Prior and Posterior Inflation for Ensemble Filters: Theoretical Formulation and Application to Community Atmosphere Model

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2019 SIAM Conference on Computational Science and Engineering [Spokane, WA]

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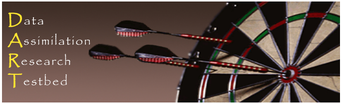

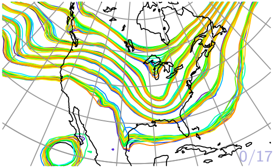

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1.1 Drawbacks (Errors) in Ensemble Filters

- **Sampling Errors:** results from using a limited ensemble size. Causes underestimation of the true variance
- **Model Errors:** *biased* model produces ensemble predictions that are typically far from the observations. Big discrepancies between the model's prediction and the observations may lead to an ensemble collapse

1.1 Drawbacks (Errors) in Ensemble Filters

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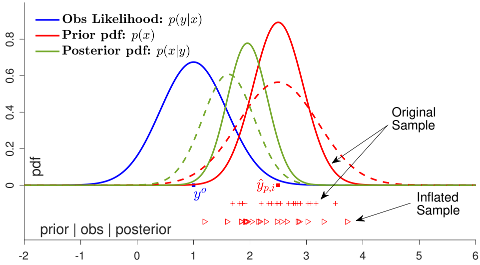
Remedy: Preserve the mean and increase the ensemble variance through "inflation"

- multiplicative
- additive
- obs. space



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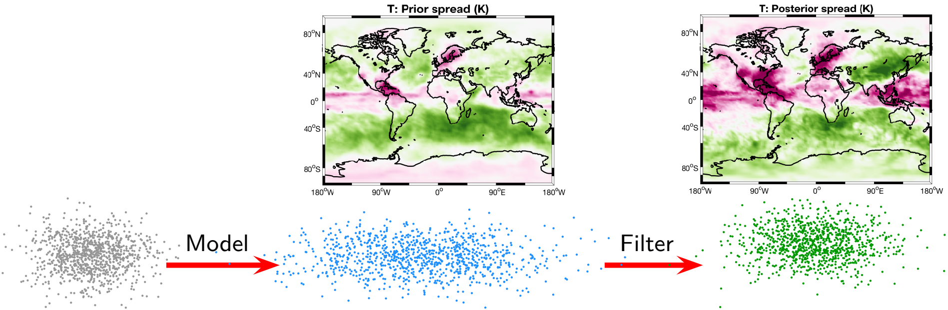

initial ensemble

prior

posterior

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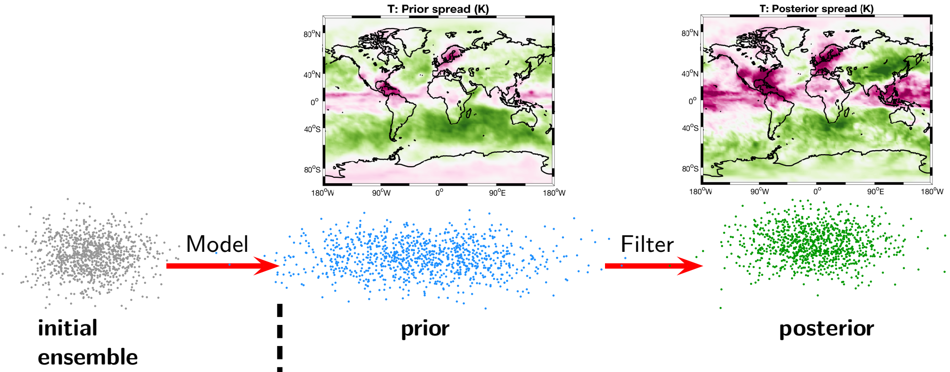

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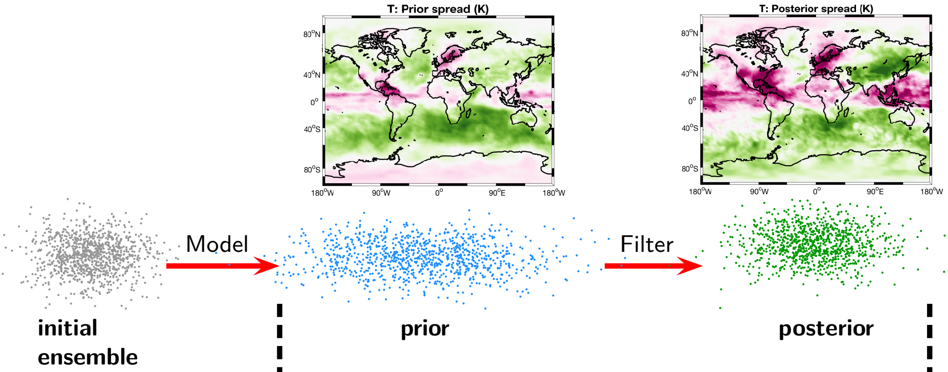

