Data
Assimilation
Research
Testbed

A new adaptive hybrid ensemble Kalman filter and optimal interpolation

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NATIONAL CENTER FOR ATMOSPHERIC RESEARCH

1. Preliminaries

- Prior distribution $p\left(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{Y}_{k-1}\right) \sim \mathcal{N}\left(\mathbf{x}_k^f, \mathbf{P}_k^f\right)$

Mean: $$\mathbf{x}_k^f = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_k^{f,i}, \qquad i = 1, 2, \ldots, N \tag{1}$$

Covariance: $$\mathbf{P}_k^f = \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathbf{x}_k^{f,i} - \mathbf{x}_k^f\right)\left(\mathbf{x}_k^{f,i} - \mathbf{x}_k^f\right)^T \tag{2}$$

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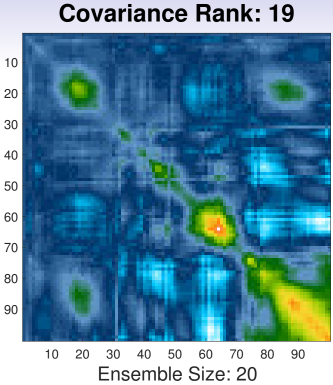
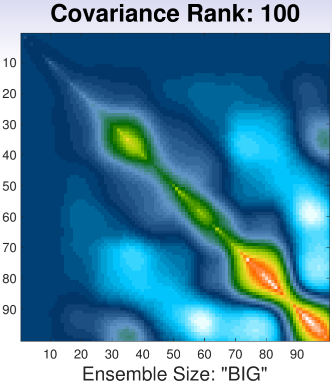
- The ensemble Kalman filter (EnKF) provides reliable background error covariances for large ensemble sizes
- (*For now*) we can't afford large ensembles especially in earth systems
- The use of small ensembles
  - ▷ causes the EnKF to be rank-deficient,
  - ▷ background variances are underestimated, and
  - ▷ generally results in low-quality forecasts


Covariance Rank: 19
10
20
30
40
50
60
70
80
90
10 20 30 40 50 60 70 80 90
Ensemble Size: 20


Covariance Rank: 100
10
20
30
40
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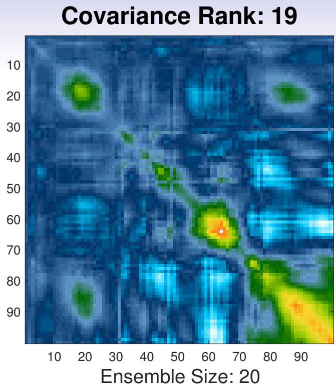
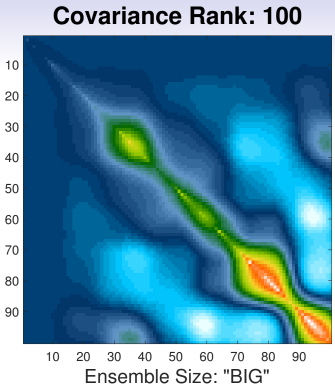
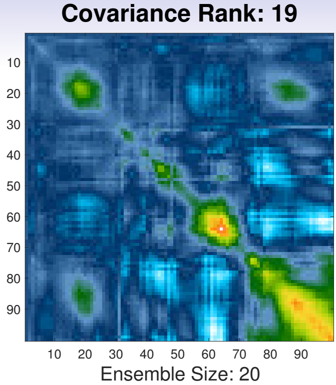
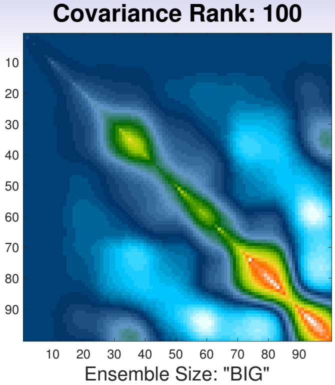
60
70
80
90
10 20 30 40 50 60 70 80 90
Ensemble Size: "BIG"





