Gravity-wave refraction by three-dimensional winds: new effects and a new parametrization scheme

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Gravity waves evolve locally but they affect the circulation globally. Think locally act globally.
Wave capture and wave–vortex duality

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Inertia-gravity waves

undulating material stratification surfaces
(isentropes/isopycnals)
surfaces are flat at rest

linear particle trajectories

\[ w' \propto \exp(i[kx + mz - \hat{\omega}t]) \]

scale-free dispersion relation
\[
\hat{\omega}^2 = \left( N^2 - f^2 \right) \frac{k^2}{k^2 + m^2} + f^2
\]
\[ f^2 \leq \hat{\omega}^2 \leq N^2 \]

Momentum flux \[ u'w' < 0 \] (unlike surface waves)

Internal waves make a significant contribution to atmospheric angular-momentum fluxes... convergence yields wave drag

Oliver Bühler, Courant Institute, New York University, 2005
Wave breaking

Large-amplitude waves overturn and break nonlinearly, leading to 3d turbulence, irreversible mixing, and convergence of wave momentum flux.

before after

Figure 3: Diffusive scheme applied to the unstable buoyancy profile: $b(x, z) = 0.3 \exp[-((x-0.5)^2 + (z-0.5)^2)/0.09] \cos[2\pi(x+z)]$. (a) (b) are surface plots of the potential temperature distribution before and after mixing. (c) and (d) give potential temperature profiles on the transects $x = 0.6$ and $z = 0.6$ respectively. The blue curve is the profile before mixing, and the green curve is the profile after mixing.

Marcus Roper,
GFD report 2005

Oliver Bühler, Courant Institute, New York University, 2005
The classical gravity wave drag picture

Ray-tracing theory for steady wavetrain as function of altitude $z$

Horizontal mean-flow inhomogeneity is ignored ("co-dimension 1")

Vertical flux of horizontal pseudomomentum equals vertical flux of horizontal mean momentum

Vertical flux of horizontal pseudomomentum is constant unless waves are dissipating or breaking

This results in the classical force balance: there is an equal-and-opposite action at a distance between mountain drag and wave drag
Wave drag on a rotating planet "gyroscopic pumping"

Convergence of angular-momentum flux acts as effective force on zonal-mean flow

Retrograde force
if \( k/\hat{\omega} < 0 \)

Retrograde force
-> poleward motion

All Rossby waves
Winter gravity waves

Prograde force
if \( k/\hat{\omega} > 0 \)

Prograde force
-> equatorward motion

Summer gravity waves

Oliver Bühler, Courant Institute, New York University, 2005
Vertical propagation of topographic Rossby and gravity waves

**GW** \( f^2 \leq \hat{\omega}^2 \leq N^2 \)  \( RW \) \( \hat{\omega} \leq 0 \)  Absolute frequency: \( \omega = \hat{\omega} + Uk \)

Topographic waves have zero absolute frequency: \( \omega = 0 \)

Topographic waves are retrograde if \( U>0 \) near surface

Vertical critical layer occurs where shear pushes intrinsic frequency towards lower limit

Summer zero-wind surface acts to filter topographic waves

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Fig. 1.4. Schematic latitude-height section of zonal mean zonal wind (m s\(^{-1}\)) for solstice conditions; W and E designate centers of westerly (from the west) and easterly (from the east) winds, respectively. (Courtesy of R. J. Reed.)
Wave-driven global circulation

Prograde and retrograde
mesospheric gravity and Rossby-wave drag:
Murgatroyd--Singleton circulation
from summer to winter pole

Retrograde stratospheric Rossby-wave drag:
Brewer--Dobson circulation from equator to pole

Summer polar mesosphere is sunniest place on Earth
Also the coldest (-163°Celsius)

Fermi review,
ME McIntyre 1993
New York University, 2005
Rough guide to gravity wave scales

To resolve a scale need 10 grid points across it

<table>
<thead>
<tr>
<th>Gravity wave scales (excludes breaking details)</th>
<th>Model resolution (grid size * 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Horizontal scale</strong></td>
<td>10 - 1000 km</td>
</tr>
<tr>
<td><strong>Vertical scale</strong></td>
<td>0.1 - 10 km</td>
</tr>
<tr>
<td><strong>Time scale</strong></td>
<td>10 mins - 1 day</td>
</tr>
</tbody>
</table>

GWs parametrized because need a factor 100 higher resolution.
How long will that take?

