Systematic Stochastic Modeling of Climate Variability

Christian Franzke

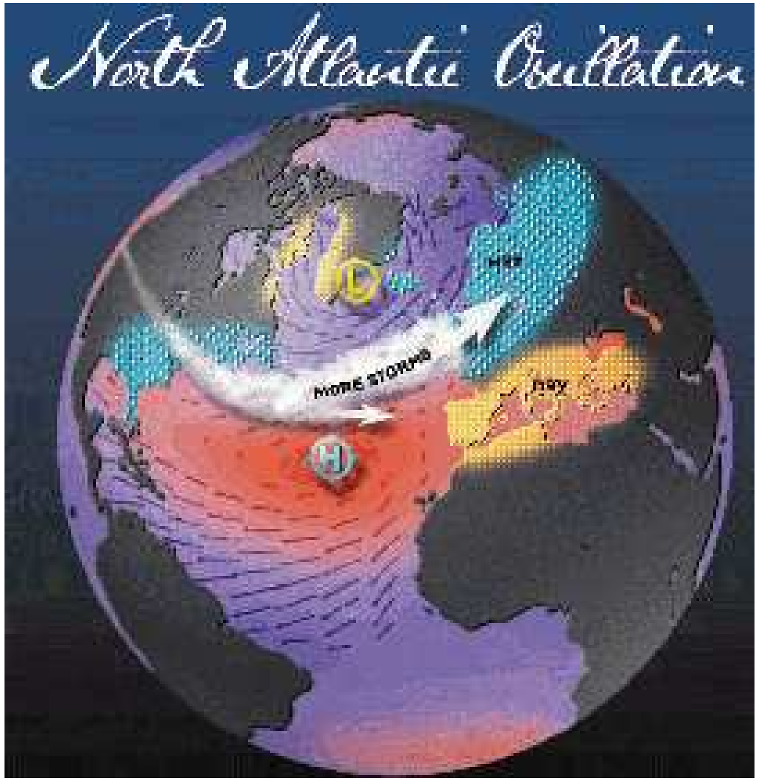
Andrew J. Majda, and Eric Vanden-Eijnden

Courant Institute of Mathematical Sciences
New York University

Time scales

- The climate system has a wide range of time scales for important physical processes
  - Boundary layer processes and convection on an hourly time scale
  - Synoptic weather systems on a daily time scale
  - Extratropical low-frequency variability on intraseasonal and interannual time scales
  - Oscillations of the coupled atmosphere-ocean system (El Nino-Southern Oscillation) on interannual to decadal time scales
- Due to the nonlinearity of the equations of motion all of these processes interact with each other

North Atlantic Oscillation



Source: www.ldeo.columbia.edu/NAO

- NAO has strong impact on regional climate and surface weather
- NAO is related to global warming
- NAO is used for studies of past climates

Why stochastic modeling?

- Time step in numerical GCMs is determined by fastest time scale which is resolved (which is usually also the smallest resolved spatial scale)
- Can we treat the fast components as a stochastic process?
  - Computationally more efficient
    - Long range weather and climate predictions
    - Estimating climate sensitivity to changes in forcing
  - Novel way to parameterize unresolved degrees of freedom

Why reduced models?

- State of the Art GCMs have $\sim 10^6 - 10^7$ degrees of freedom.
  This makes it hard to understand why something happens
- Reduced models help to understand complex models by simplifying and capturing the essence of a phenomenon

Outline

- The goal is to derive a reduced model for only the most important teleconnection patterns
- Due to the nonlinearity of the equations the unresolved modes interact with the resolved modes (Teleconnection patterns): Closure problem
- These neglected interactions are accounted for in a systematic way by using **stochastic** methods

Equations of motion

- Symbolic form of equations of motion

$$\frac{\partial \mathbf{x}}{\partial t} = F + L\mathbf{x} + B(\mathbf{x}, \mathbf{x})$$

- EOF expansion

$$\mathbf{x} = \sum a_i(t)\mathbf{e}_i + \overline{\mathbf{x}}$$

Expansion in Empirical Orthogonal Functions

- EOF expansion

$$\dot{a}_i = H_i + \sum_j L_{ij}\, a_j(t) + \sum_{jk} B_{ijk}\, a_j(t)\, a_k(t)$$

  Total energy norm EOFs ensure conservation of total energy by the nonlinear operator.

- Separation of time scales: Now we split the reduced model into climate modes,α_i, which are slowly evolving, and non-climate modes, β_i, which evolve considerably faster than the climate modes.

Formulation of the problem

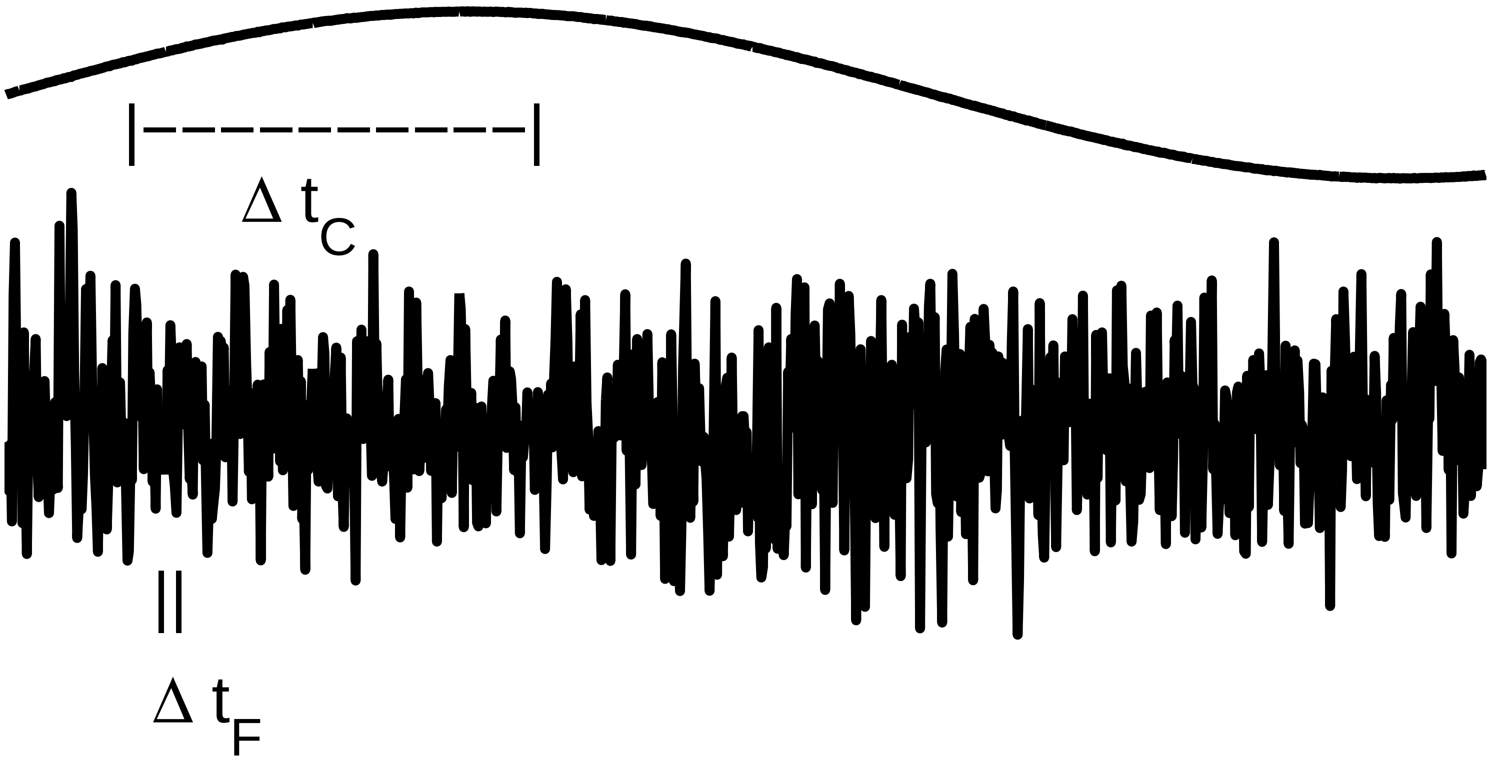
$$\frac{d\alpha(t)}{dt} = G(\alpha(t), \beta(t))$$

