Stochastic Mode–Reduction in Large Deterministic Systems

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Related Publications:
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Plan

- Mode-Elimination as a Limit of Infinite Separation of Time-Scales

- Mode-Elimination for Conservative Systems

- Example - Truncated Burgers-Hopf Equation
  - Numerical Verification of the Limiting Behavior
  - Conservative Mode-Elimination
  - Comparison with Direct Numerical Simulations
  - Balance of Terms

Essence of Mode-Reduction

Dynamical Variables:

$$\dot{Z} = f(Z)$$

Decomposition:

$$\begin{aligned} Z &= (Essential,\ Non - Essential) \\ &= (SLOW,\ FAST) \end{aligned}$$

Goal: Eliminate Fast modes; Derive Closed-Form equation for Slow Dynamics

Motivation: Interested Only in Statistical Behavior of the Slow Dynamics

Asymptotic Approach:

$$\text{Limit} \quad \frac{Time\ Scale\{FAST\}}{Time\ Scale\{SLOW\}} \rightarrow \infty$$

Rewrite the Original System

$$\dot{Z} = f(Z)$$

Decomposition:

$$Z = (SLOW,\ FAST) \equiv (X, Y)$$

Quadratic System:

$$\dot{X} = \{X, X\} + \{Y, X\} + \{Y, Y\}$$

$$\dot{Y} = \{X, X\} + \{Y, X\} + \{Y, Y\}$$

Conservation of Energy:

$$\frac{d}{dt}E = \frac{d}{dt}\left(X^2 + Y^2\right) = 2X\dot{X} + 2Y\dot{Y} =$$

$$\{X, X, X\} + \{X, X, Y\} + \{Y, Y, X\} + \{Y, Y, Y\} = 0$$

Modify the Original System

Main Idea: Introduce ε in the Equations

Preserve Conservation of Energy

Modified System

$$\begin{aligned} \dot{X} &= \{X,X\} + \frac{1}{\varepsilon}\{Y,X\} + \frac{1}{\varepsilon}\{Y,Y\} \\ \dot{Y} &= \frac{1}{\varepsilon}\{X,X\} + \frac{1}{\varepsilon}\{Y,X\} + \frac{1}{\varepsilon^2}\{Y,Y\} \end{aligned}$$

Conservation of Energy: $\dot{E} =$

$$\{X,X,X\}+\frac{1}{\varepsilon}\{X,X,Y\}+\frac{1}{\varepsilon}\{Y,Y,X\}+\frac{1}{\varepsilon^2}\{Y,Y,Y\} = 0$$

- $\varepsilon = 1$ Corresponds to the Original System
- Conserves Energy
- $\dot{Y} \sim \{Y,Y\}$
- Numerical & Analytical Approaches

Truncated Burgers-Hopf Model

Fourier-Galerkin Projection of

$$u_t + uu_x = 0$$

onto a £nite number of Fourier modes

$$u = \sum \hat{u}_k e^{ikx}, \qquad 1 \le |k| \le \Lambda$$

2Λ-dimensional system of ODEs

$$\frac{d}{dt}\hat{u}_k = -\frac{ik}{2} \sum_{p+q+k=0} \hat{u}_p^* \hat{u}_q^*$$

with Reality Condition $\hat{u}_k^* = \hat{u}_{-k}$

Main Features

- Conservation of Energy $\sum |\hat{u}_k|^2$; Equipartition
- Correlation scaling $Corr.Time\{\hat{u}_k\} \sim k^{-1}$
- Gaussian distribution in the limit $\Lambda \to \infty$
- Hamiltonian $H = \frac{1}{6} \int u_\Lambda^3 dx$

$\widehat{u}_1$ is the Slow Mode

Consider:

$$SLOW = \widehat{u}_1, \qquad FAST = \{\widehat{u}_2 \ldots \widehat{u}_\Lambda\}$$