Oliver Bühler, Courant Institute, New York University, 2005
Computer power and resolution increase

Moore’s law: computing power doubles every 18 months.

To reduce grid size by a factor 10 requires $10 \times 10 \times 10 \times 10 = 10000$ times more computing power in a three-dimensional time-dependent simulation (grid points x time steps with CFL condition).

Let $N$ by number of necessary doubling events:

$$2^N = 10^4$$

$$N \log 2 = 4 \log 10$$

$$N = 13.3$$

$$T = N \times 18 \text{ months} = 19.95 \text{ years}$$

To increase resolution by a linear factor of 10 takes 20 years

Moore is not enough.........

Oliver Bühler, Courant Institute, New York University, 2005
Multiscale-based career advice

For gravity waves need factor 100

This will take 40 years, ok for a career!
Columnar gravity wave parametrization

Parametrization is applied independently in each vertical model column

Time-dependence is ignored

Vertical wave propagation and vertical mean-flow derivatives are taken into account

Horizontal wave propagation and horizontal mean-flow derivatives are ignored: no refraction by mean flow

Many effects are neglected. Which are the important ones?

Some neglected effects are known to be important.
For instance, intermittency.
Known unknown: intermittency

Intermittency increases wave amplitudes at fixed mean wave activity flux

Wave breaking is highly amplitude-dependent

"Intermittency factor" adjusted to increase predicted amplitudes to obtain breaking

Better to model wave generation as stochastic process (future work)

Bühler, 2003 JAS

Oliver Bühler, Courant Institute, New York University, 2005

FIG. 4. Wave packets due to hypothetical intermittent wave source at $\gamma_0 = 0$. (left) Steady, nonintermittent source. (middle) Intermittent source with $\gamma = 50\%$. (right) Intermittent source with $\gamma = 25\%$. The wave amplitude increases by a factor of $\sqrt{2}$ from wave to wave, keeping the expected wave energy flux the same in all three cases.
## Unknown unknowns in parametrization

### Requirements for better parametrization

<table>
<thead>
<tr>
<th>Theoretical constraints</th>
<th>almost none; plenty of ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational constraints</td>
<td>Must be in-column &amp; inexpensive</td>
</tr>
<tr>
<td>Observational constraints</td>
<td>Must not require more information</td>
</tr>
</tbody>
</table>

*Three-dimensional refraction is a candidate*

Oliver Bühler, Courant Institute, New York University, 2005
Claim: singular geometric perturbation

Key to strong effects:
ignoring the horizontal refraction is a singular perturbation


3d wave-mean interactions exhibit new features such as remote recoil and missing forces (Bühler & McIntyre 2003) Wave-vortex duality.

What are the new effects?

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An adventure in ray tracing

Wavepackets along group-velocity ray

Wavepackets are the fundamental solutions of ray tracing

Wavetrains can be built from wavepackets

Amplitude along non-intersecting rays is determined by wave-action conservation
Warm-up example: shallow water system

Single layer of hydrostatic incompressible fluid

Variables

\[ \mathbf{x} = (x, y) \quad \mathbf{u} = (u, v) \]

Depth \( h \)

\[ \frac{Dh}{Dt} + h \nabla \cdot \mathbf{u} = 0 \]

Mass continuity

Momentum conservation

\[ \frac{Du}{Dt} + g \nabla h = 0 \]
Linear small-scale wavepacket
(Example in shallow water)

\[ u = U + u' + O(a^2) \]
\[ D_t u' + g \nabla h' = 0 \]
\[ h = H + h' + O(a^2) \]
\[ D_t h' + H \nabla \cdot u' = 0 \]

