A Systematic Multi-Scale Framework for Meteorological Modelling II: *Moist Flows*

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Thanks to ...

Motivation

Moist Aero-Thermodynamics

Meso-convective interactions revisited

Narrow columns

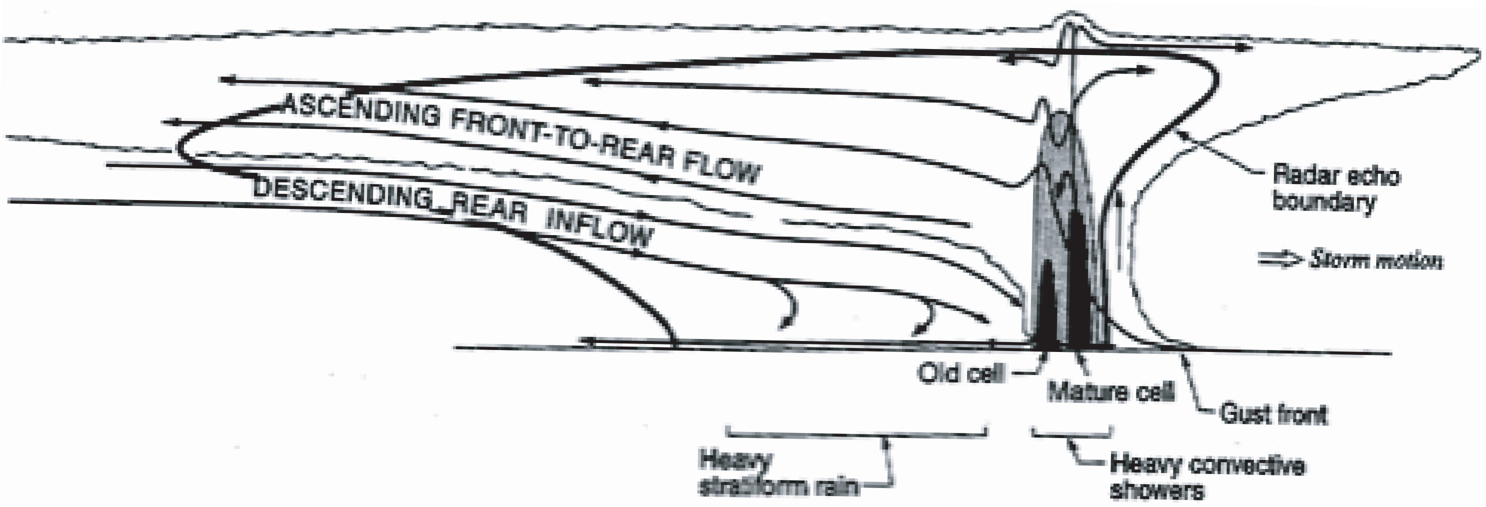
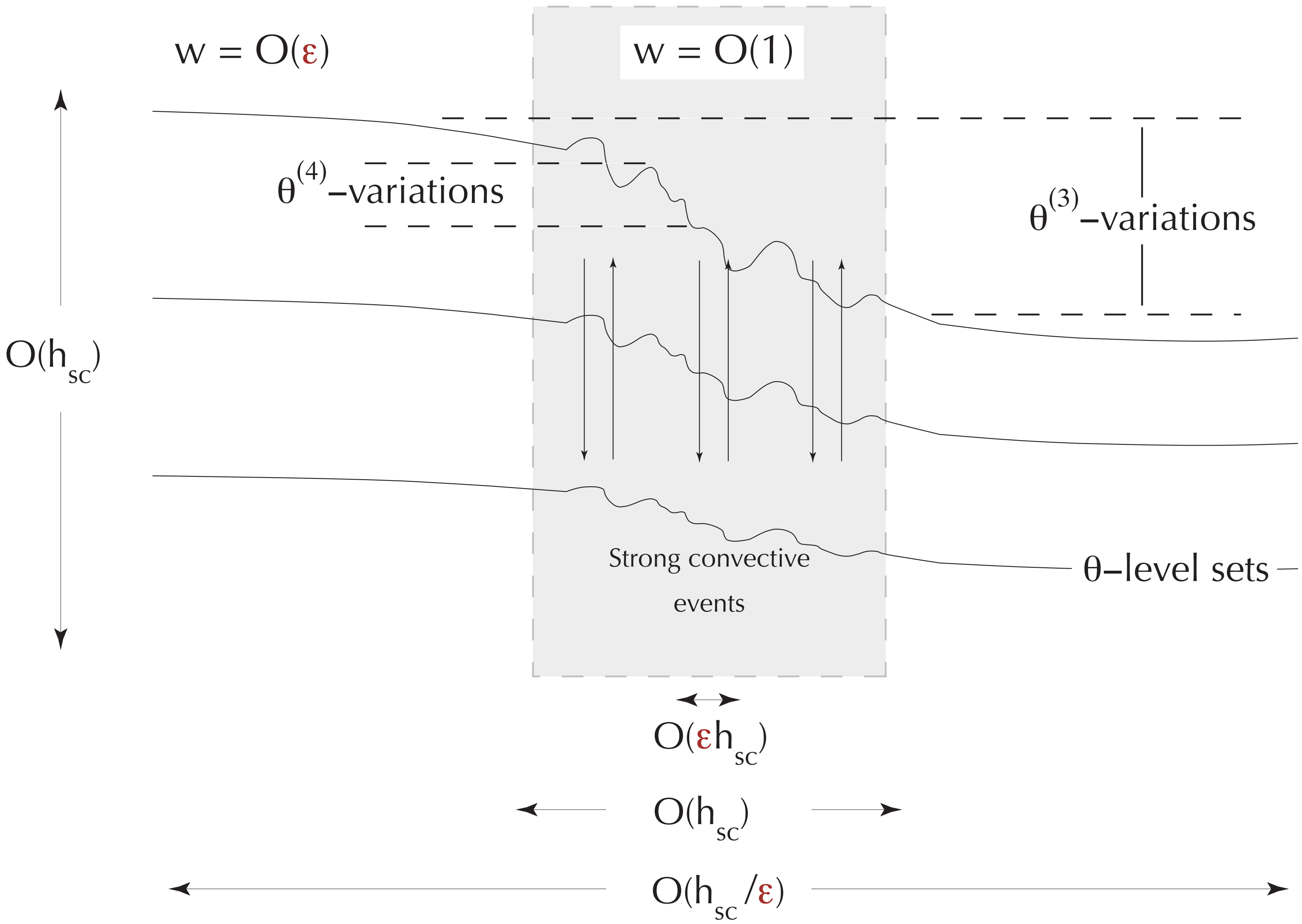