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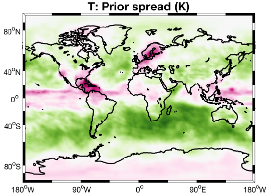

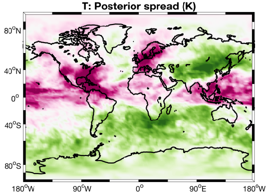

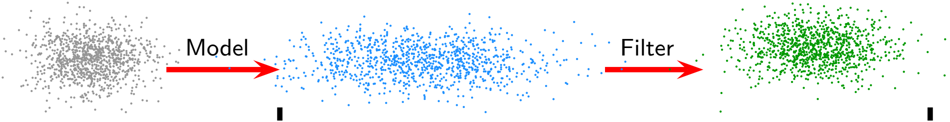

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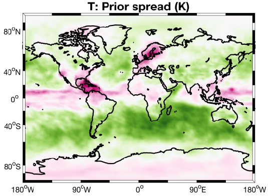
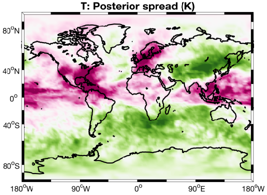
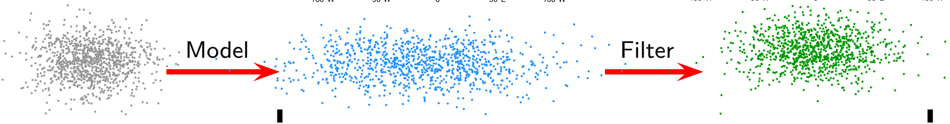
1.2 Inflate the Prior or Posterior?

⋆ How about combining both prior and posterior inflation?







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2.1 Innovation Statistics

Given a scalar variable with sample x^i and observation y_o

$$\overline{x}_{b|a} = \frac{1}{N} \sum_{i=1}^{N} x^i_{b|a}, \qquad \widehat{\sigma}^2_{b|a} = \frac{1}{N-1} \sum_{i=1}^{N} \left(x^i_{b|a} - \overline{x}_{b|a} \right)^2 \tag{1}$$

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Background and Analysis innovations:

$$d_b = y_o - h\left(\overline{x}_b\right) \approx \varepsilon_o - \varepsilon_b, \tag{2}$$

$$d_a = y_o - h\left(\overline{x}_a\right) \approx \varepsilon_o - \varepsilon_a, \tag{3}$$

- Observation Error: $\varepsilon_o \sim \mathcal{N}\left(0, \sigma^2_o\right)$
- Background Error: $\varepsilon_b \sim \mathcal{N}\left(0, \sigma^2_b\right)$
- Analysis Error: $\varepsilon_a \sim \mathcal{N}\left(0, \sigma^2_a\right)$; $\sigma^2_a \overset{\text{Kalman}}{=} \sigma^2_o\sigma^2_b/(\sigma^2_o + \sigma^2_b)$

2.2 Adaptive Prior Inflation, AI-b

Initial effort by Anderson (2009). Inflation follows a distribution, estimated recursively following Bayes'

$$p(\lambda|d_b) \propto p(\lambda) \cdot p(d_b|\lambda). \tag{4}$$

- Prior: $p(\lambda) \sim \mathcal{N}\left(\lambda_b, \sigma^2_{\lambda_b}\right)$
  - Gharamti (2018): $\boxed{IG\left[\alpha\left(\lambda_b, \sigma^2_{\lambda,b}\right), \beta\left(\lambda_b, \sigma^2_{\lambda,b}\right)\right]}$
- Likelihood: $p(d_b|\lambda) \sim \mathcal{N}(\mathbb{E}(d_b), var(d_b))$

$$\mathbb{E}(d_b) = \mathbb{E}(\varepsilon_o - \varepsilon_b) = 0, \tag{5}$$

$$var(d_b) = \mathbb{E}\left[(d_b - \mathbb{E}(d_b))^2\right] = \sigma_o^2 + \sigma_b^2 = \sigma_o^2 + \lambda_s \widehat{\sigma}_b^2 \tag{6}$$

assuming $\mathbb{E}(\varepsilon_o \varepsilon_b) = 0$.