Slowly varying wavetrain \( h' = H a(\mu x, \mu t) \exp(i\theta), \quad \mu \ll 1 \)

\[ k = \nabla \theta, \quad \omega = -\theta_t, \quad \omega = U \cdot k + \hat{\omega} \]

Dispersion relation \( \hat{\omega}^2 = g H |k|^2 \)

Wavepacket:

phase lines
Geometric ray tracing - phases

\[ \Omega(k, x, t) = U \cdot k + \hat{\omega} \]

\[ \frac{dx}{dt} = \frac{\partial \Omega}{\partial k} \quad \text{and} \quad \frac{dk}{dt} = -\frac{\partial \Omega}{\partial x} \]

Group velocity

\[ u_g = \frac{dx}{dt} = U + \hat{u}_g \]

Ray time derivative

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + (u_g \cdot \nabla) \]

Simple case \( \hat{\omega}(k) \):

Wavenumber changes due to background inhomogeneity --> refraction

\[ \frac{dk_i}{dt} = -\frac{\partial U_j}{\partial x_i} k_j \]

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Physical ray tracing - amplitudes

“Mean” is the average over rapidly varying wave phase

\[ h = \overline{h} + h' \]
\[ \overline{h}' = 0 \]

Wave energy
\[ E = \frac{1}{2} H \left( \overline{u'^2} + \frac{g h'^2}{H} \right) \]

Wave action
\[ A = \frac{E}{\omega} \]

\[ \frac{\partial A}{\partial t} + \nabla \cdot (A u_g) = 0 \]

Amplitude prediction from wave action conservation

Another important wave property:

**Pseudomomentum**
\[ \mathbf{p} = k A \]

Pseudomomentum changes with wavenumber due to refraction

Oliver Bühler, Courant Institute, New York University, 2005
Understanding wavenumber refraction

Wavenumber

\[
\frac{dk_i}{dt} = -\frac{\partial U_j}{\partial x_i} k_j \quad \text{is equivalent to} \quad \frac{dk}{dt} = -\nabla U \cdot k
\]

Passive tracer

\[\phi(x, t) \text{ such that} \]

\[D_t \phi = 0\]

\(k\) and \(\nabla \phi\) evolve similarly

(i.e. wave phase and passive tracer evolve similarly)

Intrinsic difference

\[u_g = \frac{dx}{dt} = U + \hat{u}_g\]

\[\frac{d}{dt} - D_t = (\hat{u}_g \cdot \nabla)\]

measures the misfit

Oliver Bühler, Courant Institute, New York University, 2005
Wavepacket exposed to pure strain in analogy with passive advection

\[ p = k A \]

Wavepacket is squeezed in \( x \) and stretched in \( y \).
Action is constant

Wavenumber vector \( k \) is increases in size

Pseudomomentum \( p \) increases as well

Is there a “Batchelor” regime for wave phase?

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Not in shallow water...

Answer is no in shallow water with sub-critical steady background flow

\[ \omega = U \cdot k + \omega = \text{const.} \]
\[ \hat{\omega} = \sqrt{gH} |k| \]

\[ |k| \leq \frac{\omega}{\sqrt{gH} - |U|} \]

wavenumbers are bounded unless

\[ U^2 > gH \] (geophysically less relevant regime)

Same answer for rotating shallow water, but not for 3d flow!
Boussinesq system

(no Coriolis force in talk, but in paper)

Variables
\[ x = (x, y, z) \quad u = (u, v, w) \]

Incompressible
\[ \nabla \cdot u = 0 \]

Momentum conservation
\[ \frac{Du}{Dt} + \nabla P = b \hat{z} \quad \text{buoyancy acceleration} \]

Stratification
\[ \frac{D}{Dt} (b + N^2 z) = \frac{Db}{Dt} + N^2 w = 0 \]

constant value defines 3d stratification surfaces
Plane Boussinesq gravity waves

\[ \nabla \cdot u = 0 \quad \text{implies} \quad k \cdot u' = 0 \]

Dispersion relation

\[ \hat{\omega}^2 = N^2 \frac{k^2 + l^2}{k^2 + l^2 + m^2} \]