Time-Scale Separation:

$$\frac{Corr.Time\{SLOW\}}{Corr.Time\{FAST\}} = 2$$

Modified System:

$$\frac{d}{dt}\widehat{u}_1 = -\frac{i}{2\varepsilon} \sum_{\substack{p+q+1=0 \\ 2\le|p|,|q|\le\Lambda}} \widehat{u}_p^* \widehat{u}_q^*,$$

$$\frac{d}{dt}\widehat{u}_k = -\frac{ik}{2\varepsilon}\left[\widehat{u}_{k+1}\widehat{u}_1^* + \widehat{u}_{k-1}\widehat{u}_1\right] - \frac{ik}{2\varepsilon^2} \sum_{\substack{k+p+q=0 \\ 2\le|p|,|q|\le\Lambda}} \widehat{u}_p^* \widehat{u}_q^*$$

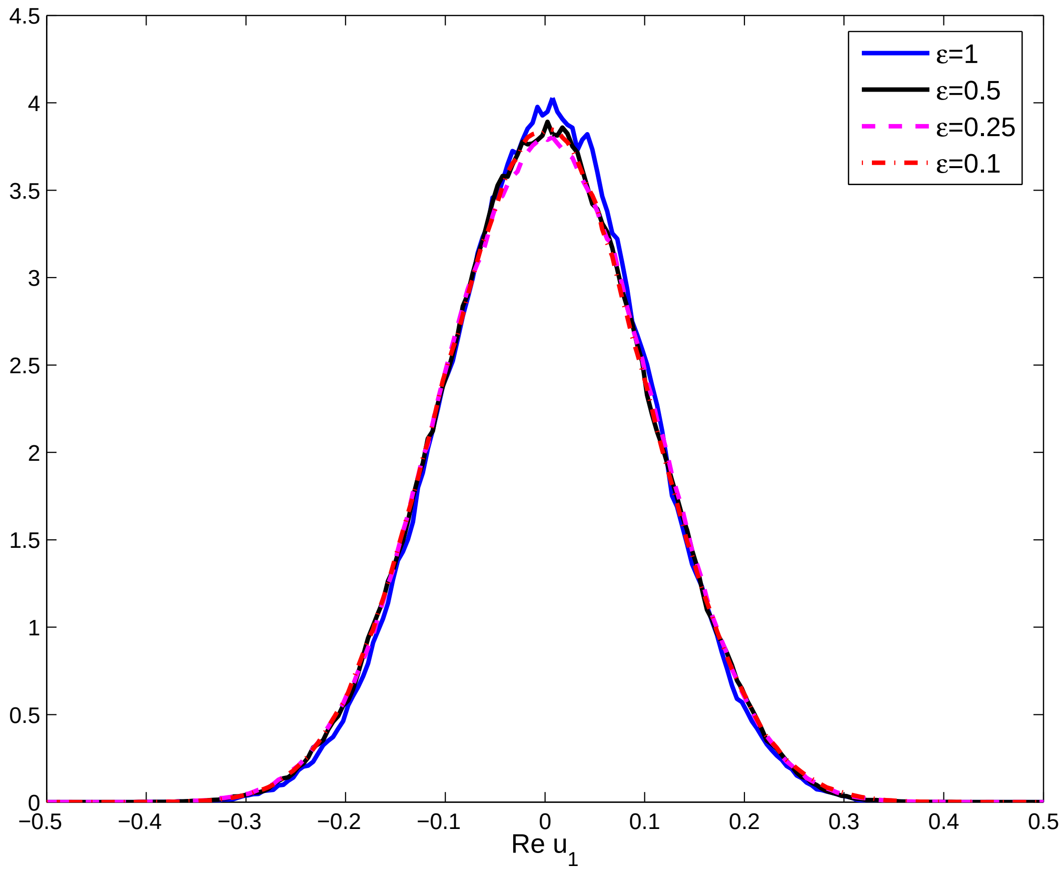
Numerical Approach

Goal: Verify Existence of the Reduced Dynamics

Also: Understand the “shape” of the Limit

Approach: Simulate Modified System with

$$\varepsilon = 1,\ 0.5,\ 0.25,\ 0.1$$



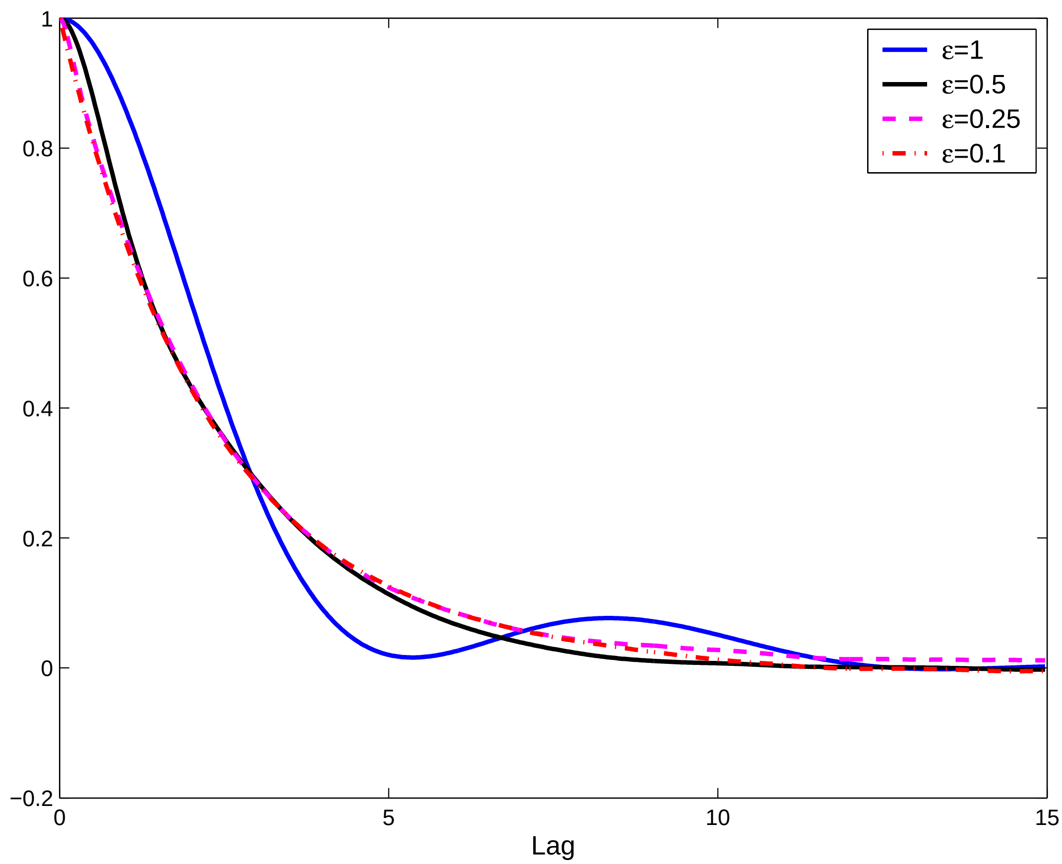
PDF of $Re\ \widehat{u}_1$

Numerical Approach

More Severe Test: Behavior of the Two-Point Statistics

