Conservative Finite Volume Schemes for Meteorological Applications

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Motivation

Balancing gravity

Momentum conservation vs. Coriolis

A low-Mach / low-Froude conservative scheme
  low Mach number Godunov-Type scheme
  robust, exact projection
  low-Froude extension
\[ u_t + u \cdot \nabla u + w u_z + \nabla \pi = S_u \]
\[ w_t + u \cdot \nabla w + w w_z + \pi_z = -\theta' + S_w \]
\[ \theta' + u \cdot \nabla \theta' + w \theta'_z = S_{\theta} \]
\[ \nabla \cdot (\rho_0 u) + (\rho_0 w)_z = 0 \]
\[ \theta = 1 + \varepsilon^4 \theta'(x, z, t) + o(\varepsilon^4) \]

Anelastic Boussinesque Model

\[ \frac{\partial T}{\partial t} + \mathbf{u}(0) \cdot \nabla q = 0 \]

\[ q = \zeta^{(0)} + \Omega_0 \beta \eta + \frac{\Omega_0}{\rho(0)} \frac{\partial}{\partial z} \left( \frac{\rho(0)}{\partial \mathbf{x}} \frac{\partial \theta^{(3)}}{\partial \mathbf{x}} \right) \]
\[ \zeta^{(0)} = \nabla^2 \pi^{(3)}, \quad \theta^{(3)} = -\frac{\partial \pi^{(3)}}{\partial z}, \quad u^{(0)} = \frac{1}{\Omega} \mathbf{k} \times \nabla \pi^{(3)} \]

Quasi-geostrophic theory

\[ \frac{\partial Q_T}{\partial t} + \nabla \cdot \mathbf{F}_T = S_T \]
\[ \frac{\partial Q_q}{\partial t} + \nabla \cdot \mathbf{F}_q = S_q \]
\[ Q_q = \int_{\rho}^\rho \rho \exp \left( \frac{\rho - \rho_0}{\rho_0} \right) \theta \, dx \]
\[ \mathbf{F}_T = \int_{\rho}^\rho \mathbf{F} \mathbf{u} \, \exp \left( \frac{\rho - \rho_0}{\rho_0} \right) \theta \, dx \]
\[ \mathbf{F}_q = \int_{\rho}^\rho \mathbf{F} \mathbf{u} \, \exp \left( \frac{\rho - \rho_0}{\rho_0} \right) \theta \, dx \]
\[ T = T(0, x) + \Gamma(x, x) \left( \min(0, H_T - \kappa) \right), \quad \eta = \eta(0, x) \exp \left( \frac{1 - \kappa}{\kappa} \right) \]
\[ \rho = \rho_0 \exp \left( -\frac{\kappa}{\rho_0} \right), \quad p = p_0 \exp \left( -\frac{\kappa}{\rho_0} \right) + \rho(0, x) \exp \left( \frac{1 - \kappa}{\kappa} \right) \]
\[ \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_0, \quad f \mathbf{k} \times \mathbf{u}_0 = -\nabla p, \quad \mathbf{u}_0 = \mathbf{u} \nabla p \]

V. Petrovskov et al., CLIMBER-2 ..., Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

\[ 10000 \text{ km} / 1 \text{ season} \]

Scales in the Atmosphere
Three-dimensional compressible flow equations

\[ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + \frac{1}{M^2} \nabla p + \frac{1}{R \Omega} \mathbf{\Omega} \times \rho \mathbf{v} = S_{\rho \mathbf{v}} - \frac{1}{F \mathbf{r}^2} \rho g \mathbf{k} \]

\[ (\rho e)_t + \nabla \cdot (\mathbf{v} [\rho e + p]) = S_{\rho e} \]

\[ (\rho Y_j)_t + \nabla \cdot (\rho Y_j \mathbf{v}) = S_{\rho Y_j} \]

\[ (\rho e) = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \mathbf{v}^2 + \rho \sum_{j=1}^{N} Q_j Y_j \]

Many asymptotic regimes — a single numerical scheme?

The “Truth”
Why conservation?

The integral conservation law

\[
\left. \int_{t_1}^{t_2} \Phi \, dx \right|_{\Omega} = - \int_{t_1}^{t_2} \oint_{\partial \Omega} f_\Phi \cdot n \, d\sigma \, dt
\]

is uniformly valid for arbitrary control volumes

Mathematically sound subgrid closures
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Spurious winds over steep orography

- atmosphere at rest
- 3000 m mountain
- 3D compressible inviscid flow eqs.
- standard finite volume scheme
- $128 \times 32$ grid cells
- velocities after about 60 min

Various finite difference / finite volume schemes produce comparable results.
The background pressure / temperature structure of the atmosphere is (almost) everywhere dominated by hydrostatic balance.