FIG. 1. Conceptual model of a multicell squall line with trailing stratiform precipitation. The storm is viewed in a cross section perpendicular to the convective line (adapted from Houze et al. 1989).

(from: Pandya & Durran (1996))


w = O(ε)
w = O(1)
θ(4)–variations
θ(3)–variations
O(hsc)
Strong convective
events
θ–level sets
O(εhsc)
O(hsc)
O(hsc/ε)

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Mass, momentum, energy balances

$$
\begin{aligned}
\rho_t &+ \nabla_\parallel \cdot (\rho \boldsymbol{u}) + (\rho w)_z &&= 0 \\
\boldsymbol{u}_t &+ \boldsymbol{u} \cdot \nabla_\parallel \boldsymbol{u} + w \boldsymbol{u}_z + \frac{1}{\mathbf{Ro_B}} (\boldsymbol{\Omega} \times \boldsymbol{v})_\parallel + \frac{1}{\mathbf{M}^2} \frac{1}{\rho} \nabla_\parallel p &&= \boldsymbol{D_u} \\
w_t &+ \boldsymbol{u} \cdot \nabla_\parallel w + w w_z + \frac{1}{\mathbf{Ro_B}} (\boldsymbol{\Omega} \times \boldsymbol{v})_\perp + \frac{1}{\mathbf{M}^2} \frac{1}{\rho} p_z &&= D_w - \frac{1}{\overline{\mathbf{Fr}}^2} \\
\theta_t &+ \boldsymbol{u} \cdot \nabla_\parallel \theta + w \theta_z &&= D_\theta + \boldsymbol{S_\theta}
\end{aligned}
$$

Moisture balances

$$
\begin{aligned}
q_{\mathrm{v},t} &+ \boldsymbol{u} \cdot \nabla_\parallel q_\mathrm{v} + w q_{\mathrm{v},z} &&= (\boldsymbol{C_\mathrm{ev}} - \boldsymbol{C_\mathrm{d}}) &&+ D_{q_\mathrm{v}} \\
q_{\mathrm{c},t} &+ \boldsymbol{u} \cdot \nabla_\parallel q_\mathrm{c} + w q_{\mathrm{c},z} &&= (\boldsymbol{C_\mathrm{d}} - \boldsymbol{C_\mathrm{ac}} - \boldsymbol{C_\mathrm{cr}}) &&+ D_{q_\mathrm{c}} \\
q_{\mathrm{r},t} &+ \boldsymbol{u} \cdot \nabla_\parallel q_\mathrm{r} + w q_{\mathrm{r},z} + \tfrac{1}{\rho}(\rho q_\mathrm{r} V_\mathrm{T})_z &&= (\boldsymbol{C_\mathrm{ac}} + \boldsymbol{C_\mathrm{cr}} - \boldsymbol{C_\mathrm{ev}}) &&+ D_{q_\mathrm{r}} \,.
\end{aligned}
$$

$$
\rho\theta = p^{\frac{1}{\gamma}} \qquad\qquad \boldsymbol{S_\theta} = \frac{\boldsymbol{\gamma - 1}}{\boldsymbol{\gamma}} \frac{\theta}{p} \boldsymbol{L^*} \, (\boldsymbol{C_\mathrm{d}} - \boldsymbol{C_\mathrm{ev}}) \,.
$$

$$q_{\mathrm{vs}} = \frac{\boldsymbol{q}^*_{\mathbf{vs}}}{p} \exp\left(\boldsymbol{A}^* \frac{T - \boldsymbol{T}^*_{\mathbf{0}}}{\boldsymbol{T}^*_{\mathbf{1}} + (T - \boldsymbol{T}^*_{\mathbf{0}})}\right)$$

$$T(\theta, p) = \theta\, p^{\frac{\gamma-1}{\gamma}}\,.$$

$$\boldsymbol{C}_{\mathbf{d}} = \boldsymbol{C}^*_{\mathbf{d}}\,(q_{\mathrm{v}} - q_{\mathrm{vs}})\, H(q_{\mathrm{c}})(q_{\mathrm{c}} + \boldsymbol{q}_{\mathbf{c}}{}^*_{\boldsymbol{n}})$$

$$\boldsymbol{C}_{\mathbf{ev}} = \boldsymbol{C}^*_{\mathbf{ev}}\,\frac{p}{\rho}\,(q_{\mathrm{vs}} - q_{\mathrm{v}})\; H(q_{\mathrm{r}})q_{\mathrm{r}}^{1/2}$$

$$\boldsymbol{C}_{\mathbf{cr}} = \boldsymbol{C}^*_{\mathbf{cr}}\, q_{\mathrm{c}} q_{\mathrm{r}}$$

$$\boldsymbol{C}_{\mathbf{ac}} = \boldsymbol{C}^*_{\mathbf{ac}}\, \max\left(0, q_{\mathrm{c}} - \boldsymbol{q}^*_{\mathbf{c}}\right)$$