$$\langle Re\ \hat{u}_1(t) Re\ \hat{u}_1(t+s) \rangle$$

$$\varepsilon = 1,\ 0.5,\ 0.25,\ 0.1$$



Corr. Function of $Re\ \hat{u}_1$

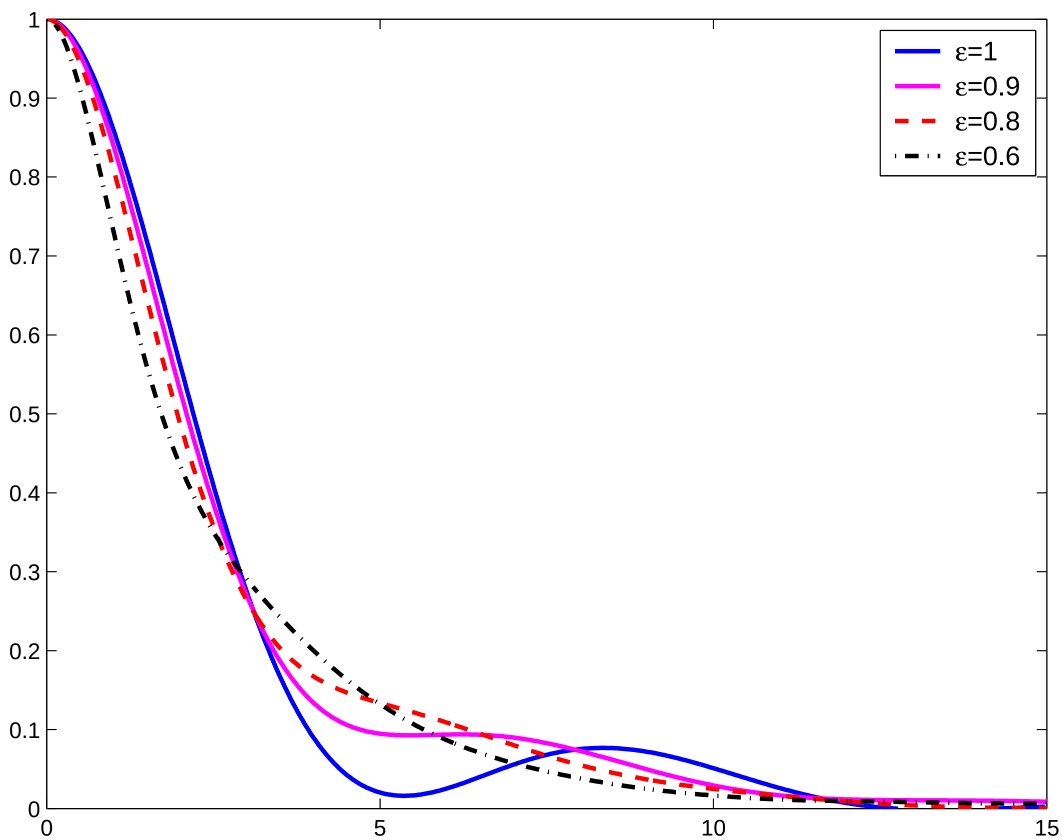
$\text{Corr.Time}_{\varepsilon=1} = 2.64, \quad \text{Corr.Time}_{\varepsilon=0.1} = 2.4$

Numerical Approach

Question: How fast the Bump Disappears

Approach: Simulate Modified System with

$$\varepsilon = 1,\ 0.9,\ 0.8,\ 0.6$$



Corr. Function of $Re\ \hat{u}_1$

Analytical Approach

Equation:

$$\dot{X} = \frac{1}{\varepsilon} f(X, Y)$$

$$\dot{Y} = \frac{1}{\varepsilon^2} g(X, Y)$$

Effective Equation:

$$dX = \bar{a}(X)dt + \bar{b}(X)dW$$

$$\bar{a} = \int_0^\infty dt \int d\mu(Y) f(X, Y(0)) \partial_X f\,(X, Y(t))$$

$$\bar{b}^2 = \int_0^\infty dt \int d\mu(Y) f(X, Y(0)) f\,(X, Y(t))$$

Overview of the Approach

Backward Equation

