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Period analysis of variable stars by robust smoothing

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Summary. The objective is to estimate the period and the light curve (or periodic function) of a variable star. Previously, several methods have been proposed to estimate the period of a variable star, but they are inaccurate especially when a data set contains outliers. We use a smoothing spline regression to estimate the light curve given a period and then find the period which minimizes the generalized cross-validation (GCV). The GCV method works well, matching an intensive visual examination of a few hundred stars, but the GCV score is still sensitive to outliers. Handling outliers in an automatic way is important when this method is applied in a 'data mining' context to a vary large star survey. Therefore, we suggest a robust method which minimizes a robust cross-validation criterion induced by a robust smoothing spline regression. Once the period has been determined, a nonparametric method is used to estimate the light curve. A real example and a simulation study suggest that the robust cross-validation and GCV methods are superior to existing methods.

Keywords: Generalized cross-validation; Period; Periodic function; Robust spline regression; Smoothing spline regression

1. Introduction

Variable stars are stars whose brightness changes over time. The class of periodic variable stars are stars whose maxima and minima brightness recur at constant time intervals. The variability of brightness allows for the classification of stars into different groups, according to information on physical properties such as magnitude, the range of period and light curve shape—a plot of the brightness variation of the star in time. It also provides important clues to the structure of the galaxies and stellar evolution (Brown and Gilliland, 1994; Gautschy and Saio, 1995, 1996; Hilditch, 2001). The primary statistical problem that is associated with classifying a variable star is to estimate its period and its light curve.

Consider a time series $\{y_i, t_i\}$,

$$y_i = f(t_i/p) + \varepsilon_i, \qquad i = 1, \ldots, n, \tag{1}$$

where y_i is the ith brightness measurement, t_i is the ith sampling time, ε_i is the ith measurement error and f is a periodic function or light curve on [0,1] ($f(t/p)$ has period p). The observations from a variable star are unequally spaced, because the data are collected only at certain times

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of night, sometimes with long interruptions. For this reason we expect that $\{t_i\}$ are unequally spaced. The basic statistical problem is to estimate both f and p.

Several methods have been developed to estimate the period of a variable star. The periodogram and least squares are the two traditional methods for estimating the period by using a simple cosine model (Deeming, 1975; Lomb, 1976; Scargle, 1982). Lafler and Kinman (1965) found the period to minimize a measure of dispersion defined by the function $\mathrm{LK}(p)$:

$$\mathrm{LK}(p) = \sum_{i=1}^{n} \{y^*_{i+1}(p) - y^*_i(p)\}^2,$$

where the y^*_i are the response values sorted by phase ($t_i \bmod(p)$). Dwortesky (1983) suggested a string length method that depends on differences in phase as well as in response. The method minimizes the string length given by

$$\mathrm{STR}(p) = \sum_{i=1}^{n} [\{y^*_{i+1}(p) - y^*_i(p)\}^2 + \{\phi^*_{i+1}(p) - \phi^*_i(p)\}^2]^{1/2},$$

where the $\phi^*_i(p)$ are the ordered phase values. Stellingwerf (1978) proposed another method based on a measure of dispersion, called phase dispersion minimization. In this method, the period is chosen to minimize the residual sum of squares of the one-way analysis of variance, after the phase interval has been divided into a number of bins and the mean response has been calculated for each bin. This particular method has gained a wide use by astronomers. Recently, Reimann (1994) suggested a nonparametric method to fit the brightness as a function of phase at a given period, using SuperSmoother, a variable span local linear smoother developed by Friedmann (1984). SuperSmoother performs three running line smooths of the data (phase and brightness) with long, medium and short span length. Cross-validation is then used to determine the span length that gives the best fit at each phase value. This method finds the period that minimizes the sum of absolute residuals obtained by SuperSmoother fitting, which is given by

$$\mathrm{AR}(p) = \frac{1}{n} \sum_{i=1}^{n} |y_i - \hat{y}_i(p)|,$$

where $\hat{y}_i(p)$ are the fitted values from SuperSmoother assuming a period p. Reimann (1994) showed through a simulation study that the cosine method with least squares and the method with SuperSmoother perform better than others. However, an obvious disadvantage with the cosine method is that it does not work well when the true light curve is not sinusoidal. Finally it is clear that any of the current methods are ineffective when the data have outliers. This lack of robustness is a practical concern for mining large data bases accumulated for light curve analysis. As one contribution of this paper, empirical results through real and numerical examples show that a properly constructed robust estimator retains high efficiency even when no outliers are present.

1.1. Data

The data sets of variable stars that are used in this paper are a derived product from the project 'Stellar astrophysics and research on exoplanets' (STARE). The primary objective of the STARE project is to use precise photometry to search for extrasolar giant planets transiting their parent stars (Charbonneau *et al.*, 2000). An important by-product of the STARE project is a survey of variable stars. For each of thousands of stars in a field, photometry data from the STARE instrument can be used to produce a light curve. Most stars are essentially constant in brightness, but about 10% of the stars are variable. Fig. 1(a) shows the brightness *versus* time (nights)

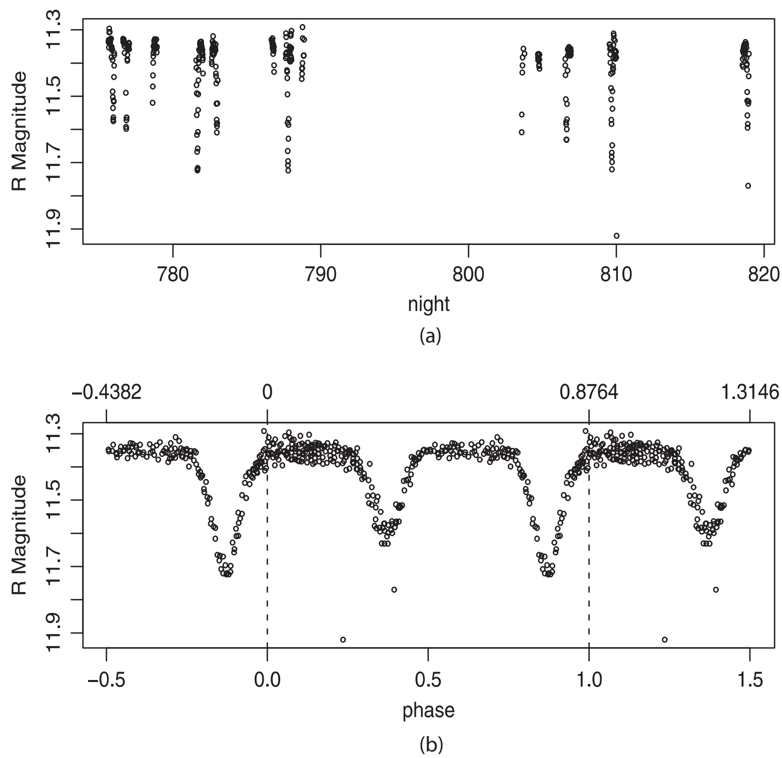


Fig. 1. (a) Brightness of a variable star (an eclipsing binary star) measured in stellar magnitude R, where $R = -2.5\ \log(F) + C$, F is the flux density from the star and C is a constant, and (b) brightness *versus* phase with $p = 0.8764$ days as period