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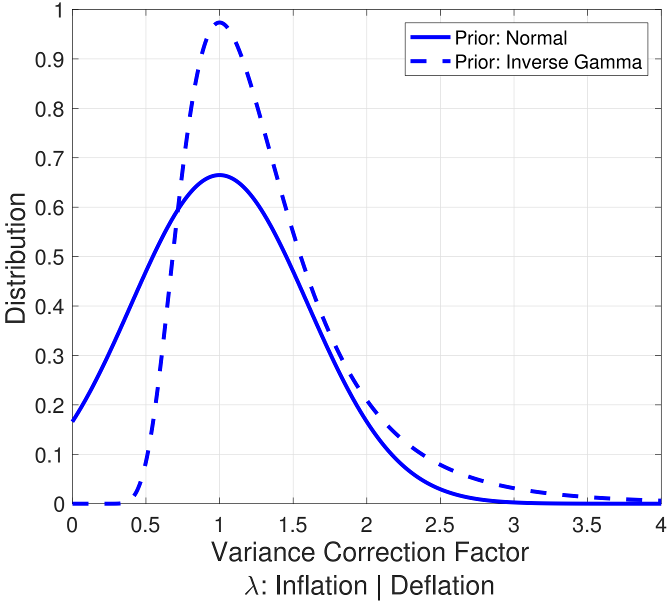

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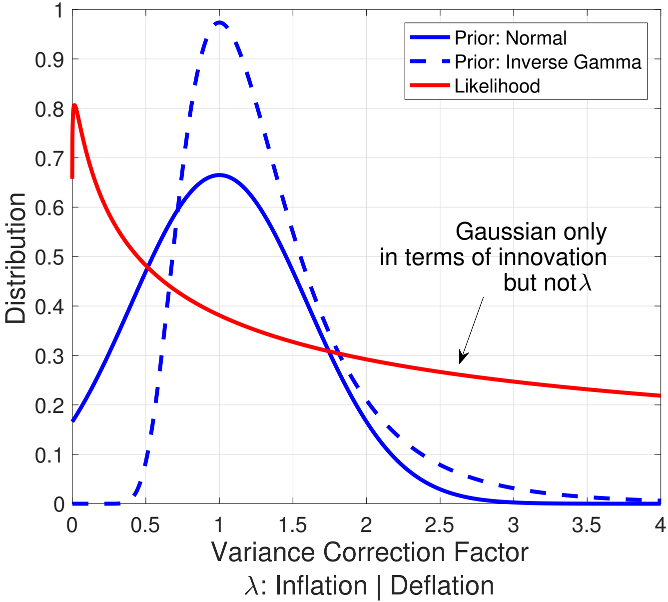

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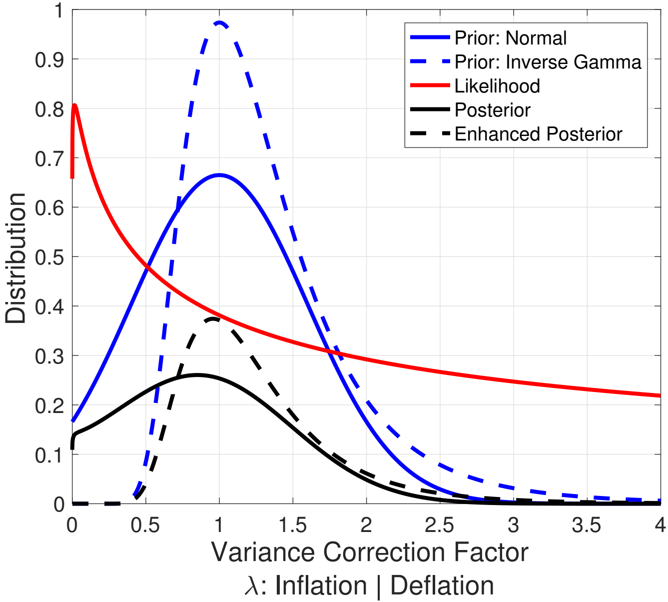

2.2 Adaptive Posterior Inflation, AI-a

Follow similar Bayes' formulation:

$$p\left(\lambda|d_a\right) \propto p\left(\lambda\right) \cdot p\left(d_a|\lambda\right). \quad (7)$$