- *Holy Covariance:* $\mathbf{B} = \lim_{N \to \infty} \mathbf{P}^e$





- *Holy Covariance:* $\mathbf{B} = \lim_{N\to\infty} \mathbf{P}^e$
- Ways to fix/improve $\mathbf{P}^e$
  1. **Inflation:** increases the variance, rank stays unchanged (spatially-const)
     $\rightarrow$ Multiplicative (prior, posterior), Additive, Relaxation
  2. **Localization:** removes spurious correlations, increases the rank
     $\rightarrow$ Covariance, local analysis
  3. **Mutli-configuration|physics** ensemble

2.1 Hybrid EnKF-OI: Terminologies

- OI: Optimal Interpolation (essentially a KF with a prescribed invariant $\mathbf{P}^f$)
- Often referred to as EnKF-3DVar
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Use a background covariance in the EnKF that is an "average" (weighted sum) of a flow-dependent background error covariance estimated from an ensemble and a predefined static covariance from a 3DVar or an OI system

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- Many different hybrid forms in the literature
- Here, we adopt the following covariance-hybridizing form

$$\mathbf{P} = \alpha \mathbf{P}^e + (1 - \alpha)\mathbf{B}$$

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- Succession of transform operators, $\mathbf{B} = \mathbf{B}^{1/2}\mathbf{B}^{T/2}$

$$\mathbf{B}^{1/2} = \mathbf{U}_p\mathbf{S}\mathbf{U}_v\mathbf{U}_h \qquad (4)$$

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- Storage issue: $\mathbf{B}$ is of size $(N_x \times N_x)$ where N_x is the dimension of the state
  - The proposed adaptive scheme only requires knowledge of the historical (climatology) realizations **and not the full B!**

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$\boxed{\textbf{How to choose } \alpha?}$

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$$\mathbb{E}\left[\mathbf{d}\mathbf{d}^T\right] = \mathbf{R} + \mathbf{H}\mathbf{P}^f\mathbf{H}^T, \tag{5}$$

where $\mathbf{d} = \mathbf{y}^o - \mathbf{H}\mathbf{x}^f$. Substitute the hybrid covariance form in eq. (5):

$$\mathbb{E}\left[\mathbf{d}\mathbf{d}^T\right] = \mathbf{R} + \alpha\mathbf{H}\mathbf{P}^e\mathbf{H}^T + (1-\alpha)\mathbf{H}\mathbf{B}\mathbf{H}^T, \tag{6}$$

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- Algorithm:
  - ▷ Assume α to be a random variable
  - ▷ Start with a prior distribution for α: $p(\alpha) \sim \mathcal{N}, \mathcal{B}$, ..
  - ▷ Use the data to construct a likelihood function: $p(\mathbf{d}|\alpha)$
  - ▷ Use Bayes' rule to find an updated estimate of α:

$$p(\alpha|\mathbf{d}) \approx p(\alpha) \cdot p(\mathbf{d}|\alpha) \tag{7}$$

  - ▷ Posterior α can be used as the prior for the next DA cycle

2.4 Hybrid EnKF-OI: Illustration

Scalar example:

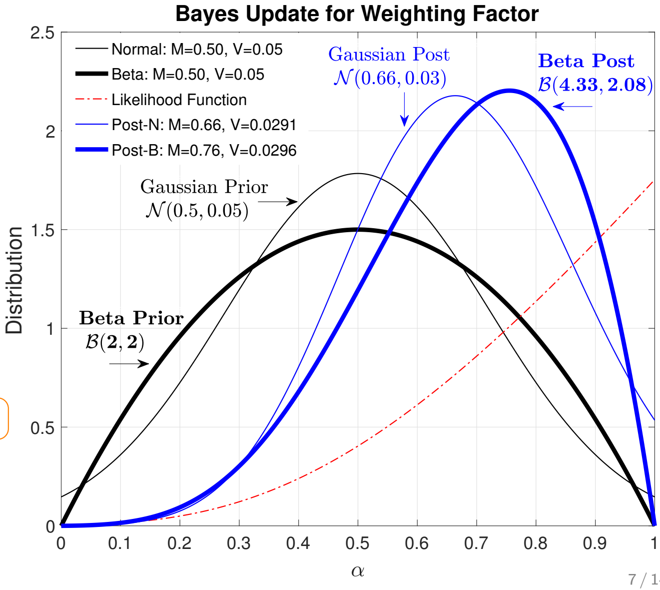
$\mathbf{P^e} \rightarrow \quad \sigma_e^2 = 0.9$