Distinguished Limits

$$\boldsymbol{q}^*_{\mathbf{vs}} = 0.021 = \boldsymbol{\varepsilon}^2\, q^{**}_{\mathrm{vs}}\,, \qquad \boldsymbol{A}^*, \boldsymbol{L}^* = (16, 30) = \frac{1}{\boldsymbol{\varepsilon}}\,(A^{**}, L^{**})\,, \qquad \boldsymbol{T}^*_{\mathbf{0}} = 1\,, \qquad \ldots$$

Multilayer Difficulty

$$q_{\mathrm{vs}} = \varepsilon^2 \frac{q_{\mathrm{vs}}^{**}}{p} \exp\left(\frac{A^{**}}{\varepsilon} \frac{T-1}{1+(T-1)-\varepsilon\, T_1^{**(1)}}\right), \qquad T = \theta\, p^{\frac{\gamma-1}{\gamma}}$$

Newtonian Limit*

$$\frac{\gamma-1}{\gamma} = \varepsilon\,\Gamma \qquad \Rightarrow \qquad T = \theta\,\left(1+\varepsilon\,\Gamma \ln p + O(\varepsilon^2)\right)$$

Weak stratification**

$$\theta = 1 + \varepsilon^2\,\Theta_2(z) + O(\varepsilon^3)$$

$$\Downarrow$$

$$\underline{q_{\mathrm{vs}} = \varepsilon^2\, q_{\mathrm{vs}}^{**}\, \exp\left(-\left[A^{**}\Gamma - 1\right]\, z\right)\, \left(1 + O(\varepsilon)\right)}\,.$$

* I. Newton (long ago), Crighton (2000), Lipps, Hemler (1982), Bannon (1995)

** Majda, Klein (2003)

$$(q_\mathrm{v}, q_\mathrm{c}, q_\mathrm{r}) = \varepsilon^2 \, (\hat{q}_\mathrm{v}, \hat{q}_\mathrm{c}, \hat{q}_\mathrm{r})$$

$$\begin{aligned}
\hat{q}_{\mathrm{v},t} + \boldsymbol{u} \cdot \nabla_\parallel \hat{q}_\mathrm{v} + w\, \hat{q}_{\mathrm{v},z} &= -\frac{1}{\varepsilon^n} \hat{C}_\mathrm{d} + \hat{C}_\mathrm{ev} + D_{\hat{q}_\mathrm{v}} \\
\hat{q}_{\mathrm{c},t} + \boldsymbol{u} \cdot \nabla_\parallel \hat{q}_\mathrm{c} + w\, \hat{q}_{\mathrm{c},z} &= +\frac{1}{\varepsilon^n} \hat{C}_\mathrm{d} - \frac{1}{\varepsilon} \hat{C}_\mathrm{cr} - \hat{C}_\mathrm{ac} + D_{\hat{q}_\mathrm{c}} \\
\hat{q}_{\mathrm{r},t} + \boldsymbol{u} \cdot \nabla_\parallel \hat{q}_\mathrm{r} + w\, \hat{q}_{\mathrm{r},z} + \tfrac{1}{\rho}(\hat{V}_\mathrm{T} \rho \hat{q}_\mathrm{r})_z &= \frac{1}{\varepsilon} \hat{C}_\mathrm{cr} - \hat{C}_\mathrm{ev} + \hat{C}_\mathrm{ac} + D_{\hat{q}_\mathrm{r}}
\end{aligned}$$