Mathematically, this follows from the distinguished limits

\[
\frac{Fr^2}{M^2} = O(1), \quad Ro \gg M^2, \quad (M \to 0)
\]

Vertical momentum balance

\[
(\rho w)_t + \nabla \cdot (\rho \mathbf{vw}) + \frac{1}{Ro} \mathbf{k} \cdot (\Omega \times \mathbf{v}) = -\left( \frac{1}{M^2} \frac{\partial p}{\partial z} + \frac{1}{Fr^2 \rho} \right).
\]

Numerical truncation errors for 30 vertical layers, 2nd order scheme

\[
(\delta w)_{\text{num.}} = O \left( \frac{1}{M^2} \Delta x^2 \right) \sim \frac{(1/30)^2}{(0.03)^2} \sim O(1)
\]

**Origin of unbalanced truncation errors**
Archimedes’ principle

\[ -\rho_i \nabla \Phi - \frac{1}{|c_i|} \int_{\partial c_i} P_i \, n \, dS = O((\Delta z)^2) \]
Archimedes' principle

\[ \frac{\partial P_i^{(0)}}{\partial z} = -\rho_i^{(0)} \]

\[ \frac{1}{|c_i|} \oint_{\partial c_i} P_i \, n \, dS \]

\[ \frac{1}{|c_i|} \oint_{\partial c_i} (P_i - P_i^{(0)}) \cdot n \, dS \]
No accumulation of unbalanced truncation errors
Inversion layers at a mid-mountain level

No accumulation of unbalanced truncation errors
Convergence vs. marginal resolution (Schaer’s Test)

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The Coriolis source-term

\[ 2 \Omega \times \rho \mathbf{v} \]

is the result of a coordinate transformation.

Standard numerical discretizations do not maintain momentum conservation.

**Idea:** Apply coordinate transformation to conservative moving grid discretization in the inertial frame.

\[ 2 \Omega \times \rho \mathbf{v} \Rightarrow \text{rotation term} + \text{flux term} \]

(work in progress with E. Audusse)
Shallow water version (1st order)

\[ h_i^{n+1} = h_i^n - \frac{\Delta t}{|C_i|} \sum_j F^h_{ij} \]

\[ q_i^{n+1} = R(\Delta t)q_i^n - h_i^{n+1}\Omega \times x_i + h_i^n R(\Delta t)\Omega \times x_i \]

\[ - \frac{\Delta t}{|C_i|} \sum_j \frac{F^q_{ij}}{|C_i|} \]

\[ - \frac{\Delta t}{|C_i|} \sum_j (F^h_{ij}\Omega \times x_{ij}) \]
Water height for stationary vortex test:

Top: Cross-wind scheme, Bottom: Standard scheme
left to right: 20 cells horizontal / 20, 40, 80 cells in the vertical
Water height for stationary vortex test:

Top: Vertical cross section, Bottom: Horizontal cross section
left to right: 20 cells horizontal / 20, 40, 80 cells in the vertical
initial data, cross-wind scheme, standard scheme

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Three zero Mach number, anelastic regimes:

1. Inkompressible flow (no gravity)

2. Ogura-Phillips-Anelastic flow \( \theta = 1 + M^2 \theta' \)

3. Buoyancy-controlled flow \( \theta = \Theta_0(z) + M^2 \theta' \)
Regime 1: Incompressible flow

Leading order result:

\[ p = P_\infty + \varepsilon^4 p' \]

Consequences:

\[ \rho e = \frac{p}{\gamma - 1} + \varepsilon^4 \frac{\rho \mathbf{v}^2}{2} \rightarrow \frac{P_\infty}{\gamma - 1} \quad (\varepsilon \rightarrow 0) \]

\[
\begin{align*}
(\rho e)_t + \nabla \cdot ([\rho e + p] \mathbf{v}) &= 0 \quad \rightarrow \quad \nabla \cdot \mathbf{v}^{(0)} = 0 \\
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad \rightarrow \quad \rho^{(0)}_t + \mathbf{v}^{(0)} \cdot \nabla \rho^{(0)} = 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + \frac{1}{\varepsilon^4} \nabla p &= 0 \quad \rightarrow \quad (\rho \mathbf{v})^{(0)}_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v})^{(0)} + \nabla p' = 0
\end{align*}
\]
Task:

Construct a scheme for low Mach number flows, which ...