$$-\frac{\partial p^{\varepsilon}}{\partial s} = \frac{1}{\varepsilon^2} L_1 p^{\varepsilon} + \frac{1}{\varepsilon} L_2 p^{\varepsilon}$$

Represent formally as Power Series

$$p^{\varepsilon} = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots$$

Collect terms and impose solvability condition

$$\frac{\partial p_0}{\partial s} = \mathbb{P} L_2 L_1^{-1} L_2 \mathbb{P} p_0$$

$$\mathbb{P} L_2 \mathbb{P} = 0$$

Derivation of the Reduced Model

Consider a general quadratic system of equations for the variables $x = \{x_i\}$ and $y = \{y_i\}$

$$\begin{cases} \dot{x}_i = \varepsilon^{-1} \sum_{j,k} m_{ijk}^{xxy} x_j y_k + \varepsilon^{-1} \sum_{j,k} m_{ijk}^{xyy} y_j y_k \\ \dot{y}_i = \varepsilon^{-1} \sum_{j,k} m_{ijk}^{yxx} x_j x_k + \varepsilon^{-1} \sum_{j,k} m_{ijk}^{yxy} x_j y_k + \varepsilon^{-2} \sum_{j,k} m_{ijk}^{yyy} y_j y_k \end{cases}$$