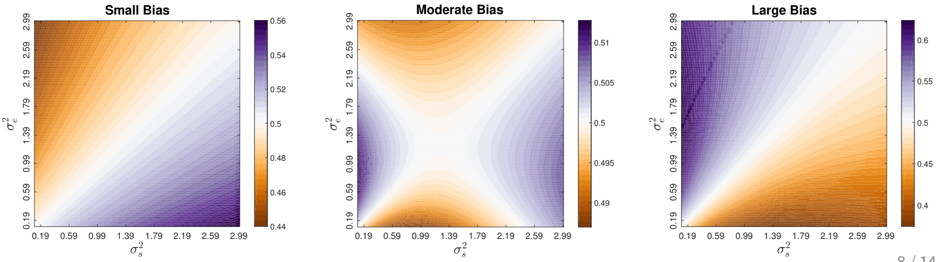
$\mathbf{B} \rightarrow \quad \sigma_s^2 = 0.2$

$\mathbf{R} \rightarrow \quad \sigma_o^2 = 0.1$

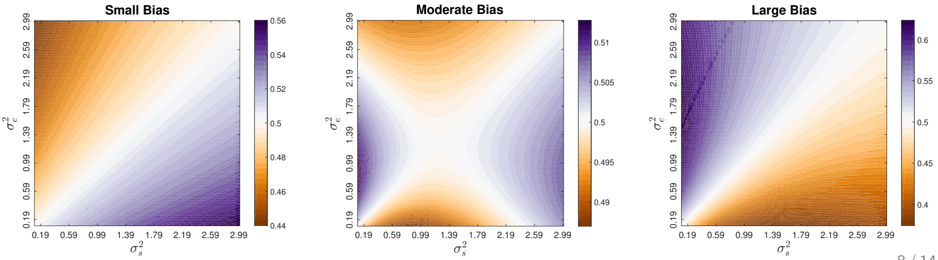
$\mathbf{d} \rightarrow \quad d = 2.5$

6 required parameters




Small Bias
Moderate Bias
Large Bias
σ_e^2
σ_s^2
0.19 0.59 0.99 1.39 1.79 2.19 2.59 2.99
0.44 0.46 0.48 0.5 0.52 0.54 0.56
0.49 0.495 0.5 0.505 0.51
0.4 0.45 0.5 0.55 0.6

Vary both σ_e^2 and σ_s^2 and fix σ_o^2



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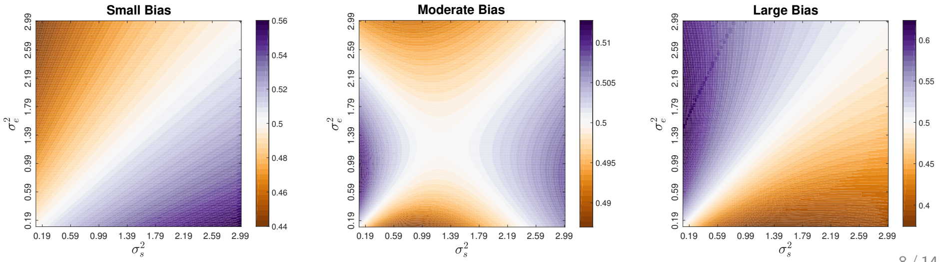
Purplish: More weight on ensemble covariance
Brownish: More weight on static covariance



Understanding the behavior of the algorithm

- When both variances match, equal weight is placed (i.e., $\alpha = 0.5$)
- Large bias: more weight on the larger variance to better fit the obs
- Small bias: good estimate; more weight on the smaller variance
- Moderate bias: alternate between the ensemble and the static variance

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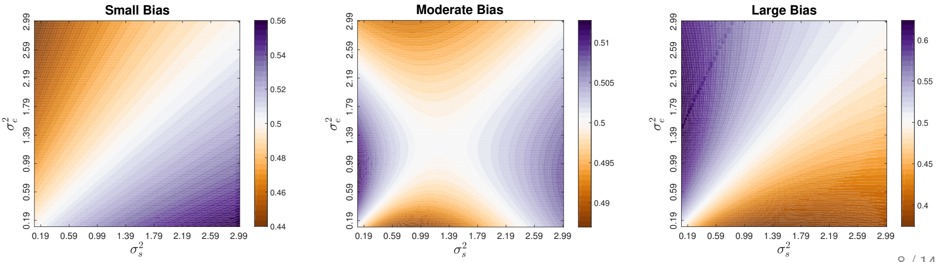