for a star classified as an eclipsing binary star. The data in Fig. 1(a) consist of 351 separate measurements of the star's brightness, taken on 13 nights contained in a 44-night interval. The precision of each measurement is about 0.01 stellar magnitude. When two stars orbit each other in the plane of the observer, the combined brightness decreases when one member of the pair eclipses the other. However, as seen in Fig. 1(a), when the observations are unequally spaced with very long interruptions, the periodicity of the variable star is not obvious. Because the brightness depends on the phase (time mod(p)), if the brightness is periodic in time with period p, then a plot of brightness *versus* phase will reveal the periodicity. Fig. 1(b) presents a plot with a potential period in the phase domain. The light curve in Fig. 1(b) is produced by folding data over the period of variability. The plot with $p = 0.8764$ (days) clearly reveals a distinct light curve of the star. A light curve generated from the correct period will be useful for classifying the star. For instance, note that from Fig. 1(b) the light curve appears to be flat between eclipses. This feature is associated with the detached (Algol) type of eclipsing binary stars.

The data that are analysed in this paper can be obtained from

`http://www.blackwellpublishing.com/rss`

1.2. Outline

In the absence of outliers, we suggest the use of the generalized cross-validation (GCV) score to estimate the period of a variable star. In Section 2, a nonparametric method based on smoothing spline regression is proposed to determine the period of a variable star which minimizes the

GCV score. However, with the recognition that a smoothing spline and thus the related GCV score are affected by outliers, we suggest a robust modification. In the robust cross-validation (RCV) method, we estimate the period to minimize the RCV score that is induced by a robust smoothing spline regression. Once the period has been determined by either the GCV or the RCV method, the light curve can be estimated by a nonparametric method such as smoothing splines or SuperSmoother. Conceptually we have found it useful to separate the smoother that is used to determine the period with that used to estimate the light curve once p has been estimated. The theoretical background of the RCV method is briefly mentioned at the end. In Section 3, we compare the GCV and the RCV methods with the existing methods by using real brightness data and a simulation study. As a related topic, we discuss a method to estimate multiple periodicity in Section 4. Some concluding remarks are made in Section 5.

2. Methodology

2.1. Estimation of period: the generalized cross-validation and the robust cross-validation method

Given a period p, $\hat{f}_\lambda(t/p)$, the periodic cubic spline, is the minimizer of

$$\frac{1}{n}\sum_{i=1}^{n}\{y_i - f(t_i/p)\}^2 + \lambda\int_{[0,1]} f''(x)^2\,\mathrm{d}x \tag{2}$$

subject to $\int f''(x)^2\,\mathrm{d}x < \infty$ and $f(0) = f(1)$, $f'(0) = f'(1)$ (Wahba, 1990).

The GCV score for estimating the period of a variable star is

$$\mathrm{GCV}(p,\lambda) = \frac{\sum_{i=1}^{n}\{y_i - \hat{f}_\lambda(t_i/p)\}^2}{n[1 - n^{-1}\,\mathrm{tr}\{A(p,\lambda)\}]^2}, \tag{3}$$

where $A(p,\lambda)$ is the smoothing matrix that is associated with the spline estimate (Hastie and Tibshirani, 1990). It is useful to let $\mathrm{GCV}(p)$ denote the minimum of $\mathrm{GCV}(p,\lambda)$ over $\lambda \in [0,\infty)$ as

$$\mathrm{GCV}(p) = \min_{\lambda}\{\mathrm{GCV}(p,\lambda)\}. \tag{4}$$

We minimize the GCV score in two steps: for each period p, $\mathrm{GCV}(p)$ is computed and then the period p^* is determined by minimizing $\mathrm{GCV}(p)$ for all p. Applying this method to the data shown in Fig. 1(a), the estimated p is obtained as 0.8762 days. This is not far from 0.8764 days that were obtained by a visual search method in Fig. 1(b). However, the GCV score (not shown) has some local minima around the global minimum. As mentioned earlier, the smoothing spline regression is a linear estimate of the data and can be severely affected by outliers. The local minima of the GCV score is apparently influenced by two outliers (determined visually) near nights 810 and 820 (Fig. 1(a)). If we compute $\mathrm{GCV}(p)$ ignoring these two outliers, then the GCV score (not shown) does not have any local minima.

Now consider a new method that adopts robust spline regression instead of the usual smoothing spline. The robust smoothing spline can be defined, by replacing the sum of squared errors in expression (2) by a different function of the errors, as follows: let $\hat{f}_\lambda(t/p)$ be the minimizer of

$$\frac{1}{n}\sum_{i=1}^{n}\rho\{y_i - f(t_i/p)\} + \lambda\int_{[0,1]} f''(x)^2\,\mathrm{d}x. \tag{5}$$

Here the function $\rho(x)$ is typically convex and increases slower than order x^2 as x becomes large. Huber's favourite is

$$\rho(x) = \begin{cases} x^2 & \text{if } |x| \leqslant C, \\ C(2|x| - C) & \text{otherwise,} \end{cases}$$

where C is a cut-off point that is usually determined from the data. For C, we follow Huber (1981) and choose $\hat{C} = 1.345$ MAD which ensures 95% efficiency with respect to the normal model in a location problem. On the basis of this characterization, we consider an idealized RCV for the smoothing parameter of robust smoothing spline regression as

$$\mathrm{RCV}^*(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \rho\{y_i - \hat{f}_{\lambda,-i}(t_i)\}, \tag{6}$$

where $\hat{f}_{\lambda,-i}(t_i)$ is the robust smoothing spline when the ith data point (t_i, y_i) is omitted. The implementation of $\mathrm{RCV}^*(\lambda)$ is not feasible, because the robust spline is a non-linear estimate and so exhaustive leave-one-out cross-validation is usually not possible. An approximation to $\mathrm{RCV}^*(\lambda)$ is needed and we propose a very effective scheme based on the concept of pseudodata. The (unobservable) pseudodata $\mathbf{z}$ are defined as

$$\mathbf{z} = \psi(\mathbf{y} - \mathbf{f})/E(\psi') + \mathbf{f}, \tag{7}$$

where $\psi = \rho'$. The pseudodata can only be constructed with knowledge of the true function. However, on the basis of this construction, Cox (1983) gave an interesting result: a robust smoothing spline fit is asymptotically equivalent to a least squares smoothing spline fit based on pseudodata. By using this fact, we suggest the approximation

$$\mathrm{RCV}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \rho\{y_i - \tilde{f}_{\lambda,-i}(t_i)\}, \tag{8}$$

where $\tilde{f}_{\lambda,-i}(t_i)$ is the least squares smoothing spline with empirical pseudodata when the ith data point (t_i, y_i) is omitted. The empirical pseudodata are defined as

$$\hat{\mathbf{z}} = \psi(\mathbf{y} - \hat{\mathbf{f}})/E(\psi') + \hat{\mathbf{f}}, \tag{9}$$

where $\hat{\mathbf{f}}$ is the robust spline applied to the full data. With our notation, the approximation for the RCV score for the variable star problem can be expressed as

$$\mathrm{RCV}(p, \lambda) = \frac{1}{n} \sum_{i=1}^{n} \rho\{y_i - \tilde{f}_{\lambda,-i}(t_i/p)\}, \tag{10}$$