$$\begin{aligned}
\hat{C}_\mathrm{d} &= \underline{C_\mathrm{d}^{**} \, (\hat{q}_\mathrm{v} - \hat{q}_\mathrm{vs}) \, H(\hat{q}_\mathrm{c})(\hat{q}_\mathrm{c} + \varepsilon q_{\mathrm{c}n}^{**})} \\
\hat{C}_\mathrm{ev} &= C_\mathrm{ev}^{**} \, (\hat{q}_\mathrm{vs} - \hat{q}_\mathrm{v}) \, H(\hat{q}_\mathrm{r}) \hat{q}_\mathrm{r}^{\frac{1}{2}} \\
\hat{C}_\mathrm{cr} &= C_\mathrm{cr}^{**} \, \hat{q}_\mathrm{c} \hat{q}_\mathrm{r} \\
\hat{C}_\mathrm{ac} &= C_\mathrm{ac}^{**} \, \max\left(0, \hat{q}_\mathrm{c} - \varepsilon \, q_\mathrm{c}^{**}\right) \, .
\end{aligned}$$

$$\underline{\hat{S}_\theta = \varepsilon^2 \, \Gamma L^{**} \, \frac{\theta}{p} \left[\frac{1}{\varepsilon^n} \hat{C}_\mathrm{d} - \hat{C}_\mathrm{ev} \right]}$$

Motivation

Moist Aero-Thermodynamics

Meso-convective interactions revisited

Narrow columns

$$\mathbf{U}(\boldsymbol{x}, z, t; \boldsymbol{\varepsilon}) = \sum_i \boldsymbol{\varepsilon}^i \, \mathbf{U}^{(i)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t),$$

$$\boldsymbol{\xi} = \boldsymbol{\varepsilon} \, \boldsymbol{x}$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{10\,\mathrm{km}}$$

Mesoscale dynamics

$$
\begin{aligned}
\overline{\boldsymbol{u}}_t + \nabla_{\boldsymbol{\xi}}\, \pi' &= -\partial_z \left(\frac{\overline{S_\theta'' \boldsymbol{u}}}{d\Theta_2/dz} \right) , \\
\theta_t' + \overline{w'} \frac{d\Theta_2}{dz} &= \overline{S_\theta'} \, , \\
\partial_z \pi' &= \theta' \, , \\
\rho_0 \nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{u}} + \partial_z \left(\rho_0 \overline{w'} \right) &= 0 \, .
\end{aligned}
$$

Convective scale dynamics

Anelastic* for near-moist adiabatic stratification

WTG otherwise $\Rightarrow$ S_θ''

* essentially

Sublinear growth conditions

Mass:

$$\nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(0)} + (\rho w)_z^{(0)} \;=\; 0$$

$$\nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u})^{(1)} + (\rho w)_z^{(1)} \;=\; -\nabla_{\boldsymbol{\xi}} \cdot (\rho \boldsymbol{u})^{(0)}$$

Horizontal momentum:

$$\nabla_{\boldsymbol{x}} p^{(0)} \;=\; 0$$

$$\nabla_{\boldsymbol{x}} p^{(j+1)} + \nabla_{\boldsymbol{\xi}} p^{(j)} \;=\; 0 \qquad (j \in \{0, 1, 2\})$$

$$(\rho \boldsymbol{u})_t^{(0)} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \circ \boldsymbol{u})^{(0)} + (\rho w \boldsymbol{u})_z^{(0)} + \nabla_{\boldsymbol{x}} p^{(4)} + \nabla_{\boldsymbol{\xi}} p^{(3)} \;=\; \boldsymbol{D}_{\rho \boldsymbol{u}}^{(0)} \,.$$

Vertical momentum:

$$\partial_z p^{(j)} \;=\; -\rho^{(j)} \qquad (j \in \{0, 1, 2, 3\})$$

Sublinear growth conditions

Potential temperature:

$$\theta = 1 + \varepsilon^2 \Theta_2(z) + \varepsilon^3 \underline{\Theta^{(3)}(\boldsymbol{\xi}, z, t)} + \varepsilon^3 \theta^{(4)}(\boldsymbol{x}, \boldsymbol{\xi}, z, t) \ldots .$$

$$\begin{aligned} S_\theta^{(0)} &= 0\,, \\ S_\theta^{(1)} &= 0\,, \\ w^{(0)} \frac{d\Theta_2}{dz} &= S_\theta^{(2)}\,, \\ \theta_t^{(3)} + w^{(0)}\theta_z^{(3)} + w^{(1)} \frac{d\Theta_2}{dz} &= S_\theta^{(3)}\,, \end{aligned}$$