2.5 Hybrid EnKF-OI: Adaptive in space

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  - Biases are not homogenous in space
  - Heterogenous observation networks (densely observed regions tend to have small ensemble spread)

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- State variables may be hybridized differently, why?
  - ▷ Biases are not homogenous in space
  - ▷ Heterogenous observation networks (densely observed regions tend to have small ensemble spread)
- Need to assimilate observations serially. For each observation:
  - ▷ Compute correlation coefficient between the observed prior ensemble, y^f, and all state variables:

$$\rho_j = \text{correlation}\left(y^f, x_j^f\right),$$

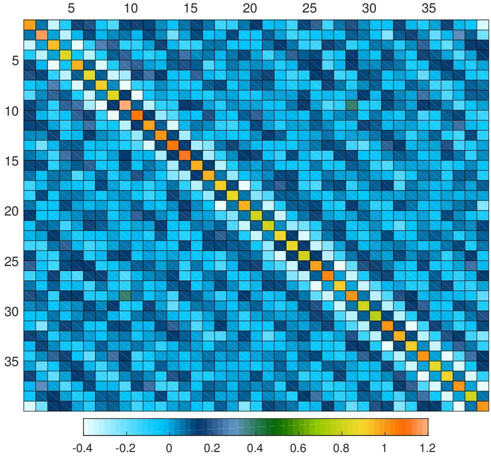
where the hybrid weighting factor is assumed to have the same correlation field (Anderson 2009, El Gharamti 2018). Thus,

$$d^2 = \sigma_o^2 + \rho_j \alpha \sigma_e^2 + (1 - \rho_j \alpha)\sigma_s^2$$

  - ▷ Find the posterior based on the modified likelihood and associated prior

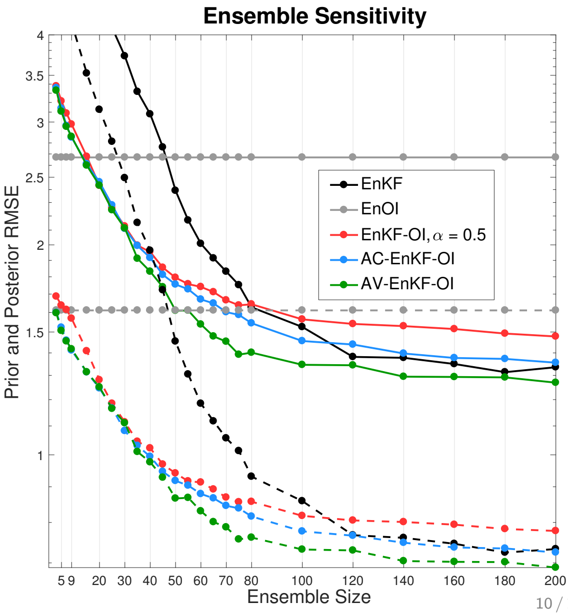
3.1 Experiments using L96: Ensemble Size

- L96: 40 variables
- Observe every other variable (total of 20)
- Observe every 5 time steps ($dt = 0.05$)
- $\mathbf{R} = 1$
- $\mathbf{B}$ Climatological run (1000 realizations)
- No inflation
- No localization
- $p(\alpha) \sim \mathcal{N}(0.5, 0.1)$



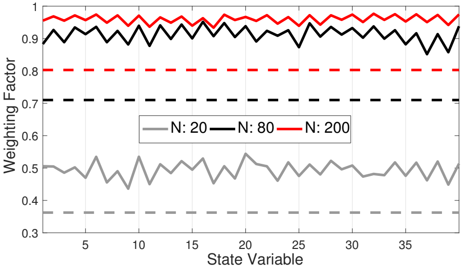
3.1 Experiments using L96: Ensemble Size