Volume Preserving => Consider Liouville Equation

$$-\frac{\partial p^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2} L_1 p^\varepsilon + \frac{1}{\varepsilon} L_2 p^\varepsilon$$

$$L_1 = \sum_{i,j,k} m_{ijk}^{yyy} y_j y_k \frac{\partial}{\partial y_i}$$

$$L_2 = \left[\sum_{i,j,k} m_{ijk}^{xxy} x_j y_k + \sum_{i,j,k} m_{ijk}^{xyy} y_j y_k \right] \frac{\partial}{\partial x_i} +$$

$$\left[\sum_{i,j,k} m_{ijk}^{yxx} x_j x_k + \sum_{i,j,k} m_{ijk}^{yxy} x_j y_k \right] \frac{\partial}{\partial y_i}$$

Derivation of the Reduced Model

Compute $\bar{L} = \mathbb{P} L_2 L_1^{-1} L_2 \mathbb{P}$

$\mathbb{P}$ - IM of the fast subsystem on $E^{fast} = E - |x|^2$

L_1^{-1} - Shift fast variables in time; Integrate for all times

$$\bar{L} = \sum_i B_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j} \frac{\partial}{\partial x_i} D_{ij}(x) \frac{\partial}{\partial x_j}$$

Where

$$D_{ij}(x) = \mathcal{E}^{1/2}(x) \int_0^\infty dt \int d\mu_N(y) P_i(y(0)) P_j\,(y(t))$$

$$P_i(y) = \sum_{j,k} m_{ijk}^{xxy} x_j y_k + \mathcal{E}^{1/2}(x) \sum_{j,k} m_{ijk}^{xyy} y_j y_k$$

$$\mathcal{E}(x) := (E - |x|^2)/N$$

$$B_i(x) = -(1 - 2N^{-1}) \mathcal{E}^{-1}(x) \sum_j D_{ij}(x) x_j$$

Reduced Model for $\widehat{u}_1$

$$d\widehat{u}_1 = B(|\widehat{u}_1|)\widehat{u}_1 dt + \sigma(|\widehat{u}_1|)dW(t)$$

Additional Assumptions: Ergodicity, Structure of the Correlation Matrix, etc.

$$B(|\widehat{u}_1|^2) = 2\sqrt{\mathcal{E}} I_2 - \frac{I_2}{\sqrt{\mathcal{E}}}|\widehat{u}_1|^2 - \left[1 + \frac{2}{N}\right]\sqrt{\mathcal{E}} I_f$$

$$\sigma^2(|\widehat{u}_1|^2) = 2\sqrt{\mathcal{E}}|\widehat{u}_1|^2 I_2 + 2\left(\sqrt{\mathcal{E}}\right)^3 I_f$$

$$\mathcal{E} = \mathcal{E}(|\widehat{u}_1|^2) = (E - |\widehat{u}_1|^2)/N$$

where

E - Total Energy of the System
N - Number of Fast Modes
I_2 - Corr.Time($\widehat{u}_2$)
I_f - Corr.Time(RHS of $\widehat{u}_1$ projected onto Fast Modes)