Question: Given the statistical behavior of $\beta(t)$ can we deduce in a effective way the statistical behavior of the solution $\alpha(t)$?

$$d\alpha(t) = G_{\text{eff}}(\alpha(t))dt + D_{\text{eff}}(\alpha(t))dW$$

W denotes Brownian motion (Stochastic Process)

Slow and fast modes



Effective equations

The MTV-theory (Majda et al. 1999 (PNAS), 2001 (CPAM), 2003 (JAS)) predicts the following functional form of the effective equations:

$$
\begin{aligned}
d\alpha &= (H^{\alpha} + L^{\alpha\alpha}\alpha + B^{\alpha\alpha\alpha}(\alpha,\alpha))dt \\
&\quad +(\tilde{H} + \tilde{L}\alpha + \tilde{B}(\alpha,\alpha) + \tilde{M}(\alpha,\alpha,\alpha))dt \\
&\quad +\sigma^{(1)}dW^{(1)} + \sigma^{(2)}(\alpha)dW^{(2)}
\end{aligned}
$$

Bare truncation

Effective equations

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$$
\begin{aligned}
d\alpha &= (H^{\alpha} + L^{\alpha\alpha}\alpha + B^{\alpha\alpha\alpha}(\alpha,\alpha))dt \\
&\quad + (\tilde{H} + \tilde{L}\alpha + \tilde{B}(\alpha,\alpha) + \tilde{M}(\alpha,\alpha,\alpha))dt \\
&\quad + \sigma^{(1)} dW^{(1)} + \sigma^{(2)}(\alpha) dW^{(2)}
\end{aligned}
$$

Deterministic correction terms
Constant forcing, linear and nonlinear

Effective equations

The MTV-theory (Majda et al. 1999 (PNAS), 2001 (CPAM), 2003 (JAS)) predicts the following functional form of the effective equations:

$$\begin{aligned} d\alpha &= (H^{\alpha} + L^{\alpha\alpha}\alpha + B^{\alpha\alpha\alpha}(\alpha,\alpha))dt \\ &\quad + (\tilde{H} + \tilde{L}\alpha + \tilde{B}(\alpha,\alpha) + \tilde{M}(\alpha,\alpha,\alpha))dt \\ &\quad + \sigma^{(1)} dW^{(1)} + \sigma^{(2)}(\alpha) dW^{(2)} \end{aligned}$$

Stochastic correction terms
Additive and multiplicative noise

Effective equations

- We assume that the dynamical system given only by the unresolved modes with interaction $B^{\beta\beta\beta}$ is ergodic and mixing and can be represented by a stochastic process
- We assume that all unresolved modes are quasi-Gaussian distributed
- Correction terms and noises are determined by minimal regression fitting of only the unresolved modes; Sole input are the variances and correlation functions of the unresolved modes (Different than regression fitting of resolved modes directly (Penland and Sardeshmukh 1995; Kravtsov et al. 2005))

Which physical processes represent the noise and correction terms?

- Linear interaction of unresolved modes with the mean state $\longrightarrow$ Additive noise and deterministic linear term
- Driving of the climate modes by unresolved modes $\longrightarrow$ Additive noise and deterministic linear term
- Advection of climate modes by unresolved modes $\longrightarrow$ Multiplicative noise and deterministic cubic term

Stochastic Mode Reduction for Atmospheric Models

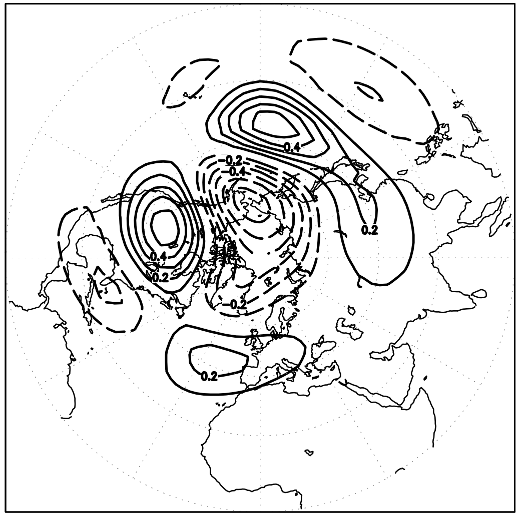
- Barotropic Model on the Sphere:
  Franzke, C., A. J. Majda and E. Vanden-Eijnden, 2005: Low-order stochastic mode reduction for a realistic barotropic model climate. J. Atmos. Sci., 1722-1745.
- Three Layer Baroclinic Model on the Sphere:
  Franzke, C. and A. J. Majda, 2006: Low-order stochastic mode reduction for a prototype atmospheric GCM. J. Atmos. Sci., 457-479.

Quasi-geostrophic model

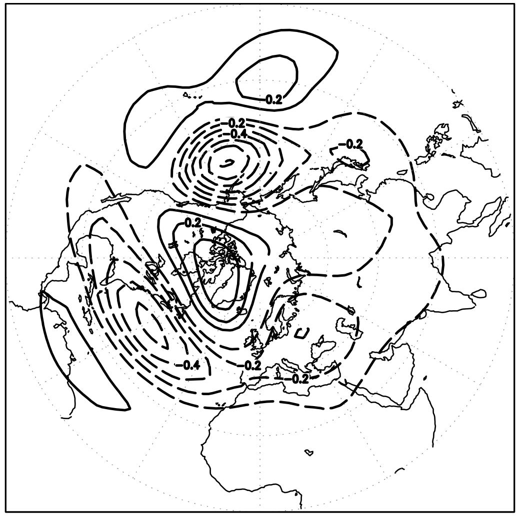
- Global spectral model (Marshall and Molteni, JAS, 1993)
- T21 resolution ($\sim 5.6^\circ \times 5.6^\circ$)
- 3 Layers
- Topography
- Forcing determined from observations