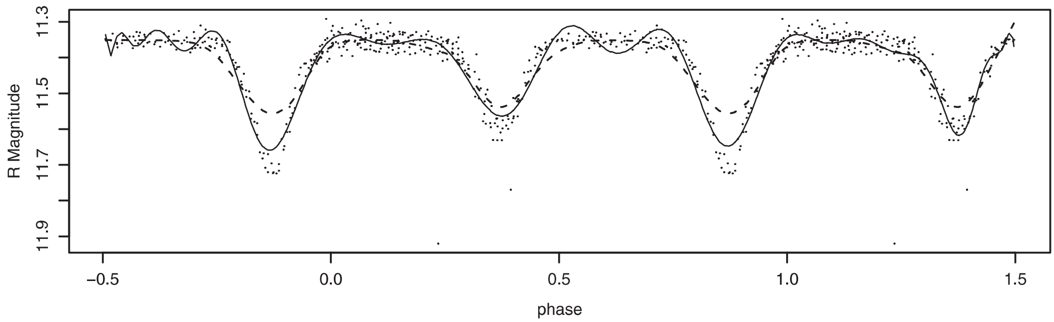
where λ is the smoothing parameter for a period p. Define $\mathrm{RCV}(p)$ as the minimum of $\mathrm{RCV}(p, \lambda)$ over $\lambda \in [0, \infty)$ for fixed p:

$$\mathrm{RCV}(p) = \min_{\lambda}\{\mathrm{RCV}(p, \lambda)\}. \tag{11}$$

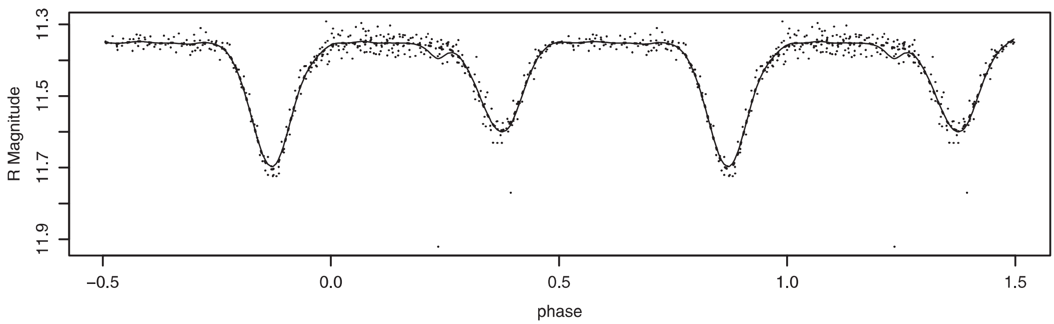
From the fact that smoothing spline regression can be severely affected by outliers, $\mathrm{RCV}(p)$ might be much less sensitive than $\mathrm{GCV}(p)$ of equation (4) with a least squares smoothing spline when data are perturbed by outliers. The $\mathrm{RCV}(p)$ score (not shown) for the data in Fig. 1(a) has a global minimum at 0.8764. Unlike ordinary GCV, the minimum is unique and smooth.

2.2. Light curve estimation

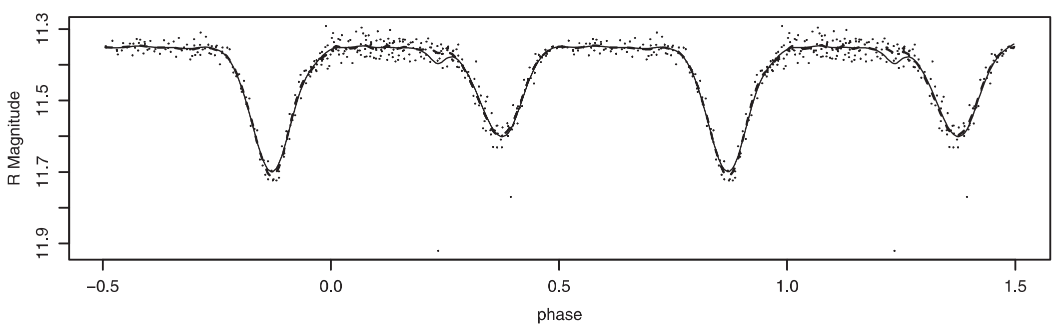
Once the period has been determined by either the GCV or the RCV method, a nonparametric curve fitting method can be used to estimate the light curve in the phase domain. Fig. 2



(a)



(b)



(c)

Fig. 2. Estimates of the light curve by various methods: (a) fits by SuperSmoother (———) and the cosine method (-------); (b) smoothing spline fit (———) and robust smoothing spline fit in Section 2.1 (-------); (c) robust smoothing spline fit (———) and robust LOESS fit (-------)

shows the estimation of the light curve of the star in Fig. 1 after the period has been determined at 0.8764 by the RCV method. Fig. 2(a) shows the estimates of the light curve by using SuperSmoother and the cosine method, Fig. 2(b) illustrates the fits by using a smoothing spline and robust smoothing spline regression and for comparison Fig. 2(c) shows the fits based on two other robust smoothing methods that are described in Section 3. All fitting methods have been used with optimal values (smoothing parameter and order) for appropriate criteria. As expected, the fit from a robust smoothing spline (the broken curve in Fig. 2(b)) provides a robust estimate relative to the two outliers. The main differences between the estimated light curve by using SuperSmoother and the estimated light curves obtained by other smoothing methods are

(a) the shape of the light curve such as the flatness between two eclipses and
(b) the difference of amplitude between the primary minimum and the secondary minimum.

The light curve (the full curve in Fig. 2(a)) by SuperSmoother has almost the same amplitudes between two minima and is rounded between eclipses, whereas the light curves that were fitted by the other smoothing methods have a different amplitude between two minima and are flat between eclipses.

The goal of estimating a light curve is not only to fit the light curve but also to obtain useful information to classify variable stars. If we classify a star as an eclipsing binary star on the basis of its light curve, further classification into a contact binary (W Ursa Majoris) type (the light curve by SuperSmoother) or a detached (Algol) type will depend on the relative amplitudes of the minima. Because SuperSmoother typically underfits the true function, it is not well suited to detect all the features of the light curve's shape that are necessary to classify the stars. Instead, as seen in Fig. 2, the smoothing spline regression captures the local structures of the true function well. Especially when the data have outliers, robust smoothing spline regression appears to be superior for this application. Note that the cosine method (the broken curve in Fig. 2(a)) can be used for estimating the light curve, but this method does not work well when the true light curve is not sinusoidal. These subjective observations are confirmed by the simulation study in Section 3.