Reduced Model for u_1

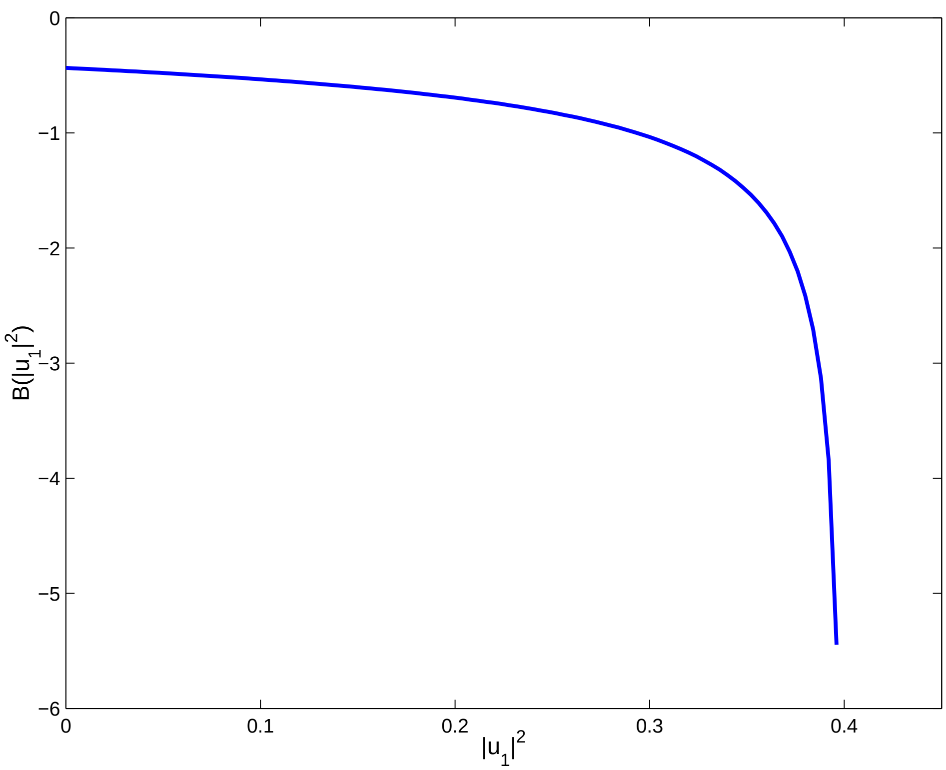
Need to Know: Moments of Fast Modes

Simplification: Can be Recast as CF of RHS $\hat{u}_1$

Approach: Compute Correlations from a single micro-canonical realization of the fast subsystem $\{y_k\}$

$$d\hat{u}_1 = B(|\hat{u}_1|)\hat{u}_1 dt + \sigma(|\hat{u}_1|)dW(t)$$

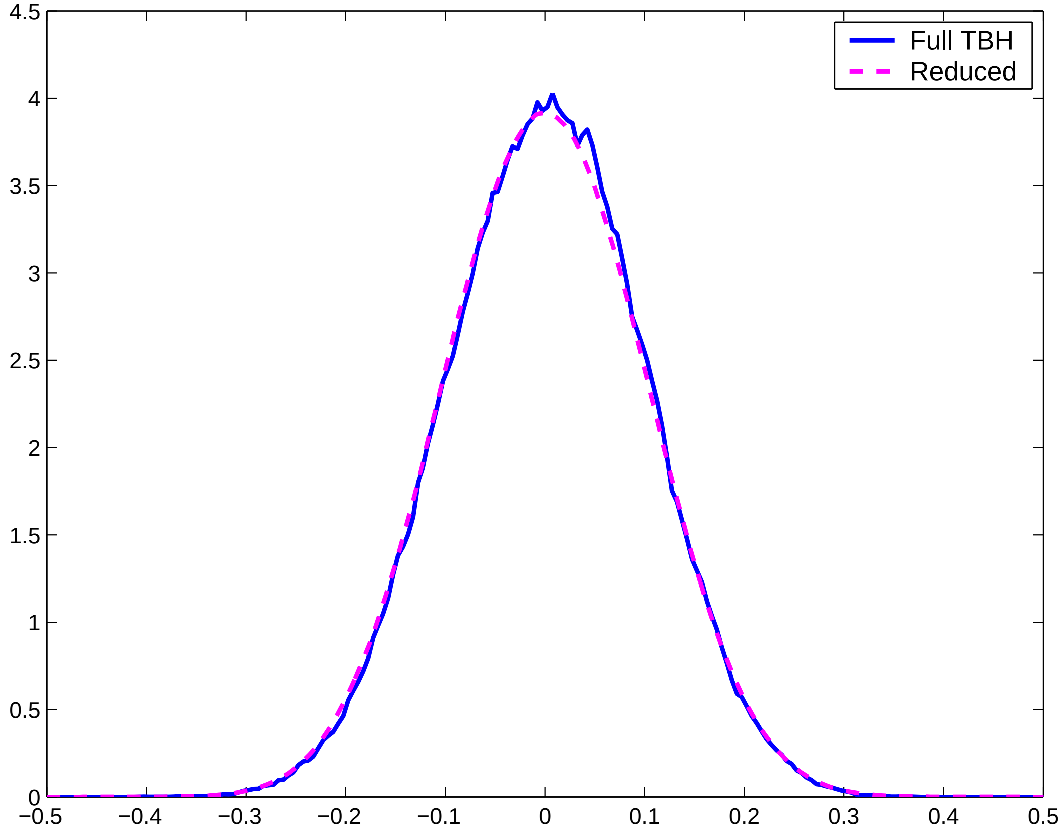
$$B = 2\sqrt{\mathcal{E}}I_2 - \frac{I_2}{\sqrt{\mathcal{E}}}|\hat{u}_1|^2 - \left[1 + \frac{2}{N}\right]\sqrt{\mathcal{E}}I_f$$



