A new adaptive hybrid ensemble Kalman filter and optimal interpolation

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1. Preliminaries

- Prior distribution $p(x_k|x_{k-1}, Y_{k-1}) \sim \mathcal{N}(x^f_k, P^f_k)$

\[\text{Mean: } x^f_k = \frac{1}{N} \sum_{i=1}^{N} x^f_{k,i}, \quad i = 1, 2, \ldots, N \quad (1)\]

\[\text{Covariance: } P^f_k = \frac{1}{N-1} \sum_{i=1}^{N} (x^f_{k,i} - x^f_k)(x^f_{k,i} - x^f_k)^T \quad (2)\]

The ensemble Kalman filter (EnKF) provides reliable background error covariances for large ensemble sizes.

For now, we can't afford large ensembles especially in earth systems. The use of small ensembles causes the EnKF to be rank-deficient, background variances are underestimated, and generally results in low-quality forecasts.
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  - causes the EnKF to be rank-deficient,
  - background variances are underestimated, and
  - generally results in low-quality forecasts
Holy Covariance: $B = \lim_{N \to \infty} P$

Ways to fix/improve $P$

1. Inflation: increases the variance, rank stays unchanged (spatially-const) → Multiplicative (prior, posterior), Additive, Relaxation

2. Localization: removes spurious correlations, increases the rank → Covariance, local analysis

3. Multi-configuration | physics ensemble

Covariance Rank: 19
Ensemble Size: 20

Covariance Rank: 100
Ensemble Size: "BIG"
\[ \mathbf{B} = \lim_{N \to \infty} \mathbf{P}^e \]

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• **Holy Covariance**: $B = \lim_{N \to \infty} P^e$

• **Ways to fix/improve** $P^e$
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  3. **Multi-configuration**|**physics** ensemble
2.1 Hybrid EnKF-OL: Terminologies

- OI: Optimal Interpolation (essentially a KF with a prescribed invariant $P^f$)
- Often referred to as EnKF-3DVar
- Initial effort by Hamill and Snyder (2000)
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What’s the idea?

Use a background covariance in the EnKF that is an “average” (weighted sum) of a flow-dependent background error covariance estimated from an ensemble and a predefined static covariance from a 3DVar or an OI system.
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- Many different hybrid forms in the literature
- Here, we adopt the following covariance-hybridizing form

$$P = \alpha P^e + (1 - \alpha)B$$
2.2 How to construct $B$

- Available from 3DVar systems
- Formed using large inventory of historical forecasts over large windows

\[ B = S \Omega S^T = \hat{S} \hat{S}^T, \]

where $\hat{S} = S \Omega^{1/2}$.

- Succession of transform operators,
  \[ B = \frac{1}{2} B_T \frac{1}{2} B_1 = U_p S U_v U_h. \]

- Storage issue: $B$ is of size $(N_x \times N_x)$ where $N_x$ is the dimension of the state.
- The proposed adaptive scheme only requires knowledge of the historical (climatology) realizations and not the full $B$!
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- Spectral decomposition is desirable

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2.3 Hybrid EnKF-OI: Adaptive Form

How to choose $\alpha$?

The ensemble statistics should satisfy:

$$E\left[dd^T\right] = R + HPH^T,$$

where $d = y_o - Hx_f$.

Substitute the hybrid covariance form in eq. (5):

$$E\left[dd^T\right] = R + \alpha HPH^T + (1 - \alpha)HBH^T,$$

$\alpha$ is a scalar coefficient.

Algorithm:

$\Delta$

Assume $\alpha$ to be a random variable

$\Delta$

Start with a prior distribution for $\alpha$:

$\Delta$

$p(\alpha) \sim N(B, ..)$

$\Delta$

Use the data to construct a likelihood function:

$\Delta$

$p(d|\alpha)$

$\Delta$

Use Bayes' rule to find an updated estimate of $\alpha$:

$$p(\alpha|d) \approx p(\alpha) \cdot p(d|\alpha)$$

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Posterior $\alpha$ can be used as the prior for the next DA cycle.
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  \[ E[dd^T] = R + \alpha HP^eH^T + (1 - \alpha)HBH^T, \]  
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2.3 Hybrid EnKF-OI: Adaptive Form

How to choose \( \alpha \)?

- The ensemble statistics should satisfy:
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  where \( d = y^o - Hx^f \). Substitute the hybrid covariance form in eq. (5):
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  \( \alpha \) is a scalar coefficient.