Wavenumber vector

\[ k = (k, l, m) \]

Frequency is independent of

\[ \kappa = \sqrt{k^2 + l^2 + m^2} \]

Group velocity magnitude

\[ |\hat{u}_g|^2 = \frac{N^2 - \hat{\omega}^2}{\kappa^2} \]

1) Unbounded wavenumber growth is possible at fixed frequency
2) Group velocity inversely proportional to wavenumber at fixed frequency

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Three-dimensional ray tracing

Horizontal background flow

\[ \mathbf{U} = (U, V, 0) \quad \text{and} \quad U_x + V_y = 0 \]

Gradient of background flow treated as steady for simplicity

\[ \nabla \mathbf{U} \equiv \begin{pmatrix} U_x & V_x & W_x \\ U_y & V_y & W_y \\ U_z & V_z & W_z \end{pmatrix} = \begin{pmatrix} U_x & V_x & 0 \\ U_y & -U_x & 0 \\ U_z & V_z & 0 \end{pmatrix} \]

\[ D = U_x^2 + \left( \frac{V_x + U_y}{2} \right)^2 - \left( \frac{V_x - U_y}{2} \right)^2 \]

positive for generic case with open 2d stream lines in group-velocity frame

Wavenumber evolution decouples into horizontal and vertical components

\[ \frac{d_g}{dt} \begin{pmatrix} k \\ l \end{pmatrix} = - \begin{pmatrix} U_x & V_x \\ U_y & -U_x \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \]

and

\[ \frac{d_g}{dt} m = -U_z k - V_z l \]

easy 2d sub-problem of advection by area-preserving flow

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Streamlines in group-velocity frame

Hyperbolic  $D>0$  Parabolic  $D=0$  Elliptic  $D<0$

The horizontal wavenumber vector aligns itself with the growing eigenvector, which is perpendicular to the extension axis.

Final orientation is independent of initial orientation: wavepacket loses memory.

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Growing mode in three dimensions

\[ \mathbf{k}_H(t) \propto (-V_x, U_x + \sqrt{D}) \exp(\sqrt{D} t) \]

\[ m(t) = -\frac{U_z k(t) + V_z l(t)}{\sqrt{D}} + O(1) \]

Intrinsic group velocity decreases as wavenumber grows; the wavepacket becomes frozen into the flow (Jones 1969, Badulin & Shriira 1993)

The reinforces the analogy between wave phase and passive tracer behaviour; reasonably to expect exponential straining to persist once it gets started

Wave capture:

Exponentially fast wave breaking scenario without critical layers!

Oliver Bühler, Courant Institute, New York University, 2005
Numerical example

Snapshots taken from numerical simulation of meandering jet stream
Interpreted based on wave straining

Plougonven & Snyder
GRL, 2005

Oliver Bühler, Courant Institute, New York University, 2005
Theory example: wave capture by blocking dipole

Upstream wind

$U$

fixed wavepacket at $t_1$

$u_g(t_1) = 0$

drifting wavepacket at $t_2 > t_1$

$u_g(t_2) > 0$

clockwise vortex

Stagnation point

counter-clockwise vortex

Strained wavepacket drifts towards stagnation point where it must break (similar to horizontal critical layer)

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Pseudomomentum surge and mean flow

\[ p = kA \]

grows exponentially during wave capture (action is conserved, but wavenumber grows)

Standard dissipative wave-mean interaction paradigm:
mean-flow momentum + wave pseudomomentum = constant.

Does the exponential pseudomomentum surge lead to a dramatic local mean-flow response?

No, because the standard paradigm does not hold for horizontal wavepacket refraction...

Need to investigate \( O(a^2) \) wave-mean interaction theory with refraction.