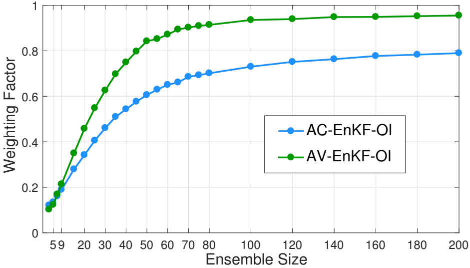
1. **EnKF**
2. **EnOI**: EnKF with fixed **B** (Hybrid; $\alpha = 0$)
3. **EnKF-OI**; $\alpha = 0.5$
4. **AC-EnKF-OI**: Adaptive spatially-Constant EnKF-OI
5. **AV-EnKF-OI**: Adaptive spatially-Varying EnKF-OI



3.1 Experiments using L96: Ensemble Size

- AC-EnKF-OI: Dashed lines
- AV-EnKF-OI: Solid lines





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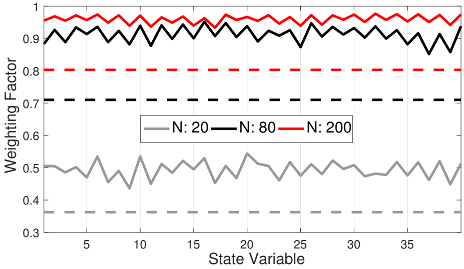
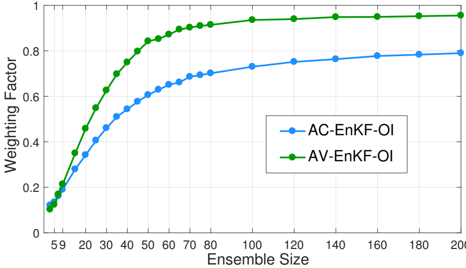
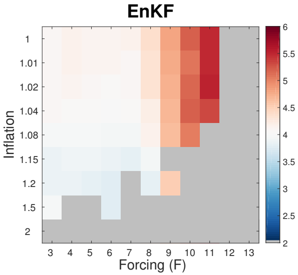


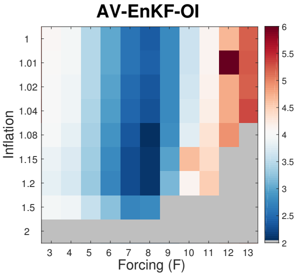
- For small ensembles, both adaptive spatially-constant and varying schemes behave the same
- Being spatially-varying, AV-EnKF-OI responds more efficiently to changes in the ensemble

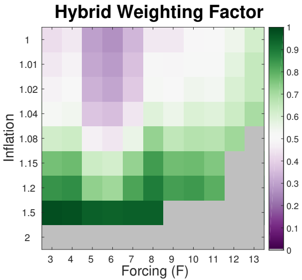


3.2 Experiments using L96: Model Error & Inflation

- Ensemble size: 20
- Model error; vary $3 \leq F \leq 13$
- $\mathbf{B}$ is generated in each case using biased F
- No localization







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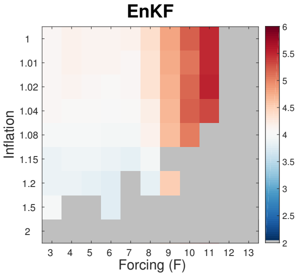
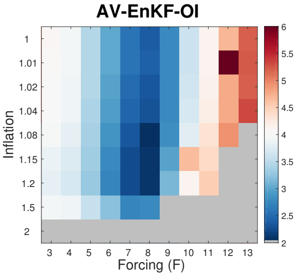
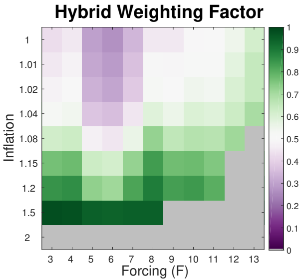




- Hybrid scheme: better stability and more accurate even in very biased conditions
- As inflation increases, adaptive α increases (more weight on the ensemble cov)