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Need to compute the likelihood, $p\left(d_a|\lambda\right) \sim \mathcal{N}\left(\mathbb{E}(d_a), var(d_a)\right)$
For each observation j:

$$\begin{aligned}
\mathbb{E}\left(d_{a,j}\right) &= \mathbb{E}\left(\varepsilon_{o,j}\right) - \mathbb{E}\left(\varepsilon_{a,j}\right) = 0, \qquad (8)\\
var\left(d_{a,j}\right) &= \mathbb{E}\left(\varepsilon_{o,j}^2\right) + \mathbb{E}\left(\varepsilon_{a,j}^2\right) - 2 \underbrace{\mathbb{E}\left(\varepsilon_{o,j}\varepsilon_{a,j}\right)}_{\neq 0 \text{ correlated errors}}, \\
&= \sigma_{o,j}^2 + \sigma_{a,j}^2 - 2\mathbb{E}\left[\left(1-k_j\right)\varepsilon_{o,j}\varepsilon_{a,j-1} + k_j\varepsilon_{o,j}^2\right], \\
&\dots \\
&= \sigma_{o,j}^2 - \sigma_{a,j}^2 \equiv \sigma_{o,j}^2 - \lambda_s\widehat{\sigma}_{a,j}^2 \qquad (9)
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$$\begin{aligned} \mathbb{E}(d_{a,j}) &= \mathbb{E}(\varepsilon_{o,j}) - \mathbb{E}(\varepsilon_{a,j}) = 0, \\ var(d_{a,j}) &= \mathbb{E}\left(\varepsilon_{o,j}^2\right) + \mathbb{E}\left(\varepsilon_{a,j}^2\right) - 2 \underbrace{\mathbb{E}(\varepsilon_{o,j}\varepsilon_{a,j})}_{\neq 0 \ \textbf{correlated errors}}, \\ &= \sigma_{o,j}^2 + \sigma_{a,j}^2 - 2\mathbb{E}\left[(1-k_j)\,\varepsilon_{o,j}\varepsilon_{a,j-1} + k_j\varepsilon_{o,j}^2\right], \\ &\dots \\ &= \sigma_{o,j}^2 - \sigma_{a,j}^2 \equiv \sigma_{o,j}^2 - \lambda_s\widehat{\sigma}_{a,j}^2 \end{aligned}$$

(8)

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thus, $p(d_a|\lambda) = (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}d_a^2(\sigma_o^2 - \lambda_s\widehat{\sigma}_a^2)^{-1}\right](\sigma_o^2 + \lambda_s\widehat{\sigma}_a^2)^{-\frac{1}{2}}$

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3. Construct the Gaussian inflation likelihood function with moments:

$$\mathbb{E}\left(\widetilde{d}_{a,j}\right) = 0, \tag{12}$$

$$var\left(\widetilde{d}_{a,j}\right) = \mathbb{E}\left(\varepsilon^2_{o,j}\right) + \mathbb{E}\left(\widetilde{\varepsilon}^2_{a,j}\right) - 2\underbrace{\mathbb{E}\left(\varepsilon_{o,j}\widetilde{\varepsilon}_{a,j}\right)}_{=0} = \sigma^2_{o,j} + \widetilde{\sigma}^2_{a,j}. \tag{13}$$

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⋆ Requires additional evaluation of eqs. (10) and (11)
⋆ Less invasive to available adaptive prior inflation code

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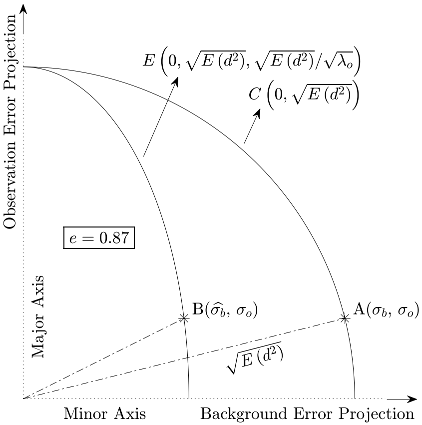

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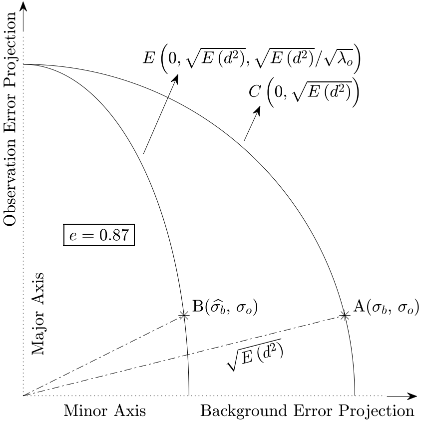

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- ensemble ones satisfy an ellipse
- eccentricity, $0 < e < 1$, a measure to determine the deviation from circle