Total energy norm EOFs

EOF1 EV 7%



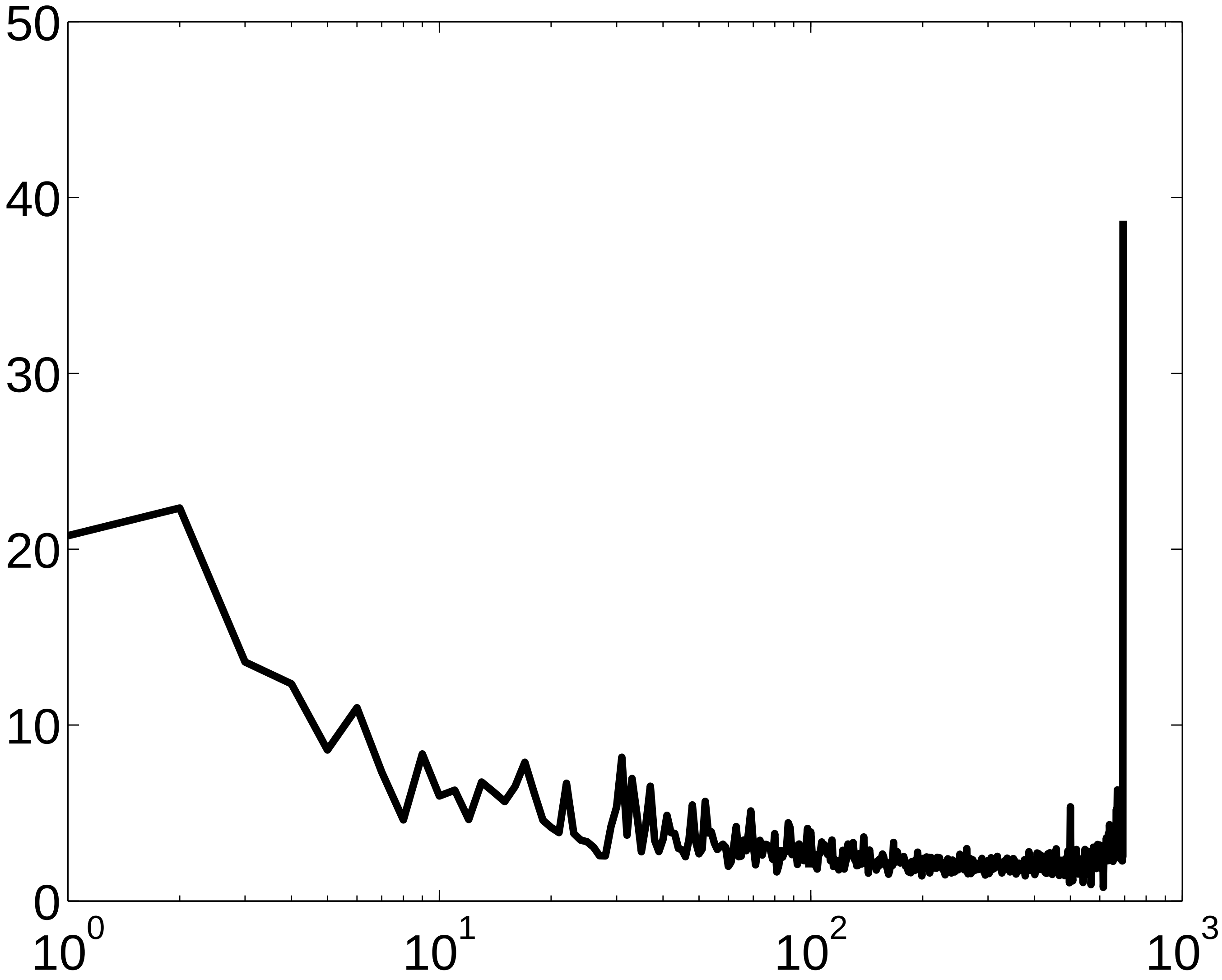
EOF2 EV 5%



Number of EOFs	PNA	NAO
4	0.25	0.24
6	0.58	0.28
8	0.65	0.57
10	0.68	0.78

Projection of EOF subspace onto observed PNA and NAO.

Autocorrelation time scale



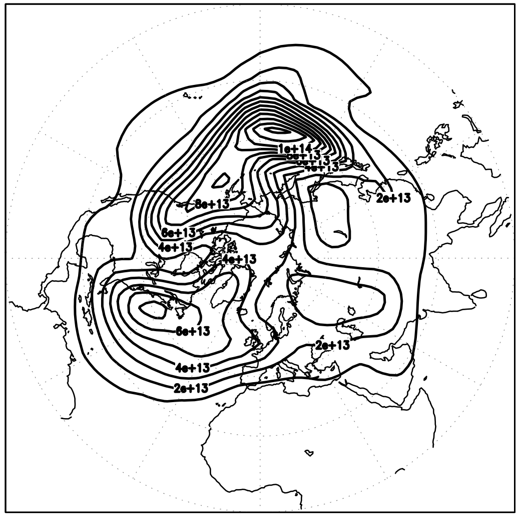
Geographical distribution

Variance: Pattern correlation

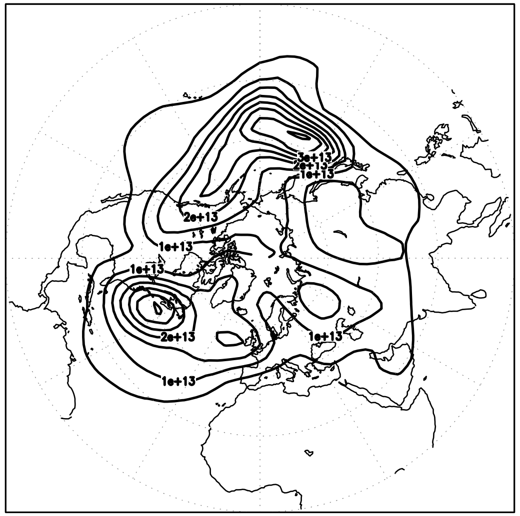
Mode	200 hPa	500 hPa
4	0.97	0.97
6	0.99	0.99
8	0.99	0.99
10	0.98	0.98

10 Modes

QG model



Stochastic model



Geographical distribution

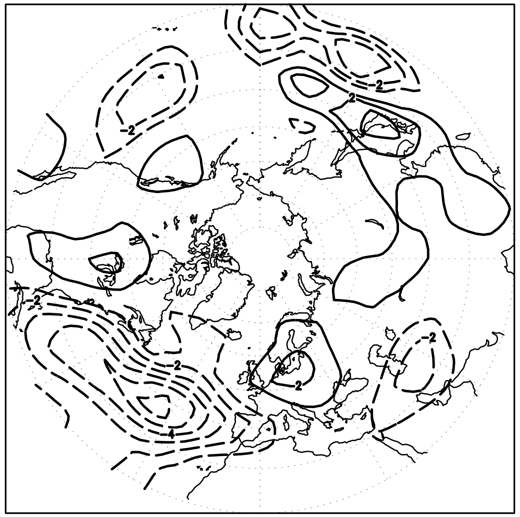
Transient eddy forcing: Pattern correlation