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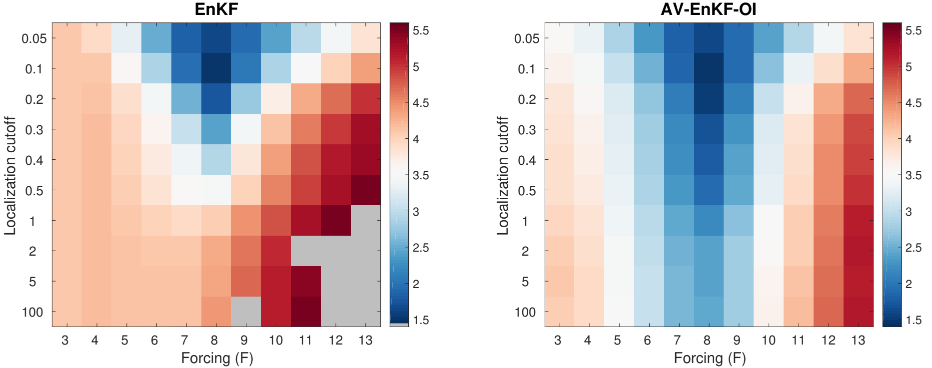

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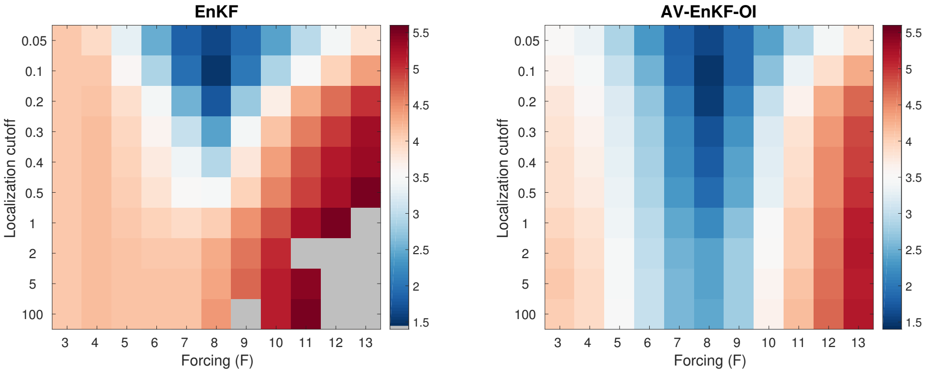

- $N_e = 20$, No inflation
- Vary both F and localization length scale
- Adaptive hybrid scheme is systematically better than the EnKF for all tested cases

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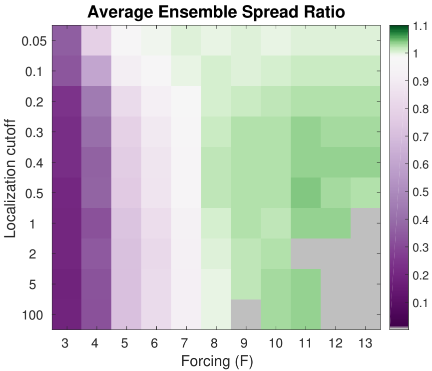

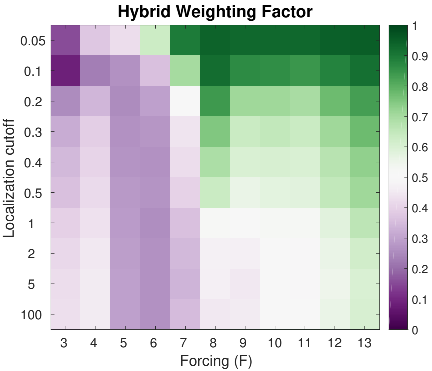

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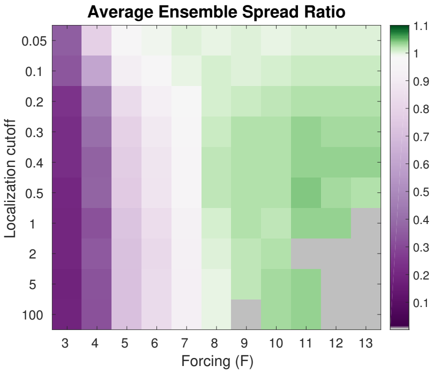