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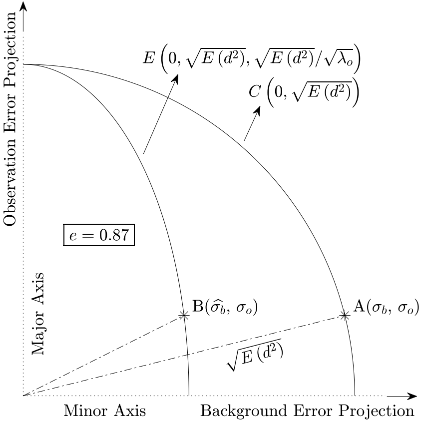

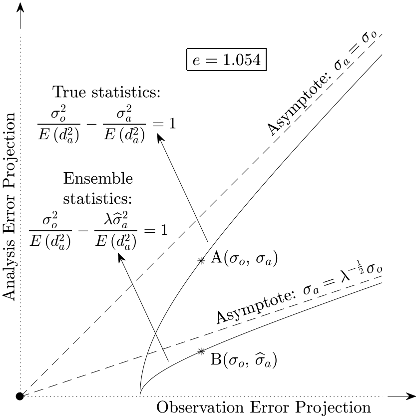

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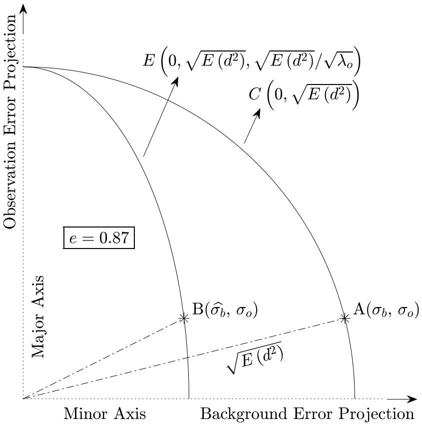

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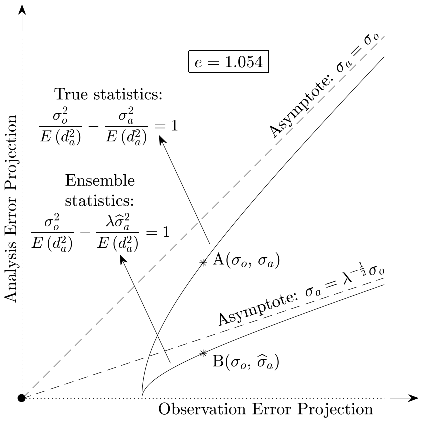
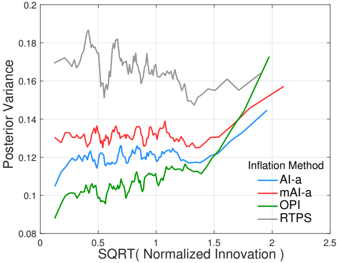


- Ideal and ensemble-based statistics follow hyperbolas
- λ determines the degree of expansion or contraction of the hyperbola
- $e > 1$ also a measure of deviation

2.5 Algorithmic Features

1. Being based on the posterior innovations, the proposed posterior inflation algorithm increases the variance proportional to the size of the innovation



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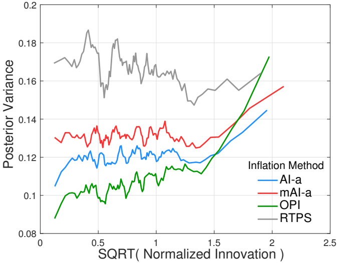
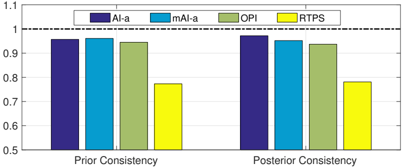


2. Consistent MSE and VAR



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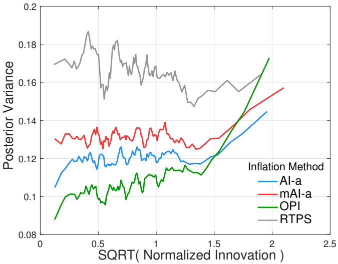