Mesoscale wave dynamics

$$
\begin{aligned}
\overline{\boldsymbol{u}}_t + \nabla_{\boldsymbol{\xi}}\, \pi^{(3)} \;&=\; -\partial_z \left(\overline{w\boldsymbol{u}}\right)\;, \\
\Theta_t^{(3)} + \overline{w^{(1)}}\frac{d\Theta_2}{dz} \;&=\; \overline{S_\theta^{(3)}}\,, \\
\partial_z \pi^{(3)} \;&=\; \Theta^{(3)}\,, \\
\rho_0 \nabla_{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{u}} + \partial_z \left(\rho_0 \overline{w^{(1)}}\right) \;&=\; 0\,.
\end{aligned}
$$

where

$$
\pi^{(3)} = \frac{p^{(3)}}{\rho_0(z)}
$$

Moisture transport

Water vapor:

$$\boldsymbol{C}_{\mathbf{d}}{}^{(i)} = 0 \qquad (i = -(n+2), ..., 1)$$

$$q_{\mathrm{v}\ t}^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(2)} + w^{(0)} q_{\mathrm{v}\ z}^{(2)} = -\boldsymbol{C}_{\mathbf{d}}{}^{(2)} + \boldsymbol{C}_{\mathbf{ev}}{}^{(2)} .$$

$$q_{\mathrm{v}\ t}^{(3)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(3)} + \boldsymbol{u}^{(1)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{v}}^{(2)} + w^{(0)} q_{\mathrm{v}\ z}^{(3)} + w^{(1)} q_{\mathrm{v}\ z}^{(2)} = -\boldsymbol{C}_{\mathbf{d}}{}^{(3)} + \boldsymbol{C}_{\mathbf{ev}}{}^{(3)} .$$

Cloud water:

$$C_{\mathrm{cr}}^{(1)} = C_{\mathrm{cr}}^{**}\, q_{\mathrm{c}}^{(2)} q_{\mathrm{r}}^{(2)} = 0 \,,$$

$$q_{\mathrm{c}\ t}^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{c}}^{(2)} + w^{(0)} q_{\mathrm{c}\ z}^{(2)} = \boldsymbol{C}_{\mathbf{d}}{}^{(2)} - C_{\mathrm{cr}}^{(2)} .$$

Rain water:

$$q_{\mathrm{r}\ t}^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{x}} q_{\mathrm{r}}^{(2)} + w^{(0)} q_{\mathrm{r}\ z}^{(2)} + \frac{1}{\rho_0} \Big(\rho_0 V_T q_{\mathrm{r}}^{(2)} \Big)_z = C_{\mathrm{cr}}^{(2)} - \boldsymbol{C}_{\mathbf{ev}}{}^{(2)} .$$

Nonlinear control of convective motions

Recall that

$$w^{(0)} \frac{d\Theta_2}{dz} = S_\theta^{(2)}$$

$$\theta_t^{(3)} + w^{(0)}\theta_z^{(3)} + w^{(1)} \frac{d\Theta_2}{dz} = S_\theta^{(3)}$$