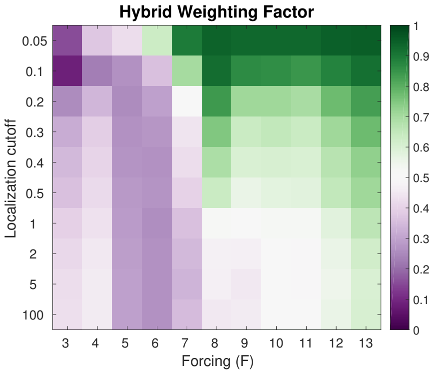

- For chaotic behaviour (i.e., $F \geq 8$): As localization increases, α increases
- Less chaotic (smaller ensemble variance): α decreases to *bring-in* variability from **B**
- Left panel: $\frac{\text{Ensemble Spread}}{\text{Hybrid Spread}}$, note the small spread in the ensemble for $F < 8$

3.4 Experiments using L96: Observation Network

- **Data Void I:** Observe the first 20 variables
- **Data Void II:** Observe the first and last 5 variables
- **Data Void III:** Observe 10 variables in the center
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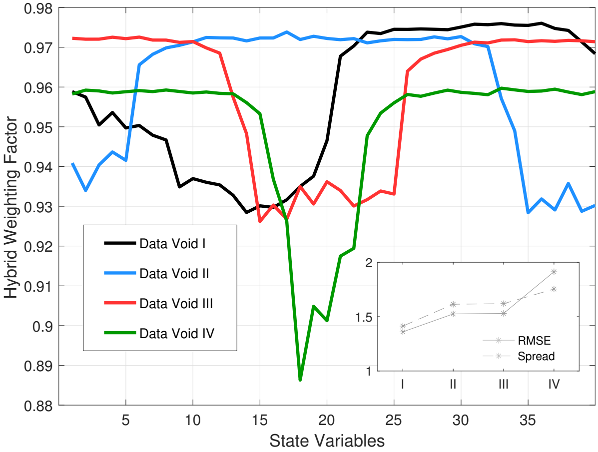

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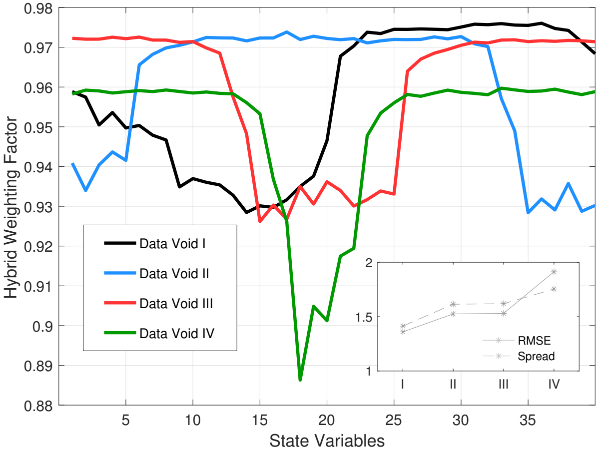

- In densely observed regions, the ensemble spread decreases. To counteract this, hybrid scheme places more on **B** to increase the variance and allow the filter to better fit the data

4. Conclusion

- Presented a new temporally and spatially varying adaptive hybrid EnKF-OI scheme
- The adaptive scheme uses the data and applies Bayes rule to determine the relative weighting between the ensemble and the static covariance
- The spatially-adaptive scheme – for now – does not support data that are not on the state grid (e.g., radiances)
- Tests using the Lorenz-96 system
- Future tests in high-order models (B-grid, CAM, WRF-Hydro ..)

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– Fair computational cost	– Storage, additional IO cost

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– Requires a large ensemble size	– Works well with fairly small ensembles
– Fair computational cost	– Storage, additional IO cost
– Strong tuning (inf, loc, ..)	– Fully adaptive, requires less inf, loc, ..

4. Conclusion

- Presented a new temporally and spatially varying adaptive hybrid EnKF-OI scheme
- The adaptive scheme uses the data and applies Bayes rule to determine the relative weighting between the ensemble and the static covariance
- The spatially-adaptive scheme – for now – does not support data that are not on the state grid (e.g., radiances)
- Tests using the Lorenz-96 system
- Future tests in high-order models (B-grid, CAM, WRF-Hydro ..)

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– Strong biases cause divergence	– More stable; able to switch to EnOI