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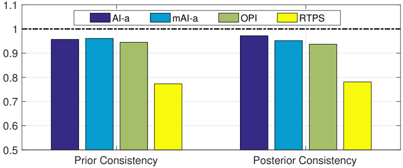
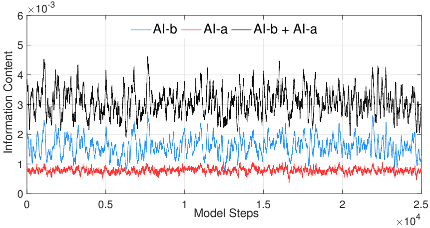
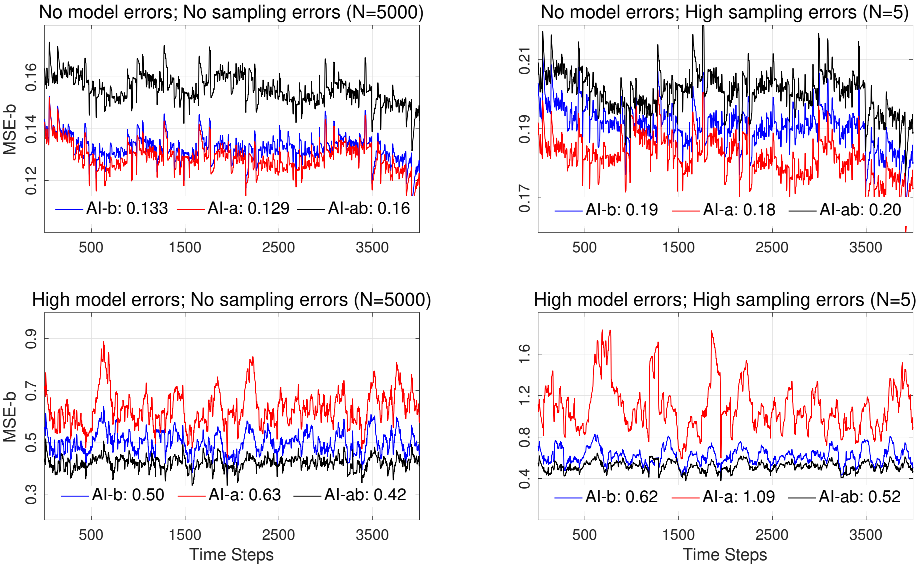


3. Information Content: $\dfrac{\partial \overline{x}_a}{\partial y_o}$

   AI-b $>$ AI-a



2.6 Lorenz 63: OSSE example



3.1 CAM (The Community Atmosphere Model)

- version: `CESM2_0_beta05`
- resolution: $1.9^\circ \times 1.9^\circ$ FV core; `LAT: 96`, `LON: 144`, `LEV: 26`
- State variables: surface pressure (PS), sensible temperature (T), wind components (U and V), specific humidity (Q), cloud liquid water (CLDLIQ) and cloud ice (CLDICE).

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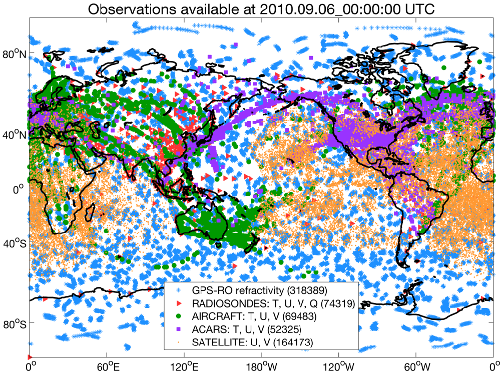
- single state spinup, 80 members ensemble initialization
- DA (EAKF) between 08.16.2010 to 09.30.2010
- data available every 6 hours: wind and temperature observations from radiosondes, ACARS and aircraft along with GPS radio occultation



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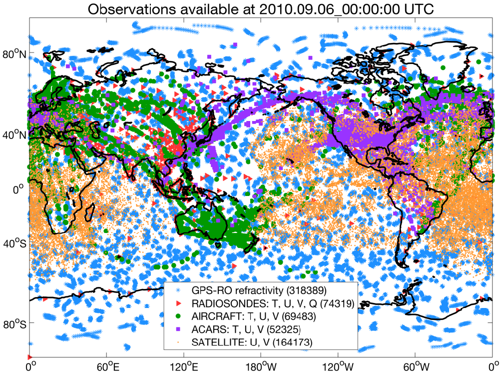
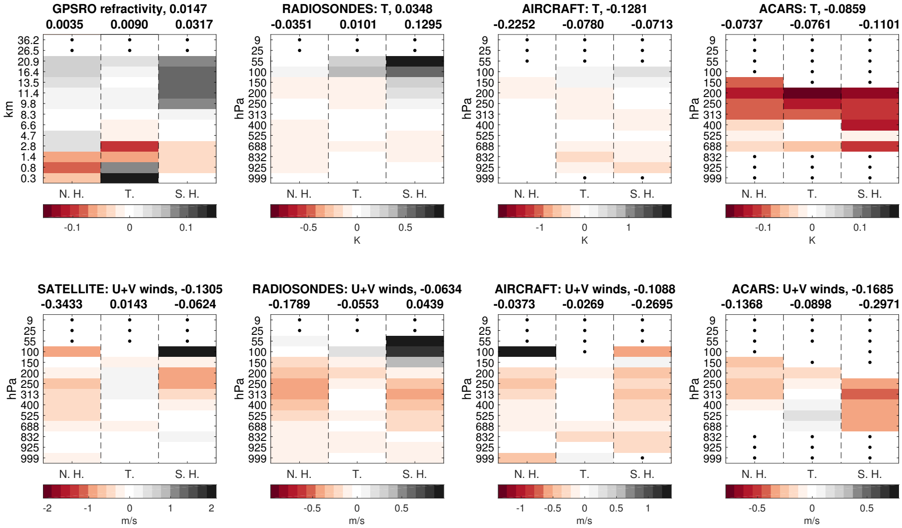
- resolution: $1.9^\circ \times 1.9^\circ$ FV core; `LAT: 96`, `LON: 144`, `LEV: 26`
- State variables: surface pressure (PS), sensible temperature (T), wind components (U and V), specific humidity (Q), cloud liquid water (CLDLIQ) and cloud ice (CLDICE).