As a topic related to estimating light curves, we suggest an approximate confidence interval for $f(t_i/p)$ with robust smoothing splines when the period p is fixed. To accomplish this we first detail an explicit equivalence between robust splines and a least squares spline based on empirical pseudodata. The robust smoothing spline fit that is described in Section 2.1 can be obtained by coupling a least squares smoothing spline with empirical pseudodata in equation (9). With empirical pseudodata $\hat{\mathbf{z}}$, consider the least squares smoothing spline problem for a fixed period p which minimizes

$$\sum_{i=1}^{n} \{\hat{z}_i - f(t_i/p)\}^2 + \lambda \mathbf{f}^{\mathrm{T}} R \mathbf{f}, \tag{12}$$

where R is a specific covariance matrix. The solution of problem (12) solves the normal equation $-2(\hat{\mathbf{z}} - \mathbf{f}) + 2\lambda R\mathbf{f} = \mathbf{0}$ and is equivalent to the normal equation of a robust smoothing spline $-\psi(\mathbf{y} - \mathbf{f}) + 2\lambda R\mathbf{f} = \mathbf{0}$ when $\mathbf{f} \equiv \hat{\mathbf{f}}$ and $E\{\psi'(\varepsilon)\} = 2$. Hence, the fit $\hat{\mathbf{f}}$ is a robust smoothing. Therefore, applying empirical pseudodata $\hat{\mathbf{z}}$ to a least squares smoothing spline produces robust smoothing. To construct a confidence interval, we apply the pseudodata concept to the confidence intervals that were proposed by Wahba (1983). The connection between a smoothing spline and a posterior mean suggests a $100(1-\alpha)\%$ confidence interval for $f(t_i/p)$ with a fixed period p as follows:

$$\hat{f}(t_i/p) \pm Z_{\alpha/2} \sqrt{\hat{\sigma}^2 \{A(\hat{\lambda})\}_{ii}}, \tag{13}$$

where

$$\hat{\sigma}^2 = \frac{\|\{I - A(\hat{\lambda})\}\hat{\mathbf{z}}\|^2}{\mathrm{tr}\{A(\hat{\lambda})\}}.$$

Fig. 3 shows a 95% confidence interval for the light curve of star 306 constructed by expression (13).

A reviewer raised some justifiable questions about whether this confidence procedure can be trusted in view of the underlying distributions being non-Gaussian. First we note that the

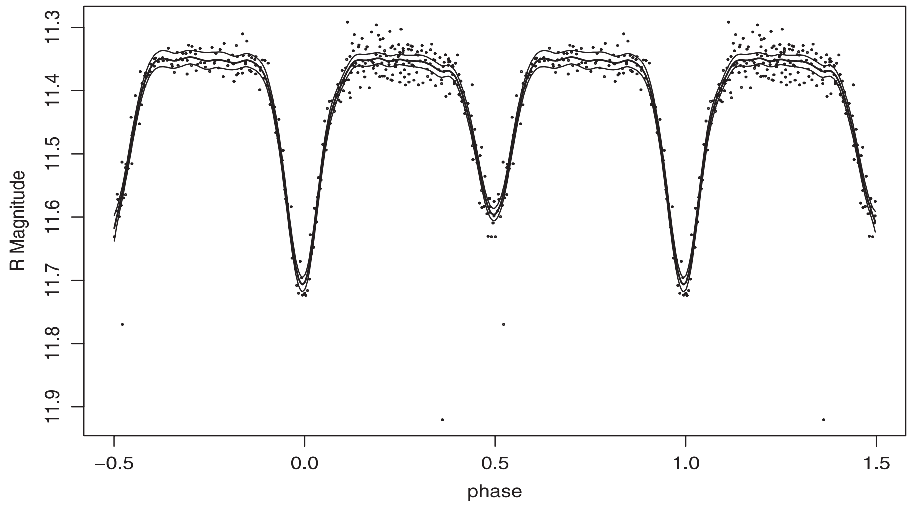


Fig. 3. 95% pointwise confidence interval

confidence interval procedure is being applied to the estimator that is derived from the empirical pseudodata and not the data in the original outlier scale. The empirical and theoretical pseudodata will always have finite moments due to the boundedness of the transformation. Thus, formulae based on first and second moments are not unreasonable. It is an open area of research for us to prove the validity of these intervals; however, we can provide some evidence based on technical results and collateral theory about why one might trust this procedure. But we should emphasize that the following discussion is far from a rigorous outline or even a sketch of a proof. The main point in our argument is to assume that the estimator based on empirical pseudodata and RCV of equation (8) approximates (i.e. is asymptotically equivalent) to the estimator using theoretical pseudodata of equation (7) and the optimal smoothing parameter. We note that Cox's results give some evidence for an asymptotic equivalence between equations (7) and (9) and the work of Hall and Jones (1990) suggests that cross-validation can provide a consistent estimate of the optimal bandwidth in the context of robust smoothing. Given the theoretical pseudodata estimator evaluated at the optimal smoothing parameter one would expect Wahba's confidence intervals to be reliable. Here we appeal to Wahba's work in this area and the fact that the spline is a special case of a geostatistics or kriging estimator. Indeed, Wahba's intervals are based on the prediction standard errors under the assumption of a particular generalized covariance. Such kriging standard errors do not depend on normality: only finite moments. In summary, although we do not have a rigorous justification of these intervals we feel that there are enough suggestions in the available theory to make them useful measures of uncertainty for the estimated function. As in any procedure that depends on underlying assumptions, care should be taken when drawing inferences. But we feel that these companion confidence intervals are much better than simply reporting an estimate without any quantification of its uncertainty.

2.3. *Theoretical motivation for RCV(λ)*

When robust smoothing spline regression is performed for estimating a light curve, the smoothing parameter λ must be selected automatically. As a selection method for λ, we believe that RCV(λ) may be useful. Note that RCV$^*(\lambda)$ mentioned here is the idealized, leave-one-out version defined as, for a fixed p,

$$\mathrm{RCV}^*(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \rho\{y_i - \hat{f}_{\lambda,-i}(t_i/p)\}.$$

This is different from our approximation defined in equations (8) and (10).