Mode	200 hPa	500 hPa
4	0.83	0.69
6	0.93	0.86
8	0.88	0.71
10	0.84	0.74

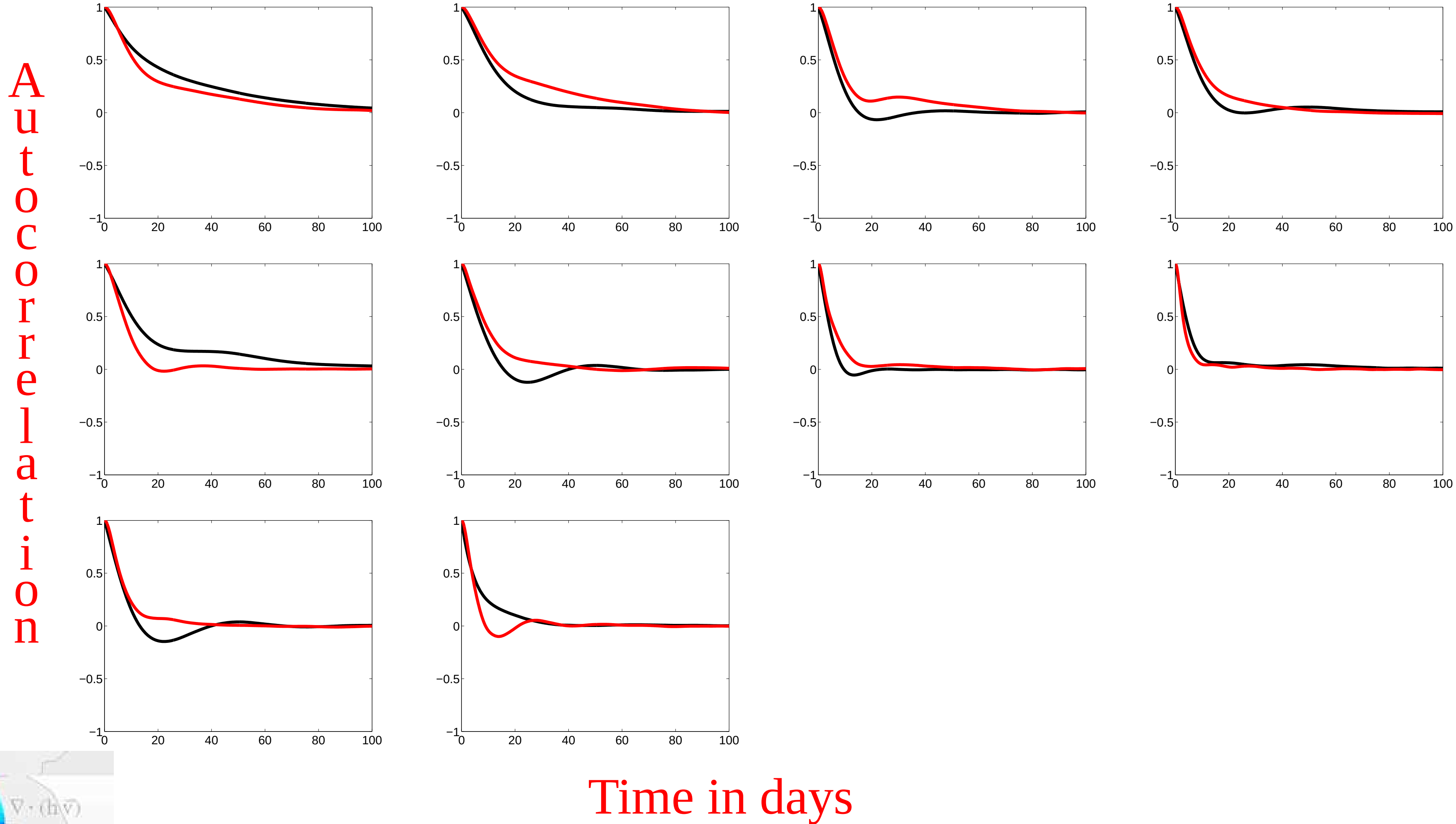
10 Modes

QG model

Stochastic model



Autocorrelation function



Black line: Stochastic model; Red line: Quasi-Geostrophic model.

Budget Analysis

- Effective equations are nonlinear with both additive and multiplicative noise
  - Linear interaction of unresolved modes with the mean state
  - Advection of climate modes by unresolved modes
- Mode reduction: $631 \longrightarrow 10$

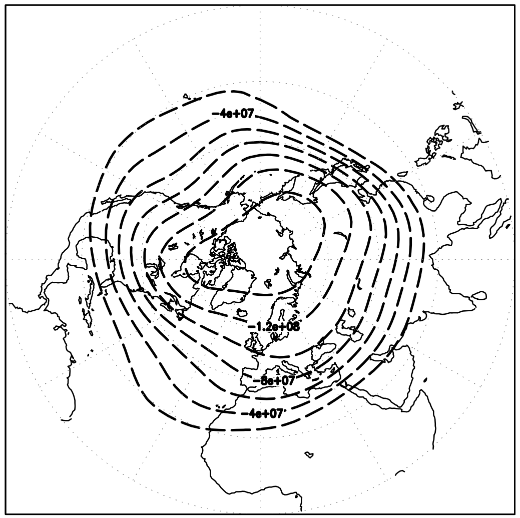
Geographical distribution

Mean: Pattern correlation

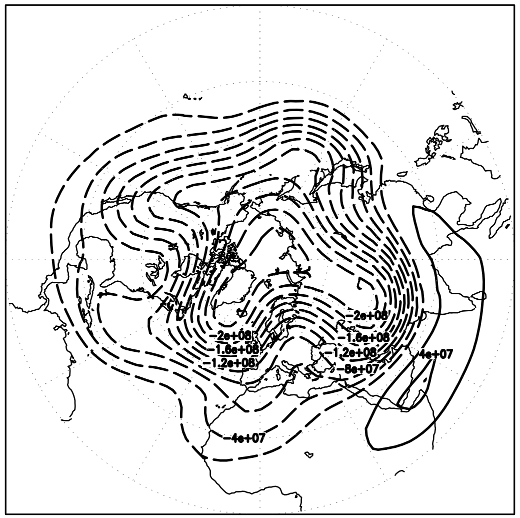
Mode	200 hPa	500 hPa
4	-0.07	-0.13
6	0.04	-0.05
8	0.18	0.13
10	0.42	0.43

10 Modes

QG model



Stochastic model



Minimally fitted stochastic model

$$
\begin{aligned}
d\alpha_i(t) = {} & \lambda_B \Big(H_i^{\alpha} + \sum_j L_{ij}^{\alpha\alpha}\alpha_j(t)dt + \sum_{jk} B_{ijk}^{\alpha\alpha\alpha}\alpha_j(t)\alpha_k(t)dt\Big) \\
& + \lambda_A^2 \sum_j \tilde{L}_{ij}^{(2)}\alpha_j(t)dt + \lambda_A\sqrt{2}\sum_j \sigma_{ij}^{(2)} dW_j^{(2)} \\
& + \lambda_M^2 \left(\sum_j \tilde{L}_{ij}^{(3)}\alpha_j(t)dt + \sum_{jkl}\tilde{M}_{ijkl}\alpha_j(t)\alpha_k(t)\alpha_l(t)dt\right) \\
& + \lambda_L^2 \left(\sum_j \tilde{L}_{ij}^{(1)}\alpha_j(t)dt\right) + \lambda_M\lambda_L \left(\tilde{H}_j^{(1)}dt + \sum_{jk}\tilde{B}_{ijk}\alpha_j(t)\alpha_k(t)dt\right) \\
& + \lambda_A\lambda_F \tilde{H}_j^{(2)}dt + \sqrt{2}\sum_j \sigma_{ij}^{(1)}(\alpha(t))dW_j^{(1)},
\end{aligned}
$$

where the nonlinear noise matrix $\sigma^{(1)}$ satisfies,

$$
\lambda_L^2 Q_{ij}^{(1)} + \lambda_L\lambda_M \sum_k U_{ijk}\alpha_k(t) + \lambda_M^2 \sum_{kl} V_{ijkl}\alpha_k(t)\alpha_l(t) = \sum_k \sigma_{ik}^{(1)}(\alpha(t))\sigma_{jk}^{(1)}(\alpha(t)).
$$