- single state spinup, 80 members ensemble initialization
- DA (EAKF) between 08.16.2010 to 09.30.2010
- data available every 6 hours: wind and temperature observations from radiosondes, ACARS and aircraft along with GPS radio occultation
- Horizontal localization cutoff: 0.15 radians (≈ 960 km)
- Vertical localization: half-width of Gaspari Cohn profile is 0.375 scale heights
- DART: latest 'Manhattan' release

3.2 Assimilation Results: AI-a vs. RTPS

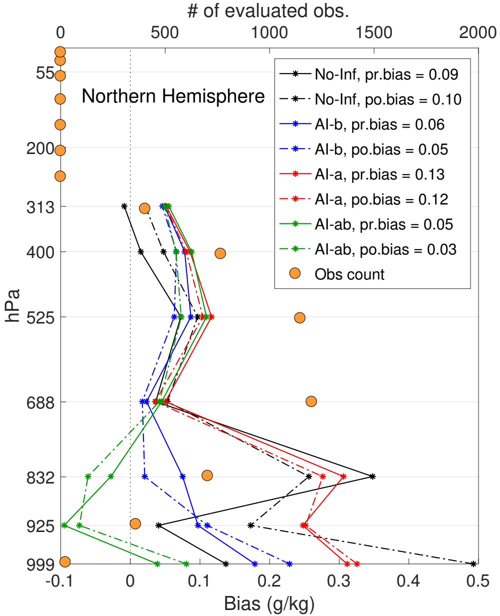


- RMSE(AI-a) - RMSE (best tuned RTPS)
- red (negative difference) means AI-a is more accurate

3.3 Assimilation Results

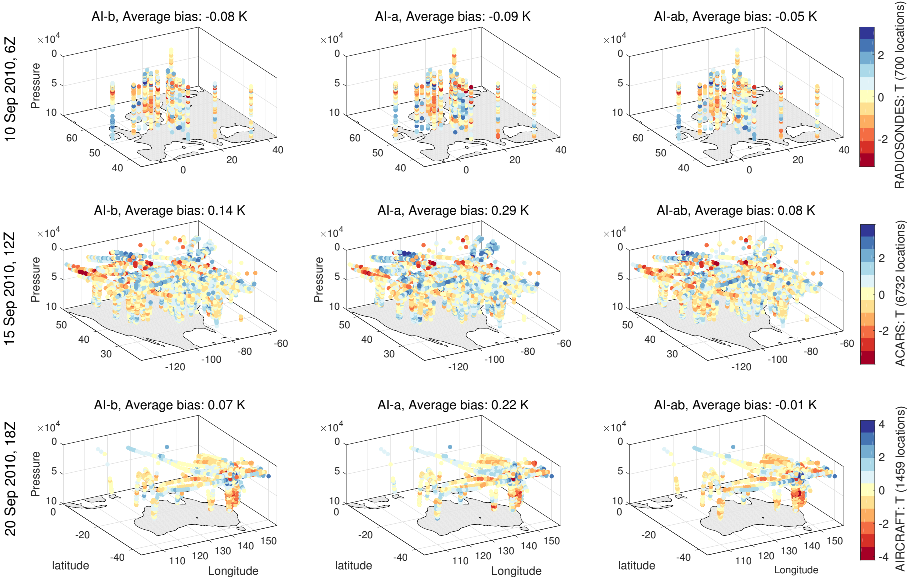
Bias Treatment

- Radiosonde humidity (Q) is not assimilated, only evaluated for verification
- Largest biases are near the surface
- AI-b is more effective than AI-a at reducing the bias
- Best performance is suggested by AI-ab (both prior and posterior are adaptively inflated)