- **Algorithm:**
  - Assume \( \alpha \) to be a random variable
  - Start with a prior distribution for \( \alpha \): \( p(\alpha) \sim N, B, \ldots \)
  - Use the data to construct a likelihood function: \( p(d|\alpha) \)
  - Use Bayes’ rule to find an updated estimate of \( \alpha \):
    \[
    p(\alpha|d) \approx p(\alpha) \cdot p(d|\alpha) 
    \]
  - Posterior \( \alpha \) can be used as the prior for the next DA cycle.
2.4 Hybrid EnKF-OI: Illustration

Scalar example:

\[ P^e \rightarrow \sigma_e^2 = 0.9 \]
\[ B \rightarrow \sigma_s^2 = 0.2 \]
\[ R \rightarrow \sigma_o^2 = 0.1 \]
\[ d \rightarrow d = 2.5 \]

6 required parameters
Understanding the behavior of the algorithm

- When both variances match, equal weight is placed (i.e., $\alpha = 0.5$).
- **Large bias**: more weight on the larger variance to better fit the observations.
- **Small bias**: good estimate; more weight on the smaller variance.
- **Moderate bias**: alternate between the ensemble and the static variance.

Vary both $\sigma^2_e$ and $\sigma^2_s$ and fix $\sigma^2_o$.

**Purplish**: More weight on ensemble covariance.
**Brownish**: More weight on static covariance.
Vary both $\sigma_e^2$ and $\sigma_s^2$ and fix $\sigma_o^2$.
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Purplish: More weight on ensemble covariance
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![Small Bias](image1)
![Moderate Bias](image2)
![Large Bias](image3)
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State variables may be hybridized differently, why?

- Biases are not homogenous in space
- Heterogeneous observation networks (densely observed regions tend to have small ensemble spread)
2.5 Hybrid EnKF-OI: Adaptive in space

- State variables may be hybridized differently, why?
  - Biases are not homogenous in space
  - Heterogenous observation networks (densely observed regions tend to have small ensemble spread)

- Need to assimilate observations serially. For each observation:
  - Compute correlation coefficient between the observed prior ensemble, $y^f$, and all state variables:
    \[
    \rho_j = \text{correlation} \left( y^f, x^f_j \right),
    \]
    where the hybrid weighting factor is assumed to have the same correlation field (Anderson 2009, El Gharamti 2018). Thus,
    \[
    d^2 = \sigma_o^2 + \rho_j \alpha \sigma_e^2 + (1 - \rho_j \alpha) \sigma_s^2
    \]
  - Find the posterior based on the modified likelihood and associated prior
3.1 Experiments using L96: Ensemble Size

- L96: 40 variables
- Observe every other variable (total of 20)
- Observe every 5 time steps ($dt = 0.05$)
- $R = 1$
- B Climatological run (1000 realizations)
- No inflation
- No localization
- $p(\alpha) \sim \mathcal{N}(0.5, 0.1)$
3.1 Experiments using L96: Ensemble Size

1. **EnKF**
2. **EnOI**: EnKF with fixed $B$ (Hybrid; $\alpha = 0$)
3. **EnKF-OI**: $\alpha = 0.5$
4. **AC-EnKF-OI**: Adaptive spatially-Constant EnKF-OI
5. **AV-EnKF-OI**: Adaptive spatially-Varying EnKF-OI
3.1 Experiments using L96: Ensemble Size

- AC-EnKF-OI: Dashed lines
- AV-EnKF-OI: Solid lines
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- AC-EnKF-OI: Dashed lines
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- For small ensembles, both adaptive spatially-constant and varying schemes behave the same
- Being spatially-varying, AV-EnKF-OI responds more efficiently to changes in the ensemble
3.2 Experiments using L96: Model Error & Inflation

- Ensemble size: 20
- Model error; vary $3 \leq F \leq 13$
- $B$ is generated in each case using biased $F$
- No localization
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- Hybrid scheme: better stability and more accurate even in very biased conditions
- As inflation increases, adaptive $\alpha$ increases (more weight on the ensemble cov)
3.3 Experiments using L96: Model Error & Localization

- Ne = 20, No inflation
- Vary both F and localization length scale
- Adaptive hybrid scheme is systematically better than the EnKF for all tested cases
  - For chaotic behaviour (i.e., $F \geq 8$): As localization increases, $\alpha$ increases
  - Less chaotic (smaller ensemble variance): $\alpha$ decreases to bring-in variability from $B$
  - Left panel: Ensemble Spread, Hybrid Spread, note the small spread in the ensemble for $F < 8$
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- **Data Void I**: Observe the first 20 variables
- **Data Void II**: Observe the first and last 5 variables
- **Data Void III**: Observe 10 variables in the center
- **Data Void IV**: Observe 5 variables in the center

In densely observed regions, the ensemble spread decreases. To counteract this, the hybrid scheme places more on the deterministic part to increase the variance and allow the filter to better fit the data.
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4. Conclusion

- Presented a new temporally and spatially varying adaptive hybrid EnKF-OI scheme
- The adaptive scheme uses the data and applies Bayes rule to determine the relative weighting between the ensemble and the static covariance
- The spatially-adaptive scheme – for now – does not support data that are not on the state grid (e.g., radiances)
- Tests using the Lorenz-96 system
- Future tests in high-order models (B-grid, CAM, WRF-Hydro ..)
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