Budget Analysis: No climate drift

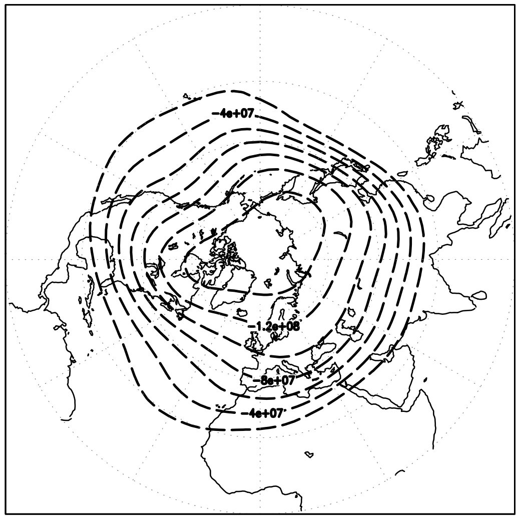
$$\lambda_B = 0.1, \lambda_M = \lambda_L = 4.0, \lambda_A = \lambda_F = 0.0$$

Mean: Pattern correlation

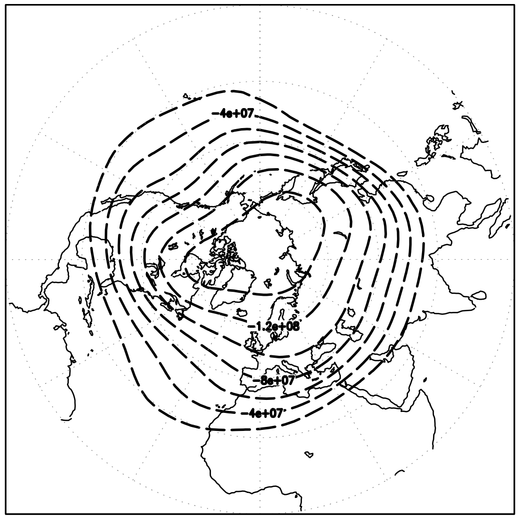
Mode	200 hPa	500 hPa
4	0.98	0.97
6	0.97	0.99
8	0.99	0.98
10	0.99	0.99

10 Modes

QG model

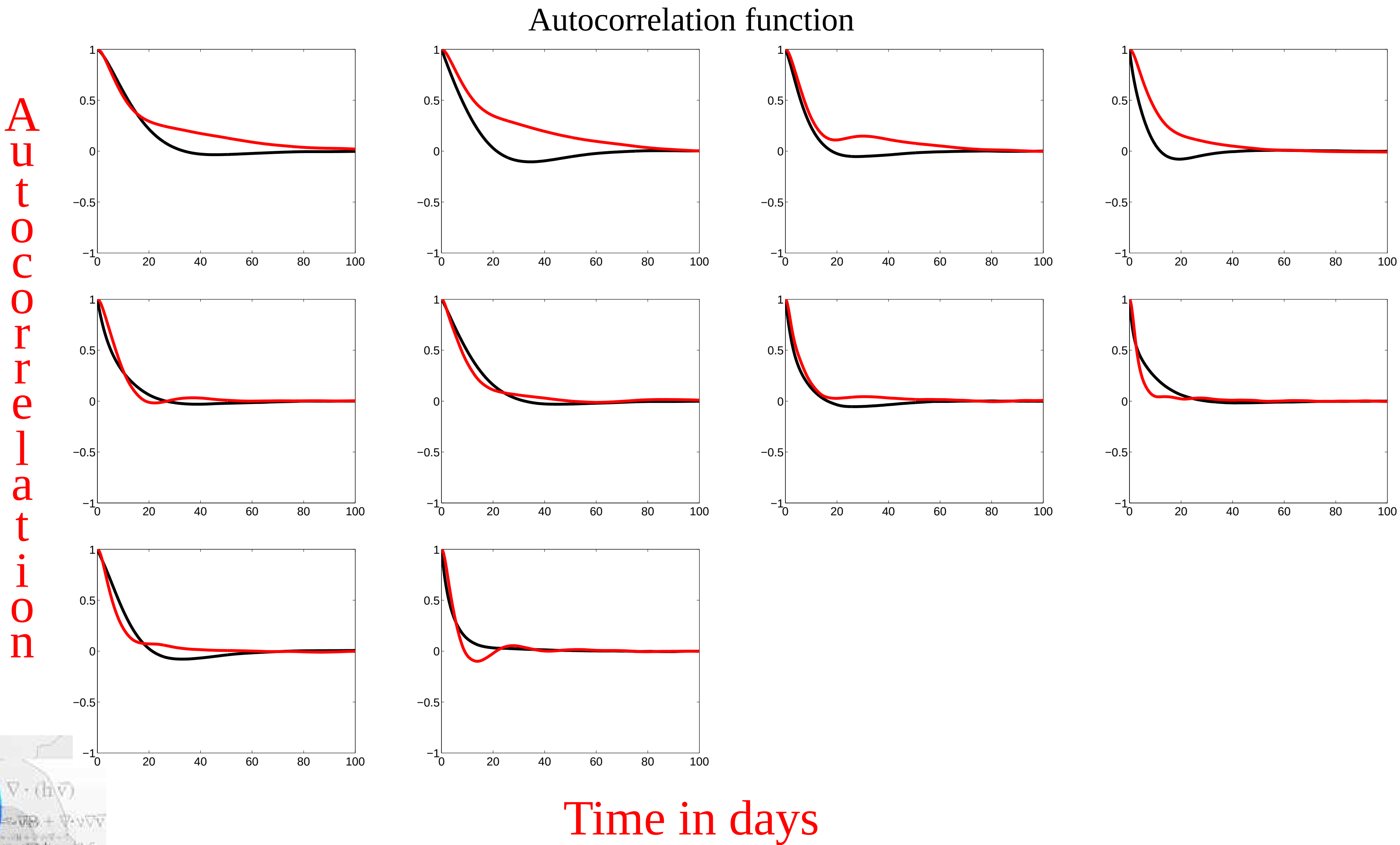


Stochastic model



Budget Analysis: No climate drift

$$\lambda_B = 0.1, \lambda_M = \lambda_L = 4.0, \lambda_A = \lambda_F = 0.0$$



Black line: Stochastic model; Red line: Quasi-Geostrophic model.

Summary and Conclusions

- Systematic procedure for stochastic parameterization of unresolved degrees of freedom
- Systematic derivation of reduced models of climate variability
- Reduced models simulate the climate statistics well
- Barotropic model: Effective linear equation with additive noise stemming from the augmented linearity (Mode reduction: $231 \rightarrow 4$)
- Baroclinic model: Multiplicative triads and augmented linearity are important correction terms (Mode reduction: $693 \rightarrow 10$)

Distinct Metastable Atmospheric Regimes Despite Nearly Gaussian Statistics: A Paradigm Model

Christian Franzke

Andrew Majda, Alexander Fischer, and Daan Crommelin

Courant Institute of Mathematical Sciences, New York University

Metastable Atmospheric Regimes: A Paradigm Model

- The structure of the low-frequency regime transitions among persistent teleconnection patterns (e.g. NAO and PNA) is of central importance for both long-range weather prediction and climate change projection
- Multiple extrema in the PDF's are usually associated with regime behavior
- BUT: Long integrations of GCM's show nearly Gaussian statistics
- Are there distinct atmospheric regimes despite nearly Gaussian statistics?
- Previous approaches include Multivariate PDFs (Kimoto and Ghil 1993; JAS), Cluster analysis (Cheng and Wallace 1993; JAS) and Gaussian mixtures (Smyth et al. 1999; JAS)
- In this talk: Objective regime identification through HMM
- Paradigm model:
  - Barotropic flow over topography
  - Low-frequency waves: Blocked and Zonal states
  - Nearly Gaussian behavior
  - Truncated low-order model is Charney-DeVore model (1979; JAS)

Metastable Atmospheric Regimes: A Paradigm Model

The barotropic quasi-geostrophic equations with a large scale zonal mean flow U on a $2\pi \times 2\pi$ periodic domain are given by

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = 0$$

$$q = \Delta\psi + h$$

$$\frac{dU}{dt} = \frac{1}{4\pi^2}\int h\frac{\partial \psi}{\partial x}dxdy$$

The model is truncated at $|\mathbf{k}|^2 \leq 17$ (57 degrees of freedom).

