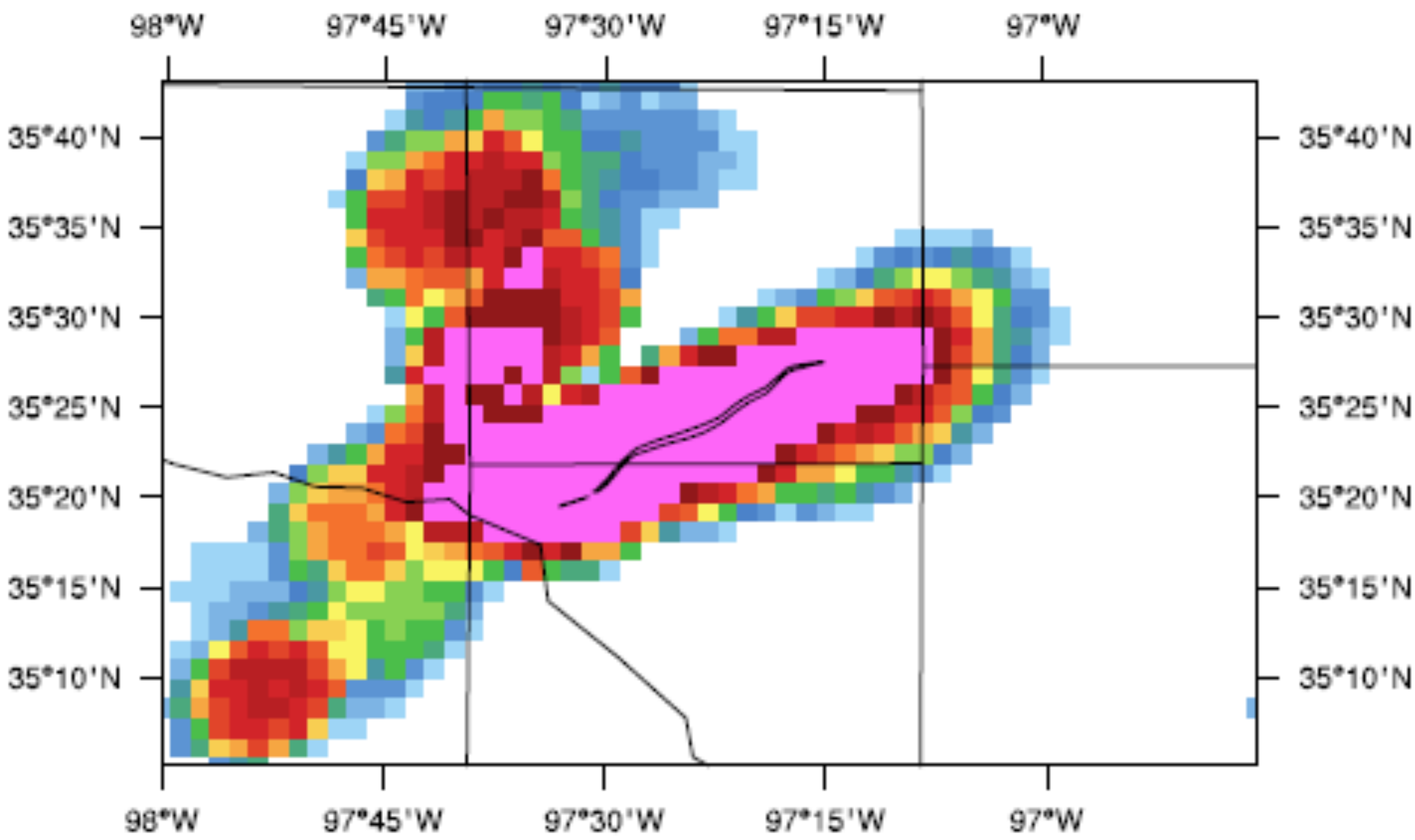




Ref and vorticity at 1 km

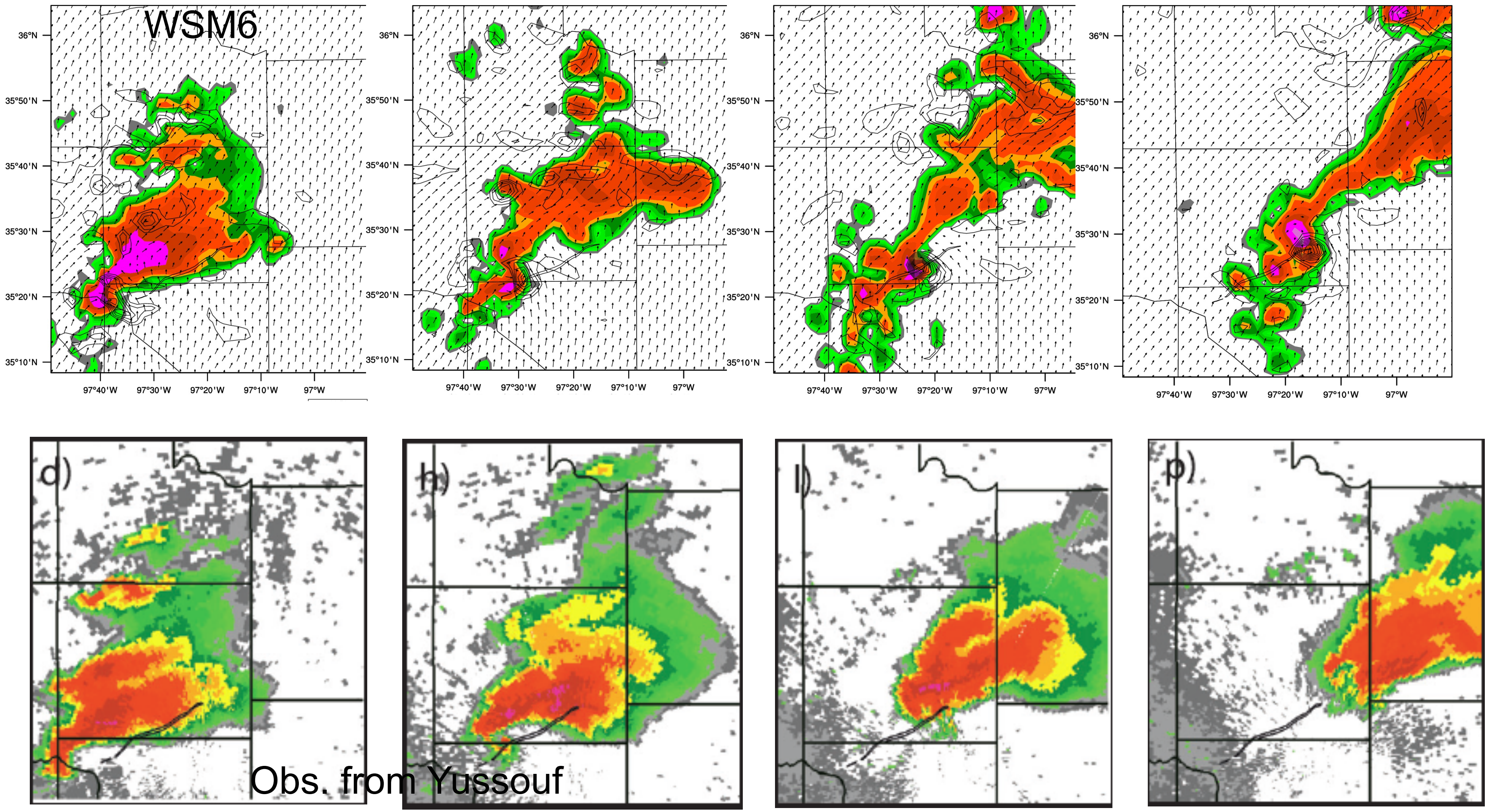
W at 4 km

Prob. fcst. starting 2200 UTC: GSI-hybrid



GSI hybrid extended with different microphysics schemes for reflectivity assimilation

2200 UTC | 2215 UTC | 2230 UTC | 2245 UTC



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Challenges for hybrid DA

- ❑Assimilating observations in a system that resolves multiple scales.
- ❑Effective methods to sample model error in ensemble background.
- ❑Assimilating advanced/new observations: e.g., cloudy radiance observations.
- ❑Correct location and field alignment errors for storm scale DA.
- ❑Variational constraint for different scales.
- ❑Improving static covariance for storm scales.

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