Source terms

$$S_\theta^{(2)} = \frac{\Gamma L^{**}}{p_0(z)} \left(-H_{\geq}(q_{\rm v} - q_{\rm vs}) w^{(0)} \frac{dq_{\rm vs}^{(2)}}{dz} + H_{>}(q_{\rm vs} - q_{\rm v}) \, \boldsymbol{C_{\rm ev}}^{(2)} \right)$$

$$S_\theta^{(3)} = \frac{\Gamma L^{**}}{p_0(z)} \left(-H_{\geq}(q_{\rm v} - q_{\rm vs}) \left(w^{(0)} \frac{dq_{\rm vs}^{(3)}}{dz} + w^{(1)} \frac{dq_{\rm vs}^{(2)}}{dz} \right) + H_{>}(q_{\rm vs} - q_{\rm v}) \, \boldsymbol{C_{\rm ev}}^{(3)} \right) + \hat{S}_\theta^{(3)}$$

Nonlinear control of convective motions / undersaturated

Strongly undersaturated Air $(q_{\rm vs}^{(2)} - q_{\rm v}^{(2)} > 0)$

$$w^{(0)} = -\frac{\Gamma L^{**} \boldsymbol{C}_{\mathbf{ev}}^{**}}{(p_0 d\Theta_2/dz)(z)} \left(q_{\rm vs}^{(2)}(z) - q_{\rm v}^{(2)} \right) q_{\rm r}^{(2)\frac{1}{2}} \qquad \text{for} \qquad q_{\rm v}^{(2)} < q_{\rm vs}^{(2)}(z)\,.$$

Weakly undersaturated Air $(q_{\rm vs}^{(2)}(z) - q_{\rm v}^{(2)} \equiv 0$ **but** $q_{\rm vs}^{(3)}(z) - q_{\rm v}^{(3)} > 0)$

$$w^{(0)} = 0$$

$$w^{(1)} = \left(-\frac{\Gamma L^{**} \boldsymbol{C}_{\mathbf{ev}}^{**}}{p_0} \left(q_{\rm vs}^{(3)}(z) - q_{\rm v}^{(3)} \right) q_{\rm r}^{(2)\frac{1}{2}} + \left(\hat{S}_\theta^{(3)} - \theta_t^{(3)} \right) \right) \left(\frac{d\Theta_2}{dz} \right)^{-1} .$$

Nonlinear control of convective motions / saturated

$$w^{(0)} \frac{d\Theta_2}{dz} = -w^{(0)} \frac{\Gamma L^{**}}{p_0(z)} \frac{dq_{\rm vs}^{(2)}}{dz}$$

$$\underline{\frac{d\Theta_2}{dz} = -\frac{\Gamma L^{**}}{p_0} \frac{dq_{\rm vs}^{(2)}}{dz}} \qquad \text{or} \qquad w^{(0)} \equiv 0$$

Precipitating clouds: rapid cloudwater collection, ($C_{\rm cr}^{(1)} = C_{\rm cr}^{**} \underline{q_{\rm c}^{(2)}} \, q_{\rm r}^{(2)} = 0$ **)**

$$C_{\rm d}^{(2)} - C_{\rm cr}^{(2)} = -w^{(0)} \frac{dq_{\rm vs}^{(2)}}{dz} - C_{\rm cr}^{**} \, q_{\rm c}^{(3)} {q_{\rm r}^{(2)}}^{1/2} = 0 \qquad \text{or} \qquad w^{(0)} = -\frac{C_{\rm cr}^{**} \, q_{\rm c}^{(3)} {q_{\rm r}^{(2)}}^{1/2}}{dq_{\rm vs}^{(2)}/dz} \, .$$

$$\underline{w^{(0)} \geq 0}$$

So, how fast do we go upward, then??