Majda, A. J., I. Timofeyev and E. Vanden-Eijnden, 2003: Systematic Strategies for Stochastic Mode Reduction in Climate, J. Atmos. Sci.

Majda, A. J., C. L. Franzke, A. Fischer, and D. T. Crommelin, 2006: Distinct Metastable Atmospheric Regimes Despite Nearly Gaussian Statistics: A Paradigm Model, PNAS.

Low-order Truncation and Steady States

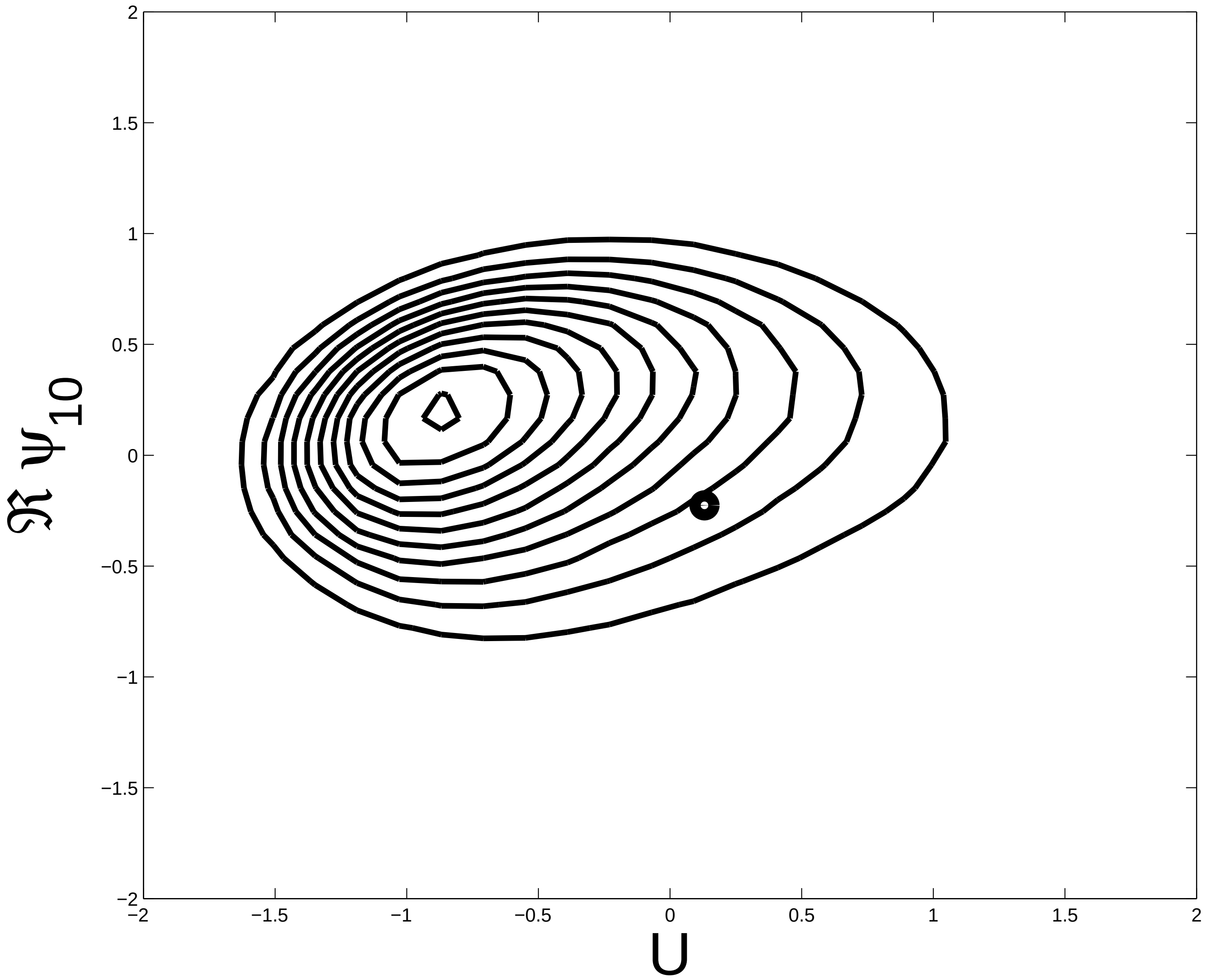
$$\begin{aligned} \psi(x,y,t) &= a(t)\sin(x) + b(t)\cos(x), \\ h(x,y) &= H(\sin(x) + \cos(x)) \end{aligned}$$

yields an exact nonlinear solution, provided U, a, and b satisfy

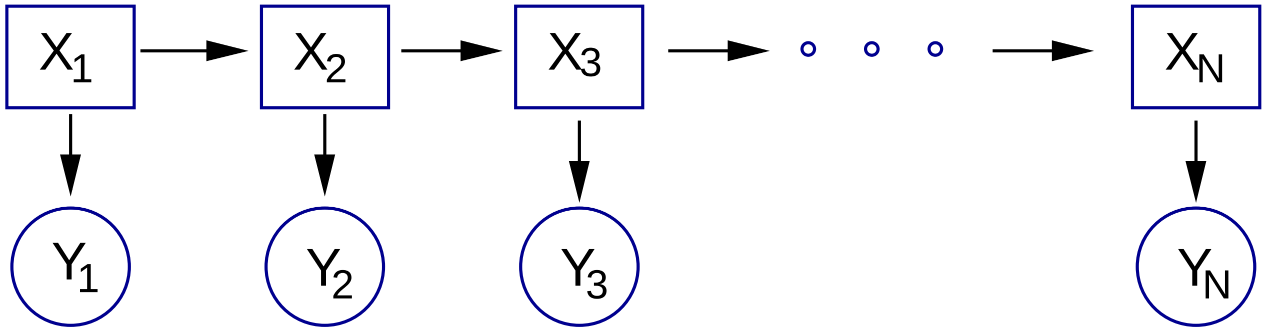
$$\begin{aligned} \dot{a} &= -UH + (U-\beta)b, \\ \dot{b} &= UH - (U-\beta)a, \\ \dot{U} &= \frac{H}{2}(a-b). \end{aligned}$$

This truncated model is equivalent to the Charney-DeVore model after a 45° rotation of a and b.

Low-order Truncation and Steady States



Hidden Markov Model (HMM)



Graphical representation of a HMM. $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ are random variables, rectangles denote hidden (unobservable) random variables, circles observable ones; the arrows specify conditional independence relations.