Oliver Bühler, Courant Institute, New York University, 2005
Wave-mean interaction theory

Slowly varying Lagrangian mean flow with strong stratification is layerwise 2d, layerwise non-divergent, and at $O(a^2)$ is governed by (Bretherton 1969)

$$ \nabla \cdot \bar{u}^L = 0 \quad \text{and} \quad \bar{w}^L = 0. $$

Can show that (AM 1978, BM 98, B00, BM05)

$$ \left( \frac{\partial}{\partial t} + \bar{u}^L \cdot \nabla \right) \{ \hat{z} \cdot \nabla \times [\bar{u}^L - \mathbf{p}] \} = 0 $$

(Lagrangian and Eulerian mean flows are equal to leading order for Boussinesq wavepackets, but this version holds more generally)

Hence only the vertical curl of pseudomomentum affects the mean flow

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Neglect intrinsic group velocity

\[ p = kA = \nabla \theta A \quad \text{and therefore} \]

\[ \hat{z} \cdot \nabla \times p = lA_x - kA_y \propto \]

\[ = dA \, d\theta = \text{const.}. \]

as both \( A \) and the wave phase are advected by area-preserving flow

Exponential surge in pseudomomentum but not in its vertical curl
Bretherton's flow (1969)

Large-scale dipolar return flow at second order in wave amplitude

Far-field mean velocity is non-divergent and decays with square of distance to wavepacket

The impulse (ie the skew linear moment of vorticity) of this layerwise 2d flow is well defined, but not its momentum

Feynman: “children on a slide”
Wave-vortex duality

Wavepacket  
Dual vortex dipole

wave pseudomomentum = dual vortex impulse

.....suggests new thinking of interactions....

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Figure 1. Left: wavepacket exposed to pure horizontal strain contracting along the $x$-axis and extending along the $y$-axis. The wavecrests align with the extension axis and their spacing is decreased, so that the wavenumber vector $k$ points at right angles to the extension axis and grows in magnitude, as suggested by the large arrow. Right: a pair of oppositely signed vortices exposed to same strain. The arrow now indicates the vortex pair’s Kelvin impulse.

Wavepacket and vortex dipole are strained in the same way.
Pseudomomentum + impulse = conserved

Potential vorticity
\[ \bar{q}^L = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^L = \mathbf{z} \cdot \nabla \times (\mathbf{u}^L - \mathbf{p}_H) \]

Impulse
\[ I(t) = \iiint i(x, t) \, dx \, dy \, dz \]

Pseudomomentum
\[ \mathbf{p}_H \equiv \iiint \mathbf{p}_H \, dx \, dy \, dz \]

Refraction terms
\[ \frac{d\mathbf{p}_H}{dt} = - \iiint (\nabla^H \mathbf{u}^L) \cdot \mathbf{p}_H \, dx \, dy \, dz \]
\[ \frac{dI}{dt} = \iiint (\nabla^H \mathbf{u}^L) \cdot \mathbf{p}_H \, dx \, dy \, dz \]

\[ \mathbf{p}_H + I = \text{constant} \]

GLM theory used here

skew linear moment of PV

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The resolution

Dipole straining increases wavepacket pseudomomentum. Bretherton flow advects vortex dipole and reduces impulse. Both compensate and the sum of $P + I$ is conserved!

Looks precisely like dipole-dipole leap-frog interaction... no accident!

Non-local interaction at a distance
Duality and dissipation

Wavepacket  Vortex dipole

Dissipation makes dual vortex real, but

Dissipation itself does not accelerate the mean flow!

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Towards a new parametrization scheme

Joint work with John Scinocca (CCCma, Univ. Victoria, CA)

1. based on existing columnar parametrization scheme

2. wave dissipation and breaking part unchanged, only non-dissipative propagation part is changed

3. requires no new assumptions on gravity wave launch spectrum (have only weak observational constraints)

4. requires horizontal derivatives of model fields, which is a bit non-trivial in operational GCMs because of parallel architecture
The key difference

Old scheme is based on constant pseudomomentum flux

New scheme is based on constant wave action flux

pseudomomentum flux(z) = k(z) wave action flux

changes in k(z) due to horizontal refraction change the pseudomomentum flux(z) and hence produce new mean-flow forces
Wave action flux replaces pseudomomentum flux.

Refraction leads to wave-mean momentum exchanges without dissipation.

Mountain drag does not simply equal wave drag anymore (it never did).