Source terms in third order θ-equation

$$S_\theta^{(3)} = \frac{\Gamma L^{**}}{p_0} \boldsymbol{C}_{\mathbf{d}}^{(3)} + \hat{S}_\theta^{(3)}$$

$$\boldsymbol{C}_{\mathbf{d}}^{(3)} = -w^{(1)} \frac{dq_{\text{vs}}^{(2)}}{dz} - w^{(0)} \frac{dq_{\text{vs}}^{(3)}}{dz}$$

Third order θ-equation

$$\theta_t^{(3)} + w^{(0)} \theta_z^{(3)} + \underline{w^{(1)} \frac{d\Theta_2}{dz}} = -\frac{\Gamma L^{**}}{p_0} \left(\underline{w^{(1)} \frac{dq_{\text{vs}}^{(2)}}{dz}} + w^{(0)} \frac{dq_{\text{vs}}^{(3)}}{dz} \right) + \hat{S}_\theta^{(3)}$$

$$\Theta_t^{(3)} + w^{(0)} \Theta_z^{(3)} = -\frac{\Gamma L^{**}}{p_0} w^{(0)} \frac{dq_{\text{vs}}^{(3)}}{dz} + \hat{S}_\theta^{(3)}$$

Coupling with mesoscale waves

$$\underline{w_*^{(0)} = \left(\hat{S}_\theta^{(3)} - \boldsymbol{\Theta}^{(3)}{}_t \right) \left(\boldsymbol{\Theta}^{(3)}{}_z + \frac{\Gamma L^{**}}{p_0} \frac{dq_{\text{vs}}^{(3)}}{dz} \right)^{-1}} \qquad w^{(0)} = H\left(w_*^{(0)} \right)$$

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$$\mathbf{U}(\boldsymbol{x}, z, t; \boldsymbol{\varepsilon}) = \sum_i \boldsymbol{\varepsilon}^i \mathbf{U}^{(i)} (\boldsymbol{\eta}, \boldsymbol{x}, z, \tau)$$

$$\boldsymbol{\eta} = \boldsymbol{x}/\boldsymbol{\varepsilon}$$

$$\tau = t/\boldsymbol{\varepsilon}$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{10\,\mathrm{km}}\,, \qquad t = \frac{t'}{20\,\mathrm{min}}$$

Mesoscale Dynamics

$$
\begin{aligned}
\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi' &= 0 \\
\overline{w}_\tau + \pi'_z &= \overline{\theta'} \\
\overline{\theta'}_\tau + \overline{w} \frac{d\Theta_2}{dz} &= \frac{\Gamma L^{**}}{p_0} \overline{C''} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

Bulk Microscale Column Dynamics

$$
\begin{aligned}
\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) \widetilde{w} &= \widetilde{\theta'} \\
\widetilde{\theta'}_\tau + \widetilde{w} \frac{d\Theta_2}{dz} &= \frac{\Gamma L^{**}}{p_0} \widetilde{C''} \,.
\end{aligned}
$$

Coupling to Moisture Physics

$$
C'' = H(q'_{\mathrm{c}})\, C''_{\mathrm{d}} + \left[1 - H(q'_{\mathrm{c}})\right] C''_{\mathrm{ev}}
$$

Saturated Air

$$C''_{\mathrm{d}} = C^{**}_{\mathrm{d}}\, \underline{\delta q_{\mathrm{v}}^{(n^*)}}\, q'_{\mathrm{c}} \;=\; \underline{-\left[\left(\widetilde{w^{(0)}} + \overline{w^{(0)}}\right) \frac{dq''_{\mathrm{vs}}}{dz} - D''_{q_{\mathrm{v}}}\right]}$$

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q'_{\mathrm{c}} \;=\; H(q'_{\mathrm{c}})\, C''_{\mathrm{d}} - C^{**}_{\mathrm{cr}} q''_{\mathrm{r}}\, q'_{\mathrm{c}}$$

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q''_{\mathrm{r}} \;=\; 0$$

Undersaturated Air

$$C''_{\mathrm{ev}} \;=\; -C^{**}_{\mathrm{ev}}\, \left(q''_{\mathrm{vs}}(z) - q''_{\mathrm{v}}\right)\, {q''_{\mathrm{r}}}^{1/2}$$

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q''_{\mathrm{v}} \;=\; 0$$

$$\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) q''_{\mathrm{r}} \;=\; 0$$

Conclusions