Given a period p, we conjecture that the minimizer of $\mathrm{RCV}^*(\lambda)$ also minimizes in asymptotic mean-squared error between the estimate of the robust smoothing spline regression and the truth. Denote the mean-squared error as

$$\mathrm{MSE}(\hat{f}, f) = \frac{1}{n} \sum_{i=1}^{n} E(f_i - \hat{f}_i)^2.$$

A robust extension to the result of Craven and Wahba (1979) gives the following. Given a fixed p, if λ^* is the minimizer of $E\{\mathrm{RCV}^*(\lambda)\}$ over $[\lambda_n, \Lambda_n]$, then

$$\frac{\mathrm{MSE}(\hat{f}_{\lambda^*}, f)}{\min_\lambda\{\mathrm{MSE}(\hat{f}_\lambda, f)\}} \to 1, \qquad \text{as } n \to \infty. \tag{14}$$

We now include a sketch of the proof for property (14). More theoretical results for RCV^* and rigorous proofs of result (14) are in progress and will appear elsewhere. Let $\tilde{f}_\lambda$ be a least square smoothing spline fit with pseudodata in equation (7). By a Taylor series expansion of RCV^*, $E\{\mathrm{RCV}^*(\lambda)\}$ is asymptotically equivalent to $E\{\mathrm{CV}(\lambda)\}$ based on pseudodata

$$E\{\mathrm{RCV}^*(\lambda)\} \approx \frac{1}{n} \sum_{i=1}^{n} E(f_i - \tilde{f}_{\lambda,-i})^2 + \text{constant}.$$

Thus, by using the result of Craven and Wahba (1979), it can be shown that $E\{\mathrm{RCV}^*(\lambda)\} \approx \mathrm{MSE}(\tilde{f}_\lambda, f) + \text{constant}$. Finally, with Cox's (1983) result, $\hat{f}_\lambda$ inherits the same asymptotics as $\tilde{f}_\lambda$, and we conclude that

$$E\{\mathrm{RCV}^*(\lambda)\} \approx \mathrm{MSE}(\hat{f}_\lambda, f) + \text{constant}.$$

In comparison with these results, Hall and Jones (1990) discussed kernel M-estimates of the regression function by using Huber's ρ-function. They showed that least squares cross-validation results in optimal bandwidth selection (and determining C) with respect to the mean-squared error. However, we have found that it is difficult to extend their results to spline-type estimates based on pseudodata.

3. A comparison of methods

Here we report the results of our analysis of several variable stars and one numerical experiment. These experiments are designed for comparing the practical performances of the proposed approaches with some existing methods. To assess the performance of the proposed method RCV when the data are perturbed by outliers, we use two robust smoothing approaches. One is a robust LOESS method based on the assumption of symmetric errors instead of Gaussian errors. Therefore, the robust LOESS estimate is not adversely affected if the errors have a long-tailed distribution (Chambers and Hastie, 1993). The other is a robust fit of a smoothing spline by using the L_1-norm. The algorithm is an iterative reweighted smooth spline algorithm which performs a least squares smoothing spline at each step with the weights w equal to the inverse of the absolute value of the residuals for the last iteration step. Note that this robust smoothing spline is different from the robust smoothing spline fit based on the empirical pseudodata in

Section 2.1. Both robust smoothing methods are applied to find the period that minimizes the sum of square residuals obtained from fitting.

For two experiments, the following eight methods are compared:

(a) rcv, the RCV method that was proposed in Section 2.1 as the target;
(b) gcv, the GCV method that was described in Section 2.1;
(c) lk, the measure-of-dispersion procedure of Lafler and Kinman (1965);
(d) pdm, the phase dispersion minimization procedure;
(e) Fourier, the cosine method with least squares;
(f) sm, the smoothing procedure by SuperSmoother;
(g) rloess, the robust smoothing procedure by robust LOESS;
(h) rss, the robust smoothing procedure by robust smoothing splines.

All the smoothing methods were performed with some forms of smoothing parameter selection and the order for Fourier has been selected by Akaike's information criterion.

3.1. Analysis of data on variable stars

In this section, for several real variable stars, we compare the eight methods cited above. For the real eclipsing binary star (306) data shown in Fig. 1, based on the global minima from an exhaustive search, the period is estimated as 0.8762 and 0.8764 under the GCV and the RCV methods respectively. Existing methods result in almost the same period as the RCV method (Table 1). However, as shown in Table 1, we find that the estimated periods vary with different procedures for some variable stars. Fig. 4 displays light curves produced by folding data over the estimated period determined through the rcv method.

To estimate the error of the period estimate, we simply use a bootstrap method (Efron, 1979). The procedure is performed as follows: B training sets $\mathbf{y}^{*b}$, $b=1,2,\ldots,B\,(=100)$, are drawn

Table 1. Estimated period and its (bootstrap) standard deviation σ_B, given in parentheses, for several variable stars according to different methods

Method	*Estimated periods (days) and standard deviations for the following stars:*						
	306	*969*	*1164*	*1744*	*4699*	*4865*	*5954*
rcv	0.8764	2.2124	1.4630	1.3328	3.2316	2.3250	0.9808
	(0.0001)	(0.0030)	(0.0004)	(0.0015)	(0.0295)	(0.0016)	(0.0003)
gcv	0.8762	2.2134	1.4634	1.3328	3.2858	2.3256	0.9806
	(0.0003)	(0.0028)	(0.0010)	(0.0020)	(0.0168)	(0.0024)	(0.0004)
lk	0.8764	2.2148	1.4632	1.3320	3.2220	2.3240	0.9804
	(0.0001)	(0.0022)	(0.0006)	(0.0013)	(0.0332)	(0.0014)	(0.0003)
pdm	0.8764	2.2182	1.4592	1.3316	3.2966	2.3196	0.9798
	(0.0001)	(0.0020)	(0.0003)	(0.0019)	(0.0322)	(0.0018)	(0.0003)
Fourier	0.8762	2.2144	1.4626	1.3322	3.2894	2.3210	0.9804
	(0.0002)	(0.0034)	(0.0007)	(0.0011)	(0.0190)	(0.0021)	(0.0003)
sm	0.8762	2.2116	1.4626	1.3316	3.2896	2.3278	0.9802
	(0.0003)	(0.0023)	(0.0014)	(0.0009)	(0.0322)	(0.0030)	(0.0003)
rloess	0.8760	2.2134	1.4632	1.3328	3.2888	2.3278	0.9804
	(0.0004)	(0.0021)	(0.0006)	(0.0009)	(0.019)	(0.0028)	(0.0003)
rss	0.8766	2.2142	1.4632	1.3334	3.2878	2.3262	0.9808
	(0.0003)	(0.0018)	(0.0003)	(0.0043)	(0.0313)	(0.0011)	(0.0003)