- conserves mass, momentum, and total energy,
- is at least second order accurate in space and time,
- requires the solution of at most linear, scalar equations (possibly a few of them per time step)
- reduces, for $M = 0$, to the variable density, incompressible solver in Schneider et al., JCP, (1999)
- reduces, for $M = 1$, to a second order Godunov-type scheme
Explicit / implicit split for the Euler equations

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + \nabla p + \left( \frac{1}{M^2} - 1 \right) \nabla p_\ast &= 0 \\
(\rho e)_t + \nabla \cdot ([\rho e + \pi] \mathbf{v}) &= 0
\end{align*}
\]

\[
\begin{align*}
\rho e &= \frac{p}{\gamma - 1} + M^2 \frac{\rho \mathbf{v}^2}{2} \\
\pi &= \left( 1 - M^2 \right) p_\ast + M^2 p
\end{align*}
\]
Energy balance (Divergence correction for energy fluxes)

\[
M^2 \left( \frac{1}{\Delta t} \left[ \frac{\delta p_\star'}{\gamma - 1} \right] \left[ \frac{\rho v^2}{2} \right]_{n}^{**} - \mathbf{v}^{**} \cdot \nabla \delta p_\star' \right] + \left( 1 - M^2 \right) \nabla \cdot \left( \delta p_\star' \mathbf{v}^{**} \right) \right)
= - \left( \nabla \cdot \left( \rho h^* \mathbf{v}^{**} \right) - \frac{\Delta t}{2} \nabla \cdot \left( h^* \nabla \delta p \right) \right)

M = 0 \Rightarrow \text{classical linear projection step}

M \ll 1 \Rightarrow \text{linear, non-singular inclusion of weak compressibility}
Advected homentropic vortex at $M = 0.2$

Convergence rates: $v \rightarrow h^2$, $p \rightarrow h^{1.5}$, $\rho \rightarrow h^1$
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Second projection: discrete approximations of $\nabla \cdot \left( -\frac{1}{\rho} \nabla p \right) = r$

**Finite elements** (with test functions $\phi_\nu$):

$$\forall \nu : \int_{C_{i,j}} \left[ \frac{1}{\rho} \nabla p \cdot \nabla \phi_\nu - r \phi_\nu \right] dx = 0$$

where

$$p(x, y) = \sum_\mu \hat{p}_\mu \phi_\mu$$

**Finite volumes** on dual cells

$$\int_{\partial C_{i,j}} \frac{1}{\rho} \nabla p \cdot \mathbf{n} \, d\sigma - \int_{C_{i,j}} r \, dx = 0$$

**Bilinear pressure on FV-cells** $C_{i,j}$

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**Bilinear pressure distributions: FE vs. FV stencil**
Example: (robust, second order)* stencils for $\Delta p = \nabla \cdot \nabla p$

Conforming **finite elements**

Finite volumes on dual cells:


**Bilinear pressure distributions: FE vs. FV stencil**
Cell averaged updates

\[(\rho \mathbf{v})_{i,j}^{n+1} = (\rho \mathbf{v})_{i,j}^* - \Delta t \begin{pmatrix} (\partial_x p^{(2)})_{i,j} \\ (\partial_y p^{(2)})_{i,j} \end{pmatrix} \]

**Conservative** gradient approximation, e.g.,

\[\left(\partial_x p^{(2)}\right)_{i,j} \approx \frac{1}{2\Delta x} \left( [p_{i+\frac{1}{2},j+\frac{1}{2}} + p_{i+\frac{1}{2},j-\frac{1}{2}}] - [p_{i-\frac{1}{2},j+\frac{1}{2}} + p_{i-\frac{1}{2},j-\frac{1}{2}}] \right)\]
**Local updates** within cells $C_{i,j}$:

$$(\rho \mathbf{v})^{n+1}(x, y) = (\rho \mathbf{v})^*(x, y) - \Delta t \, \nabla p^{(2)}(x, y)$$

where

$$\nabla p^{(2)}(x, y) \bigg|_{i,j} = \begin{pmatrix}
(\partial_x p^{(2)})_{i,j} + (y - y_j) (\partial_{xy} p^{(2)})_{i,j} \\
(\partial_y p^{(2)})_{i,j} + (x - x_i) (\partial_{xy} p^{(2)})_{i,j}
\end{pmatrix}.$$  