Hidden Markov Model (HMM)

The conditional independence relations between X and Y are defined by the factorization

$$P(X_1, \dots, X_T, Y_1, \dots, Y_T) =$$
$$P(X_1)P(Y_1|X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})P(Y_t|X_t)$$

A HMM is defined by the following components:

- N hidden States $S = s_1, s_2, \dots, s_N$
- the observation space $V \subset \mathbb{R}^d$
- a $(N \times N)$ stochastic transition matrix $A = (a_{ij})$
- a stochastic vector $\pi = (\pi_1, \dots, \pi_N)$
- probability distributions $B_n, n = 1, \dots, N$ on V

Parameter estimation by EM and Viterbi algorithms

References: Rabiner (1989), Ghahramani (2001), Fischer et al. (2006)

HMM Analysis for Metastable Regimes

$$A^{0.2} = \begin{pmatrix} 0.985 & 0.015 \\ 0.016 & 0.984 \end{pmatrix},$$

$$B_1 = \mathcal{N}(-0.035, 0.304), B_2 = \mathcal{N}(-0.789, 0.119)$$

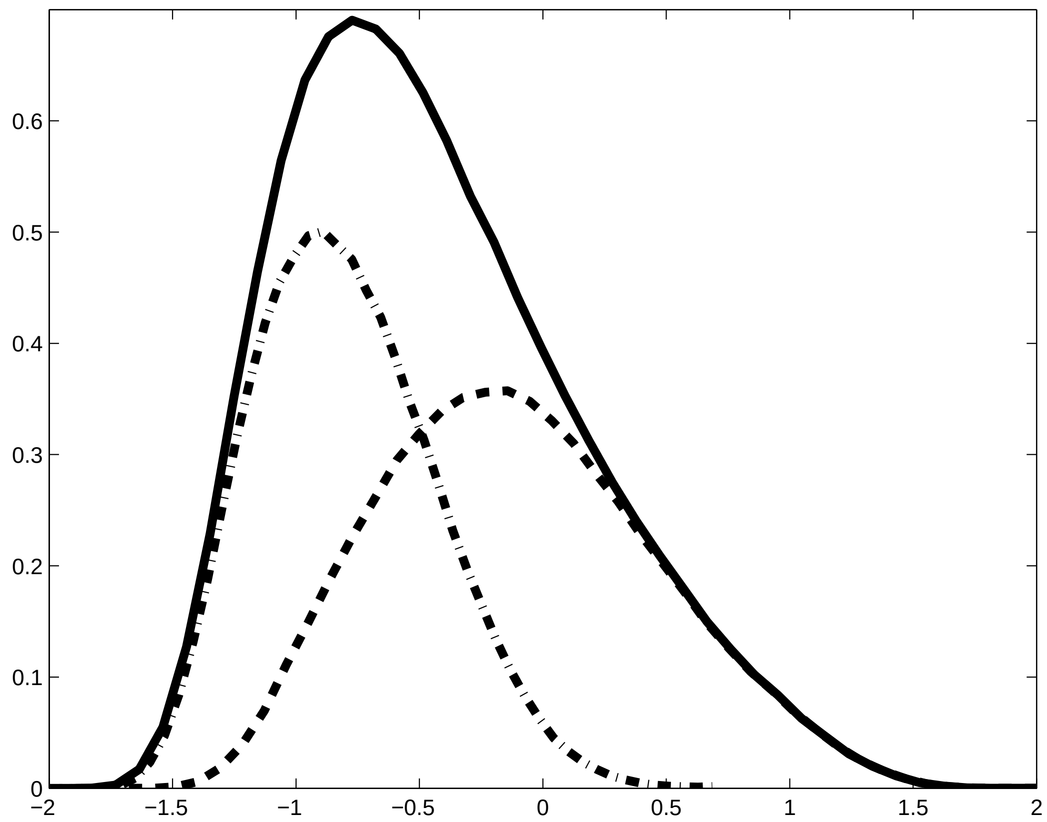
eigenvalues of $A^{0.2}$:$\lambda_1(A^{0.2}) = 1, \lambda_2(A^{0.2}) = 0.969$

$$\text{Invariant distribution of } A^{0.2}\text{:}(0.529, 0.471) \qquad (1)$$

Autocorrelation time scale of U: 5 time units

Residence time: H1 20 time units; H2 15 time units

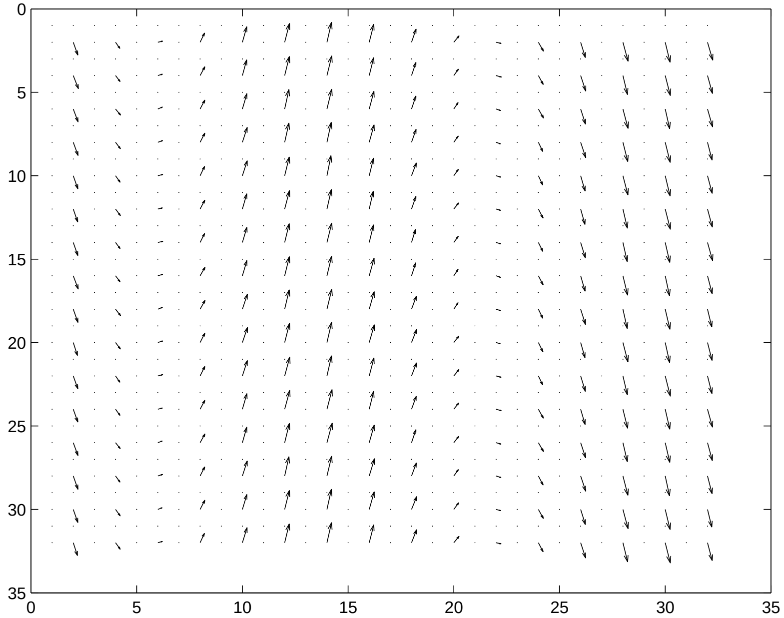
HMM Analysis for Metastable Regimes



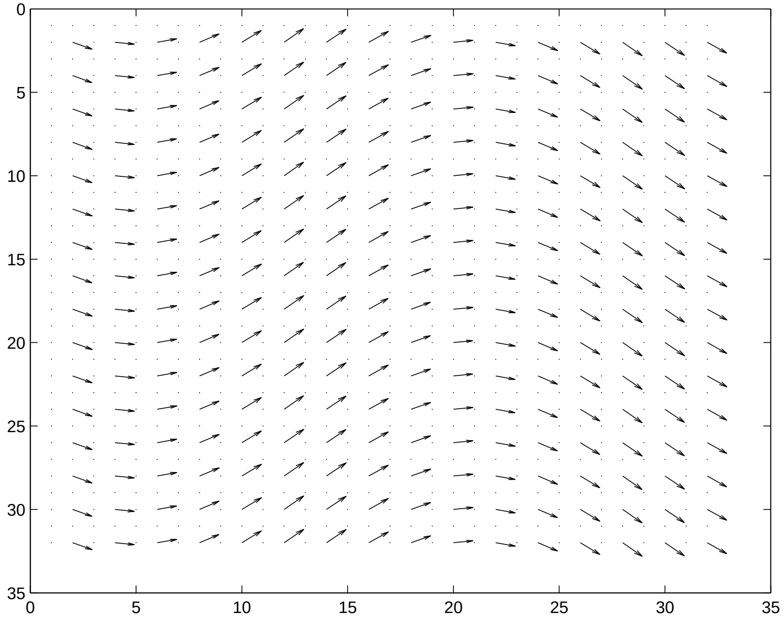
Climatological marginal PDF of U (solid line) and weighted conditional PDF's of hidden state 1 (Blocked flow, dashed line) and hidden state 2 (Zonal flow, dashed-dotted line).