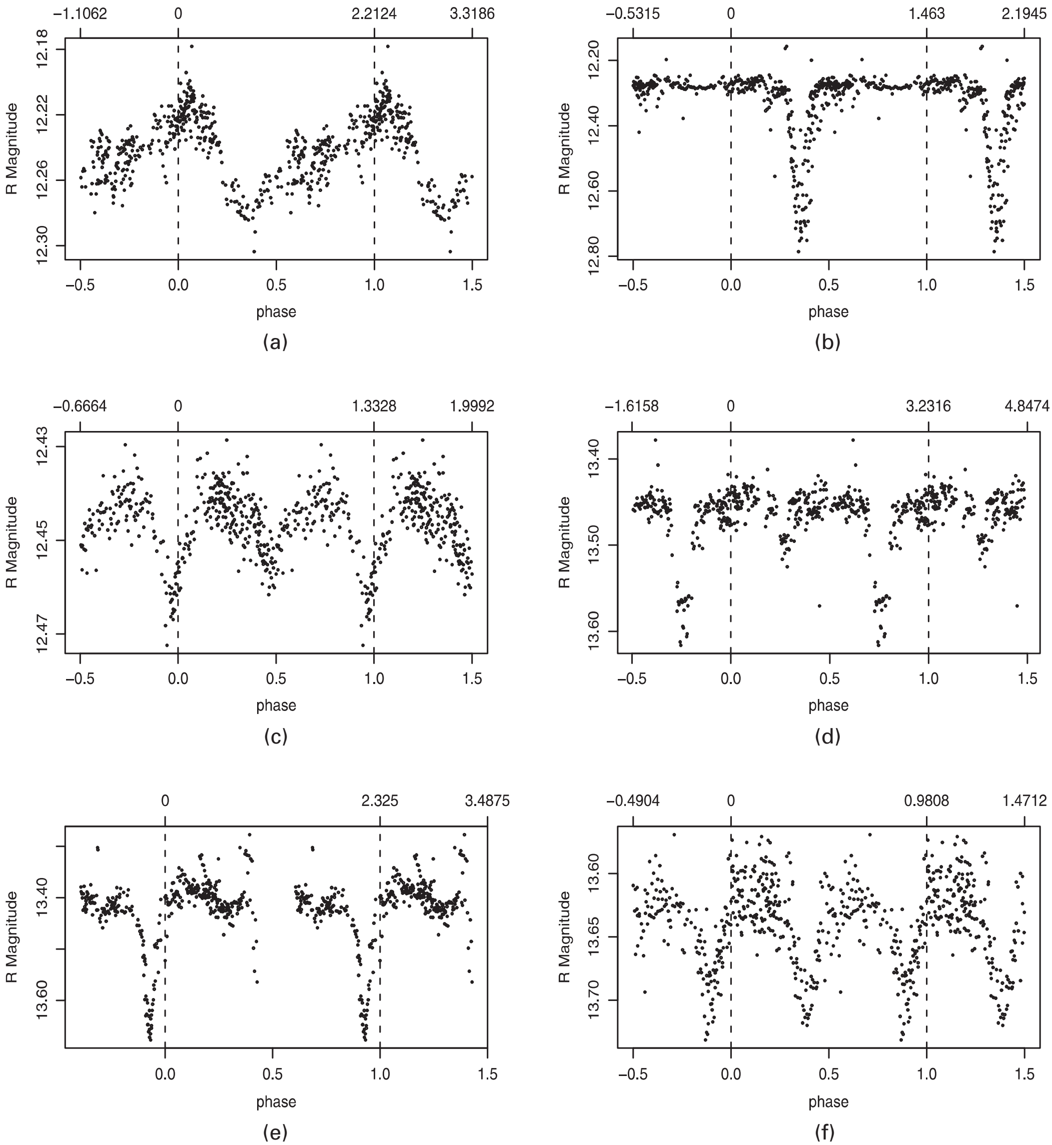


Fig. 4. Brightness *versus* phase with period estimates by the RCV method: (a) star 969; (b) star 1164; (c) star 1744; (d) star 4699; (e) star 4865; (f) star 5954

with replacement from the original data set **y**. Each sample has the same size as the original data set. The period p^{*b} is estimated from each bootstrap training set, and then we compute its standard deviation to assess the accuracy of the period estimate:

$$\sigma_B = \sqrt{\left\{ \frac{1}{B-1} \sum_{b=1}^{B} (p^{*b} - \bar{p}^*)^2 \right\}},$$

where $\bar{p}^* = \Sigma_{b=1}^{B} p^{*b}/B$. The errors of the period estimate, σ_B, for several real variable stars are given in parentheses in Table 1.

3.2. Simulation study

In this experiment the proposed RCV method is compared with existing methods through a simulation study based on two test functions and three noise levels. In summary, this is a three-factor experiment with two test functions, three levels of noise and eight methods of estimating the period. We use the two test functions that are plotted in Fig. 5. The first function is a sinusoidal signal, $f_i = 0.1\ \sin(2\pi t_i/3)$, that is a curve resembling the shape for an eclipsing binary star. The period, 3 days, is typical for stars in the data set and we also match the irregular time sampling based on brightness data from observed data. For the noise models, we consider

(a) Gaussian errors, $N(0, 0.04)$,
(b) Student t noise with 3 degrees of freedom scaled by 0.05 and
(c) a mixture of 95% $N(0, 0.04)$ and 5% $N(0, 0.3)$ at a random location.

Noise model (a) represents the case that outliers are not present, and models (b) and (c) are considered for the case where outliers are present. The noise level, 0.04, is consistent with the estimated noise level for several real eclipsing binary stars.

For the second test function, the signal is a periodic function that is similar to the actual Algol-type star as shown in Fig. 1(b). The test function has a geometric interpretation. Assume two spheres with the same radius r that are orbiting each other. We calculate overlapping and non-overlapping areas of the two spheres and equate non-overlapping areas to brightness.

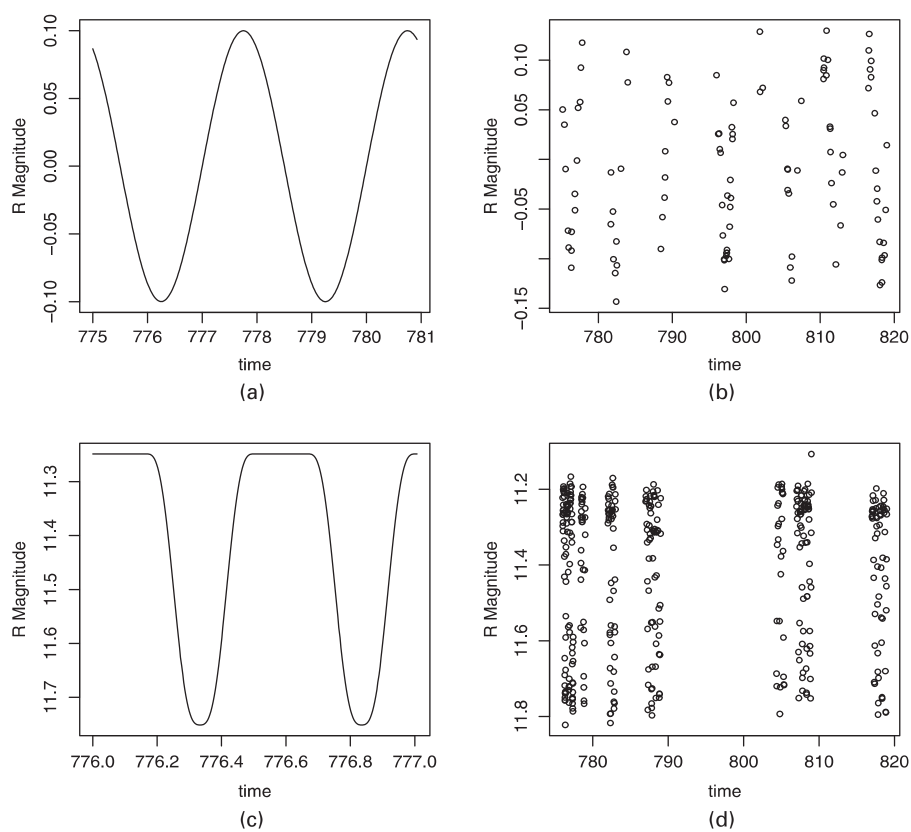


Fig. 5. Simulated sinusoidal signals with noise: (a) true light curve with period 3 days; (b) irregularly sampled brightness; (c) true light curve with period 1 day; (d) irregularly sampled brightness