**Second projection modifies piecewise linear reconstruction:**

$$\left( \partial_x (\rho \mathbf{v}) \right)_{i,j}^{n+1} = \left( \partial_x (\rho \mathbf{v})^* \right)_{i,j} - \Delta t \left( \partial_{xy} p^{(2)} \right)_{i,j}$$

$$\left( \partial_y (\rho \mathbf{u}) \right)_{i,j}^{n+1} = \left( \partial_y (\rho \mathbf{u})^* \right)_{i,j} - \Delta t \left( \partial_{xy} p^{(2)} \right)_{i,j}$$
Exact projection achieved with:

\[
(\nabla \cdot \mathbf{u})_{i,j}^{n+1} = \frac{1}{|\partial \overline{C}_{i,j}|} \oint_{\partial \overline{C}_{i,j}} \left( \frac{1}{\rho} \mathbf{m} \right)^{n+1} \cdot \mathbf{n} \, d\sigma
\]

where

\[
\mathbf{m}_{i,j}^{n+1}(x, y) = \begin{pmatrix}
(\rho u)_{i,j}^{n+1} + (x - x_i) \left( \partial_x (\rho u) \right)^*_{i,j} + (y - y_j) \left( \partial_y (\rho u) \right)^{n+1}_{i,j} \\
(\rho v)_{i,j}^{n+1} + (x - x_i) \left( \partial_x (\rho v) \right)^{n+1}_{i,j} + (y - y_j) \left( \partial_y (\rho v) \right)^*_{i,j}
\end{pmatrix}
\]
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**low-Froude extension**
**Regime 3: Buoyancy-controlled anelastic flow** $\theta = \Theta_0(z) + M^2 \theta'$

Leading order results:

$$p = P_0(z) + M^2 p' \quad \rho = R_0(z) + M^2 \rho' \quad R_0(z) = \Theta_0(z) P_0(z)^{1/\gamma}$$

Consequences:

$$\rho_e = \frac{p}{\gamma - 1} + \frac{M^2 \rho v^2}{2} + \rho gz \quad \rightarrow \quad \frac{P_0(z)}{\gamma - 1} + \frac{R_0(z) gz}{(M \rightarrow 0)}$$

$$(\rho e)_t + \nabla \cdot (\rho [\rho e + p] v) = 0 \quad \rightarrow \quad \nabla \cdot \left( v^{(0)} R_0(z) \left[ \frac{\gamma P_0}{\gamma - 1 R_0} (z) + gz \right] \right) = 0$$

$$\rho_t + \nabla \cdot (\rho v) = 0 \quad \rightarrow \quad \nabla \cdot \left( v^{(0)} R_0(z) \right) = 0$$

Unfortunately

$$\left[ \frac{\gamma P_0}{\gamma - 1 R_0} + gz \right] \equiv \Phi(z) \not\equiv \text{const.} \quad \Rightarrow \quad \nabla_\parallel \cdot u^{(0)} = 0, \quad \text{and} \quad w^{(0)} = 0$$

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**Anelastic limits**
Regime 3: Buoyancy-controlled anelastic flow $\theta = \Theta_0(z) + M^2 \theta'$

Leading order results:

$$p = P_0(z) + M^2 p' \quad \rho = R_0(z) + M^2 \rho' \quad R_0(z) = \Theta_0(z) P_0(z)^{1/\gamma}$$

Consequences (cont’d):

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \cdot \mathbf{v}) + \frac{1}{M^2}(\nabla p + \rho g) = 0 \quad \rightarrow \quad (\rho \mathbf{v})^{(0)}_t + \nabla \cdot (\rho \mathbf{v} \cdot \mathbf{v})^{(0)} + (\nabla p' + \rho') = 0$$

$(P_0, \Theta_0, p', \rho')$ must be maintained as primary variables in a numerical implementation
Generalization to anelastic regimes:

**Energy balance**  (Divergence correction for energy fluxes)

\[
M^2 \left( \frac{1}{\Delta t} \left[ \frac{\delta p_*'}{\gamma - 1} + \left( \frac{\rho v^2}{2} \right) \right]_{n} - \Delta t v^{**} \cdot \nabla \delta p_*' \right) + (1 - M^2) \nabla \cdot (\delta p_*' v^{**})
\]

\[
= - \left( \nabla \cdot (\rho h^* v^{**}) - \frac{\Delta t}{2} \nabla \cdot (h^* [\nabla \delta p_*' + \delta \rho_*' k]) \right)
\]

**Mass balance**  (Divergence correction for mass fluxes)

\[
\frac{M^2}{\Delta t} \frac{\delta \rho_*'}{\Delta t} = - \left( \nabla \cdot (\rho^* v^{**}) - \frac{\Delta t}{2} \nabla \cdot (\nabla \delta p_*' + \delta \rho_*' k) \right)
\]
Potential Temperature:

\[ \theta = \frac{\frac{1}{\gamma} p}{\rho} \]

Rising thermal plume \((\delta \theta / \theta = 0.1)\) in a 10 km neutral atmosphere
Conclusions