HMM Analysis for Metastable Regimes

a) Hidden State 1: Blocked Flow

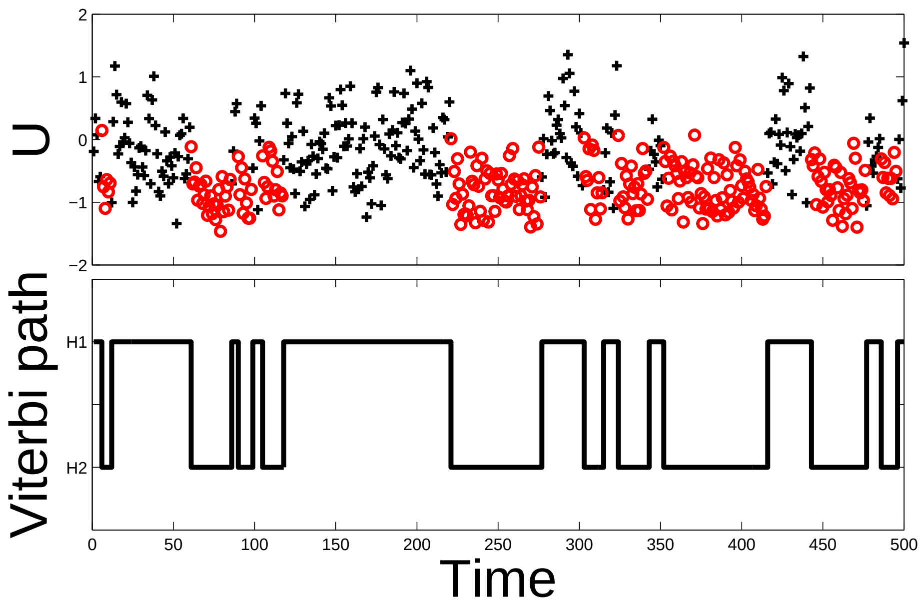


b) Hidden State 2: Zonal Flow



Velocity field conditioned on the Viterbi path of a HMM analysis in the subspace U for a) hidden state 1 (Blocked flow), and b) hidden state 2 (Zonal flow).

HMM Analysis for Metastable Regimes



U (upper panel) and Viterbi path (lower panel). For U path black crosses and red circles denote states which correspond to hidden state 1 (Blocked flow) and 2 (Zonal flow), respectively.

Why only 2 hidden states?

Analysis with 4 hidden states

eigenvalues of $A^{0.2}$:$\lambda_1(A^{0.2}) = 1, \lambda_2(A^{0.2}) = 0.972,$

$$\lambda_3(A^{0.2}) = 0.930, \lambda_4(A^{0.2}) = 0.731$$

Invariant distribution of $A^{0.2}$:$(0.137, 0.125, 0.345, 0.393)$

Predicted nonlinear reduced equation for U

$$\frac{dU}{dt} = -\gamma(U)U + \frac{\gamma'(U)}{\alpha\mu} + \sqrt{\frac{2\gamma(U)}{\alpha\mu}}\dot{W} \tag{2}$$

where $\gamma'(U) = d\gamma/dU$ and

$$\gamma(U) = 2\sum_{\mathbf{k}} \frac{\mathbf{k}_x^2 \left|H_{\mathbf{k}}\right|^2 \gamma_{\mathbf{k}}}{\gamma_{\mathbf{k}}^2 + \left(\Omega_{\mathbf{k}} - k_x(\alpha\mu)^{\frac{1}{2}}U\right)^2} \tag{3}$$

$$\begin{aligned} H_{\mathbf{k}} &= h_{\mathbf{k}}\sqrt{\frac{\mu}{|\mathbf{k}|^2(\mu + |\mathbf{k}|^2)}} \\ \Omega_{\mathbf{k}} &= \frac{k_x\beta}{|\mathbf{k}|^2} - \overline{U}k_x \end{aligned}$$

Majda, A. J., I. Timofeyev and E. Vanden-Eijnden, 2003 (JAS)

Empirical reduced equation for U

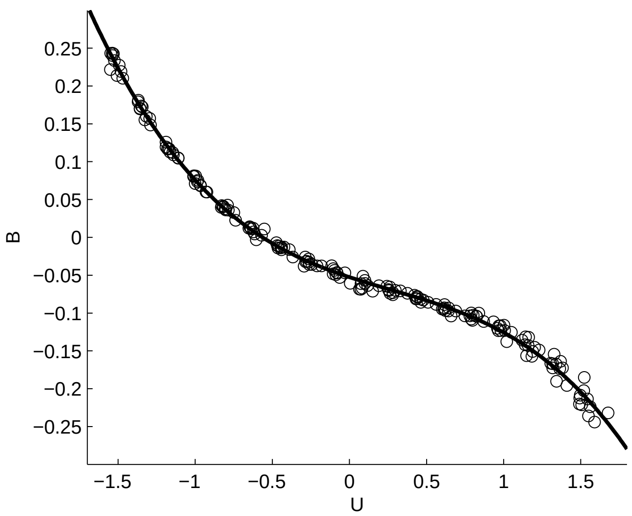
$$dU = B(U)dt + \sqrt{A(U)}dW$$

- $B(U)$ is the drift coefficient
- $\frac{A(U)}{2} > 0$ is the diffusion coefficient
- A and B are estimated from observed U
- W is Brownian motion.

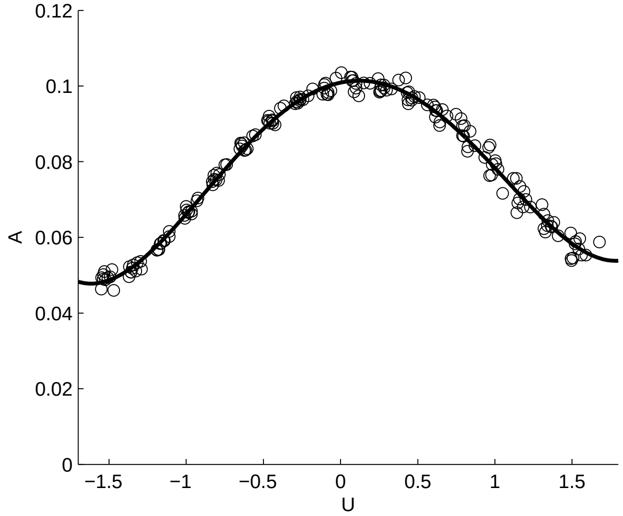
Crommelin, D. T., and E. Vanden-Eijnden (J. Comp. Phys.; Comm. Math. Sci. 2006)

Drift and Diffusion

a) Drift



b) Diffusion



Reconstructed a) Drift B and b) Diffusion A from time series variable U. The open circles are the result of the reconstruction, carried out 10 times on 10 different (non-overlapping) segments of the time series.

Metastable Regimes

$$A^{0.2} = \begin{pmatrix} 0.990 & 0.010 \\ 0.016 & 0.984 \end{pmatrix},$$

$$B_1 = \mathcal{N}(-0.748, 0.086), B_2 = \mathcal{N}(0.209, 0.200)$$

eigenvalues of $A^{0.2}$:$\lambda_2(A^{0.2}) = 0.9744$

Summary and Conclusions

- HMM are utilized for objective atmospheric regime identification
- Two regimes are identified, which correspond to blocked and zonal flow
- Low-order stochastic models capture regime behavior
- This offers potential for using reduced stochastic models for long-range predictability