The test function that is obtained in this manner is

$$f(t) = \begin{cases} \alpha(\pi r) + \beta, & a \leqslant t < b, \\ \alpha[\pi r - r^2\{\theta - \sin(\theta)\}]_{0 \leqslant t < 2\pi} + \beta, & b \leqslant t < c, \\ \alpha[\pi r - r^2\{\theta - \sin(\theta)\}]_{2\pi \leqslant t < 0} + \beta, & c \leqslant t < d, \\ \alpha(\pi r) + \beta, & d \leqslant t < e, \\ \alpha[\pi r - r^2\{\theta - \sin(\theta)\}]_{0 \leqslant t < 2\pi} + \beta, & e \leqslant t < f, \\ \alpha[\pi r - r^2\{\theta - \sin(\theta)\}]_{2\pi \leqslant t < 0} + \beta, & f \leqslant t < g, \end{cases}$$

$\alpha = 0.07$, $\beta = -11.5$ and radius $r = 1$ results in the 'true' light curve in Fig. 5(c). We select a period $g - a = 1$ day. Irregular sampling provides the brightness that is shown in Fig. 5(d). Finally, three noise scenarios are considered:

(a) Gaussian error $N(0, 0.06)$,
(b) Student t noise with 3 degrees of freedom scaled by 0.08 and
(c) a mixture of 95% $N(0, 0.06)$ and 5% $N(0, 0.8)$ at a random location.

The noise level 0.06 is a reasonable choice for an Algol-type star.

The box plots in Fig. 6 summarize the results of period estimates based on 100 replications. From the simulation results, we have the following empirical observations:

(a) the rcv and gcv methods give nearly identical results for the Gaussian noise;
(b) for the first test function (sinusoidal) and Gaussian noise, the Fourier method provides the best result with gcv and rcv;
(c) both robust procedures rcv and rloess outperform non-robust procedures when the error is $t(3)$ and a mixture of two Gaussian distributions at random locations;
(d) the rcv method always outperforms the two other robust methods, rloess and rss for $t(3)$ and the mixture noise scenario.

4. Multiple periodicity

Here we discuss the estimation of multiple periodicity of a variable star. Consider a time series $\{y_i, t_i\}$ with fixed L periodicity,

$$y_i = \sum_{l=1}^{L} f_l(t_i/p_l) + \varepsilon_i,$$

where f_l is the lth period function with period p_l. The statistical problem with the above additive model is to estimate f_l and p_l, $l = 1, 2, \ldots, L$. Since the RCV method in Section 2.1 is induced by a smoothing technique (a robust smoothing spline), we can use a backfitting algorithm which is well known in fitting nonparametric additive regression (Chambers and Hastie, 1993). The backfitting algorithm can be briefly described as follows. For simplicity, let $L = 2$, i.e. $y_i = f_1(t_i/p_1) + f_2(t_i/p_2) + \varepsilon_i$. Consider the system of two equations:

$$\begin{aligned} \mathbf{f}_1 &= S_1(\mathbf{y} - \cdot - \mathbf{f}_2), \\ \mathbf{f}_2 &= S_2(\mathbf{y} - \mathbf{f}_1 - \cdot), \end{aligned} \tag{15}$$

where the dots in the equation are place holders showing the term that is missing in each row. Here a vector $\mathbf{f}_l$ denotes the function f_l evaluated at the sampling time t_i and period p_l. S_l represents the smoother operator matrix for smoothing against t_i at the period p_l. In this case, we use robust smoothing splines in expression (5) as all the smoothers. Thus, this system solves the problem

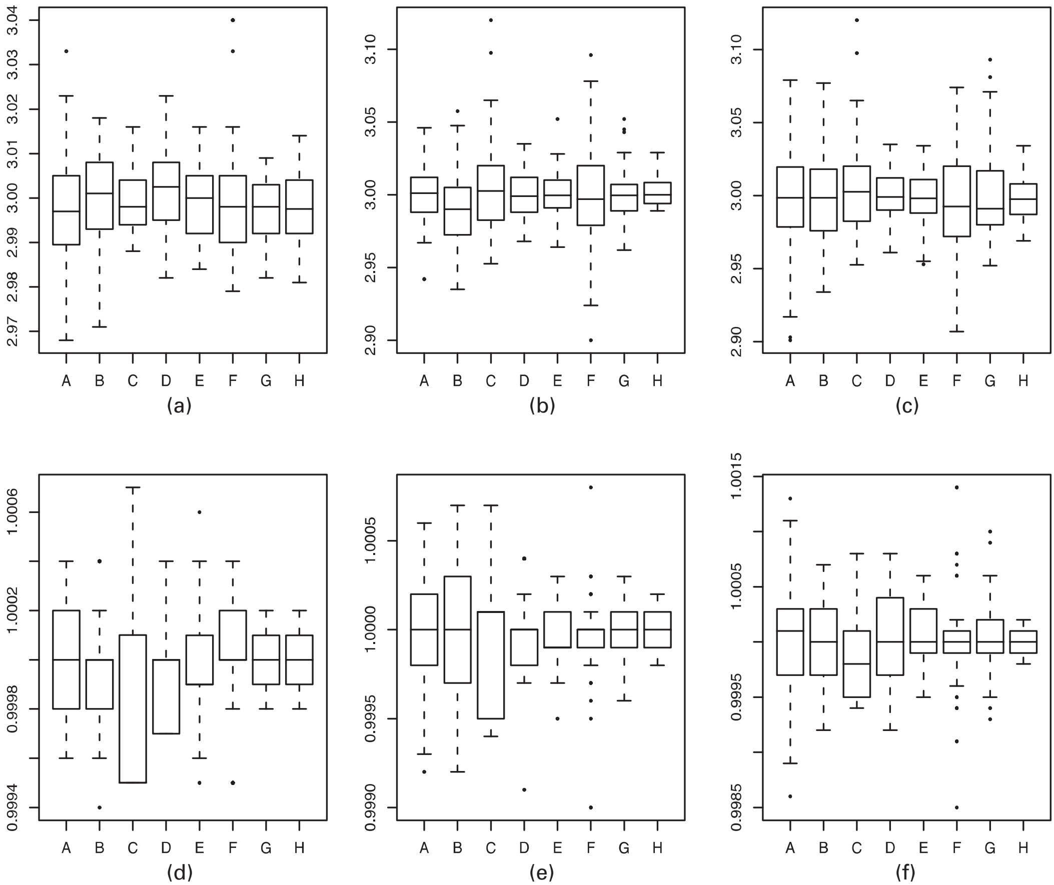


Fig. 6. Box plots of the period estimates with respect to two test functions and three noise scenarios (A, Lafler and Kinman's (1965) method; B, pdm method; C, Fourier method; D, SuperSmoother; E, rloess; F, rss; G, gcv; H, rcv): (a) test function 1, normal noise; (b) test function 1, $t(3)$ noise; (c) test function 1, mixture noise; (d) test function 2, normal noise; (e) test function 2, $t(3)$ noise; (f) test function 2, mixture noise

$$\frac{1}{n}\sum_{i=1}^{n}\rho[y_i-\{f_1(t_i/p_1)+f_2(t_i/p_2)\}]+\sum_{l=1}^{2}\lambda_l\int_{[0,1]}f_l''(t)^2\,\mathrm{d}t. \tag{16}$$

To solve this system, the Gauss–Seidel iterative method loops through the equations, substituting the most updated versions of functions on the right-hand side with each iteration. For estimating multiple periods, we use the algorithm based on backfitting as follows. Suppose that we have an initial period estimate $\hat{p}_1$.