3.3 Assimilation Results

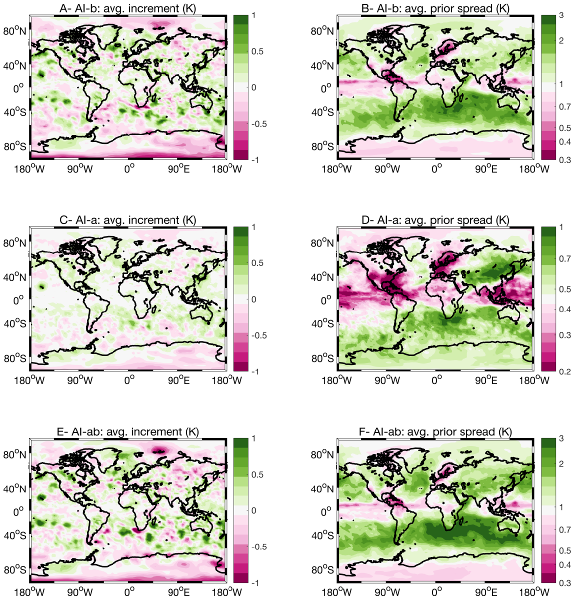
Bias Treatment cont.



3.4 Assimilation Results

Increments & Spread

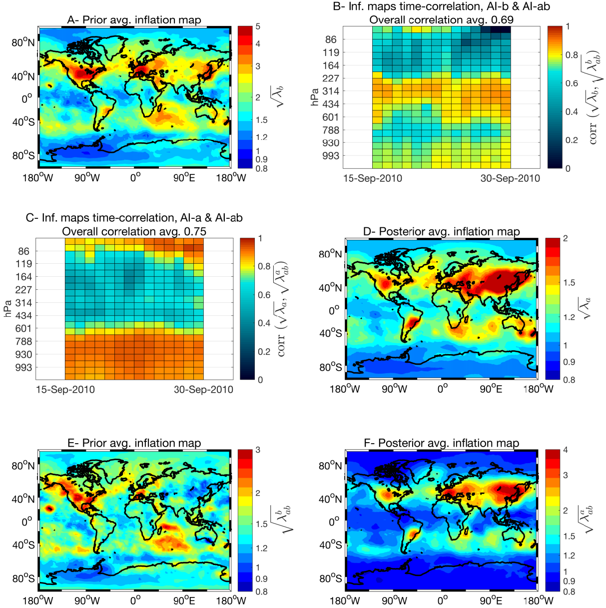
- T increments at ~ 697 hPa and average ensemble spread
- Major updates happen in the southern and northern extratropics
- Strong cooling at low latitudes; given CAM4's warming bias
- AI-a suggests smallest increments and spread (less information content compared to AI-b and AI-ab)



3.5 Assimilation Results

Inflation Fields

- Average inflation maps (panels A, D, E and F) and the time-correlation between AI-ab inflation and those of AI-b (panel B) and AI-a (panel C)
- $\sqrt{\lambda_{ab}^{b}}$ (prior inflation of AI-ab) and $\sqrt{\lambda_{ab}^{a}}$ (posterior inflation of AI-ab) are highly correlated with $\sqrt{\lambda_b}$ (prior inflation of AI-b) and $\sqrt{\lambda_a}$ (posterior inflation of AI-a), respectively
- Arctic and the Antarctic Circles experience a 20% deflation which could be attributed to the sparsity of observations



4. Conclusion

- Proposed a spatially and temporally varying adaptive posterior covariance inflation (AI-a)
- The algorithm is based on Bayes' and uses analysis innovations to update the inflation
- With no model errors, AI-a resulted in higher quality estimates than AI-b (better treatment of sampling errors)
- When model errors are dominant, as in CAM4, AI-a was found less useful
- Compelling results obtained by combining both AI-b and AI-a

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I. **Gharamti, M. E.** "Enhanced Adaptive Inflation Algorithm for Ensemble Filters." *Monthly Weather Review*, 2, 623-640

II. **Gharamti, M. E.**, Raeder, K., Anderson, J. and Wang, X. "Comparing Adaptive Prior and Posterior Inflation for Ensemble Filters Using an Atmospheric General Circulation Model." *Monthly Weather Review*, to appear