Step 1: obtain residuals by $y_i-\hat{f}_1(t_i/\hat{p}_1)$.
Step 2: estimate $\hat{p}_2$ by using the RCV method that is described in Section 2.1.
Step 3: take residuals by $y_i-\hat{f}_2(t_i/\hat{p}_2)$.
Step 4: estimate $\hat{p}_1$ by using the RCV method.
Step 5: repeat steps 1–4 until the estimates $\hat{p}_1$ and $\hat{p}_2$ converge.

Finally, we apply the algorithm to a real Delta Scuti type star (543) in the STARE database. The algorithm converges to two periods, $\hat{p}_1 = 0.03903$ days and $\hat{p}_2 = 0.04804$ days after several

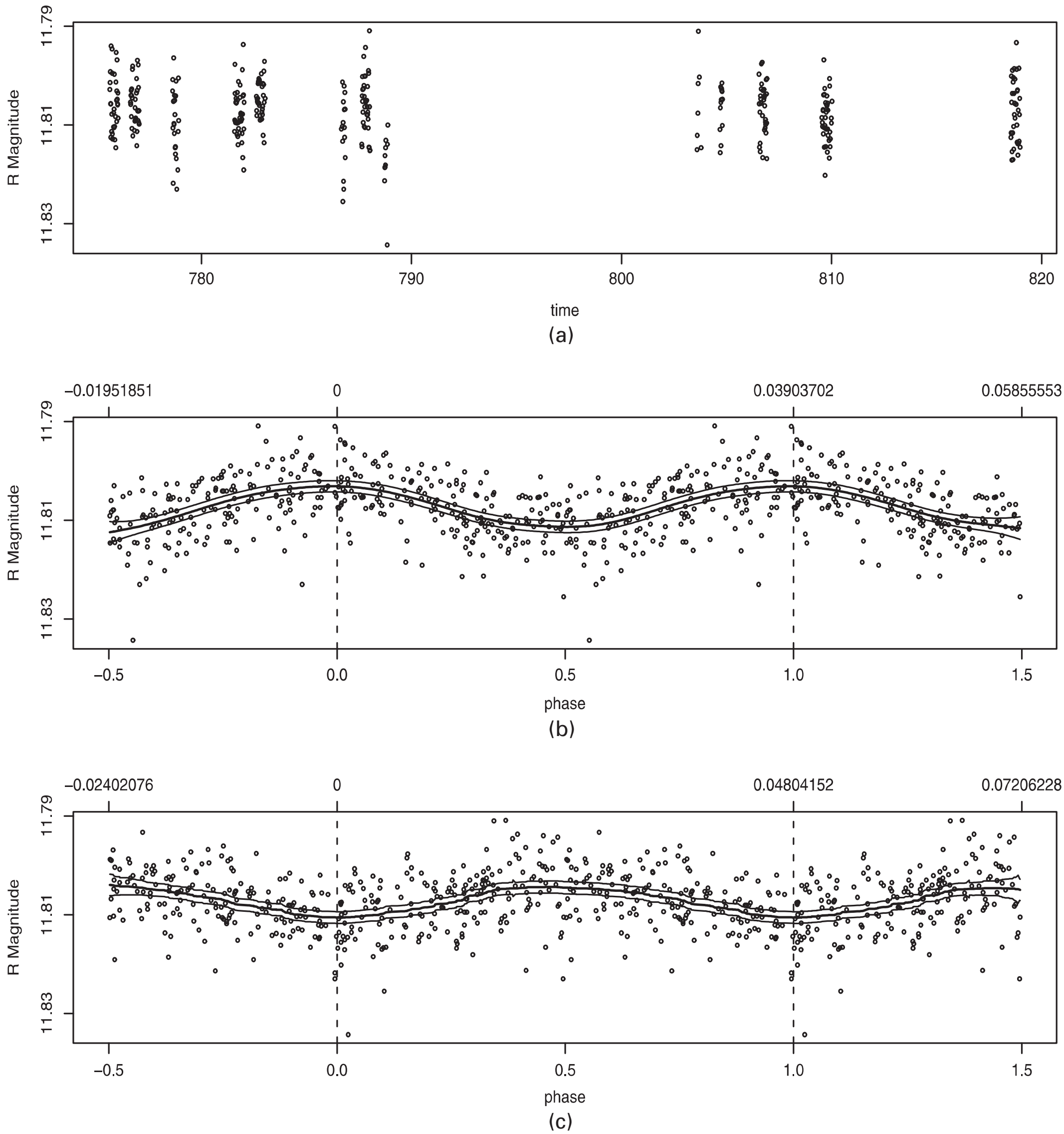


Fig. 7. (a) Brightness of a variable star with multiple periodicity, (b) brightness *versus* phase with the first period $p = 0.03903$ days and estimate of the light curve with 95% pointwise confidence interval and (c) brightness *versus* phase with the second period $p = 0.04804$ days and estimate of the light curve with 95% pointwise confidence interval

iterations. Fig. 7 shows the brightness of star 543 in the time domain and plots of brightness *versus* phase with period estimates determined by the backfitting algorithm.

5. Conclusion

We have proposed two methods to estimate the period of a variable star from the irregularly observed brightness. The GCV method is to minimize the GCV score that is generated by a smoothing spline, whereas the RCV method is based on robust smoothing spline regression as a robust version to the outliers. On the basis of actual light curve data and a simulation study, we have shown that the method proposed estimates the period more accurately than existing methods. In case the signal is perturbed by a few outliers, the RCV method followed by the robust smoothing spline regression is a useful tool for estimating the period. In general, the robust smoothing spline that is proposed in this paper provides a resistant method to outliers so the advantage of this approach might be important for the next phase of our group's scientific project. Currently, we are applying the RCV method to determine periods for a survey of approximately 6000 stars.

As a future direction for statistical research, we also note that the pseudodata idea can be fruitfully applied to thin plate smoothing spline and wavelet shrinkage to obtain robust estimators. It might also be interesting to apply the RCV method to other databases such as the 'High precision parallax collecting satellite' and 'Massive compact halo objects' databases.

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