Drift Term $B(|\hat{u}_1|)$; Total Energy $E = 0.4$
$I_2 = 0.14$, $I_f = 4.3$

Reduced Model for $\hat{u}_1$

One–Point Statistics: Perfect Agreement



PDF of Re $\hat{u}_1$

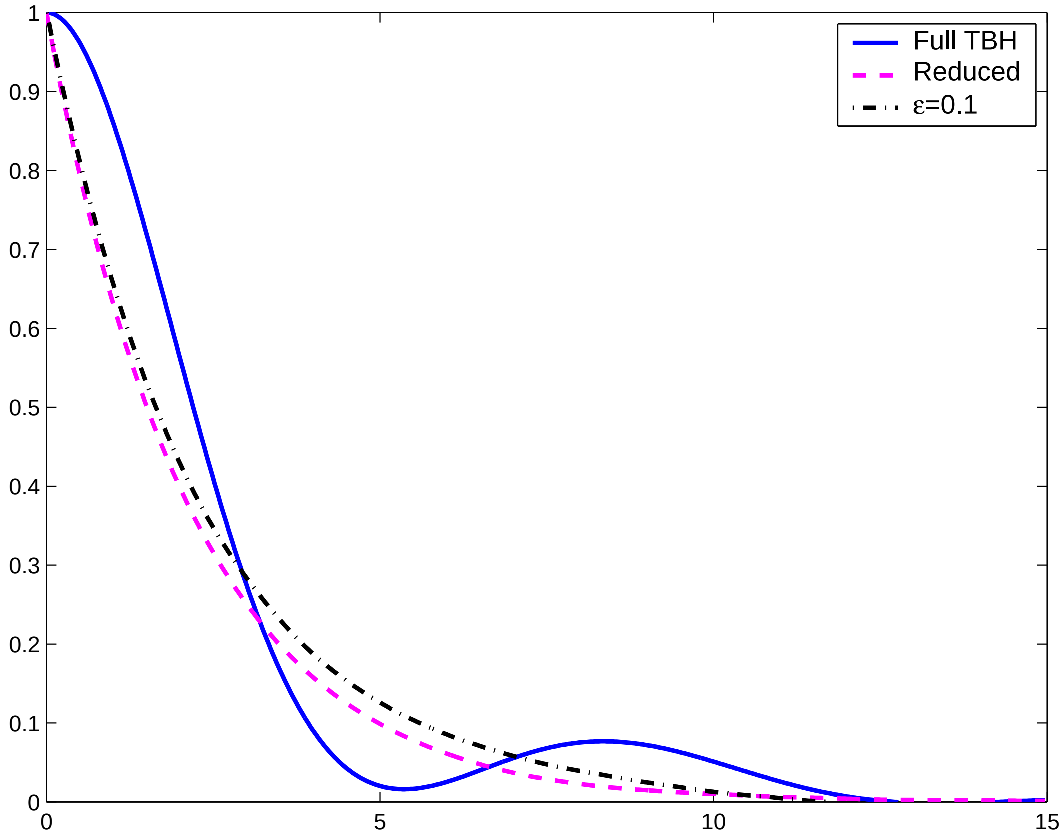
Blue - DNS with 20 Modes

Magenta - Reduced Equation

Reduced Model for $\hat{u}_1$

Two Point Statistics: Cannot Reproduce DNS Exactly

Analytical vs Numerical: Should be Identical; It's the same Limit $\varepsilon \longrightarrow 0$



Correlation Function of $\hat{u}_1$

Blue - DNS with 20 Modes

Magenta - Reduced Equation

Black - Modified Equation with $\varepsilon = 0.1$

$\widehat{u}_1$ and $\widehat{u}_2$ are the Slow Modes

Consider:

$$SLOW = \{\widehat{u}_1,\ \widehat{u}_2\}, \qquad FAST = \{\widehat{u}_3 \ldots \widehat{u}_\Lambda\}$$

Time-Scale Separation:

$$\frac{Corr.Time\{SLOW\}}{Corr.Time\{FAST\}} = \frac{3}{2}$$

Modified System:

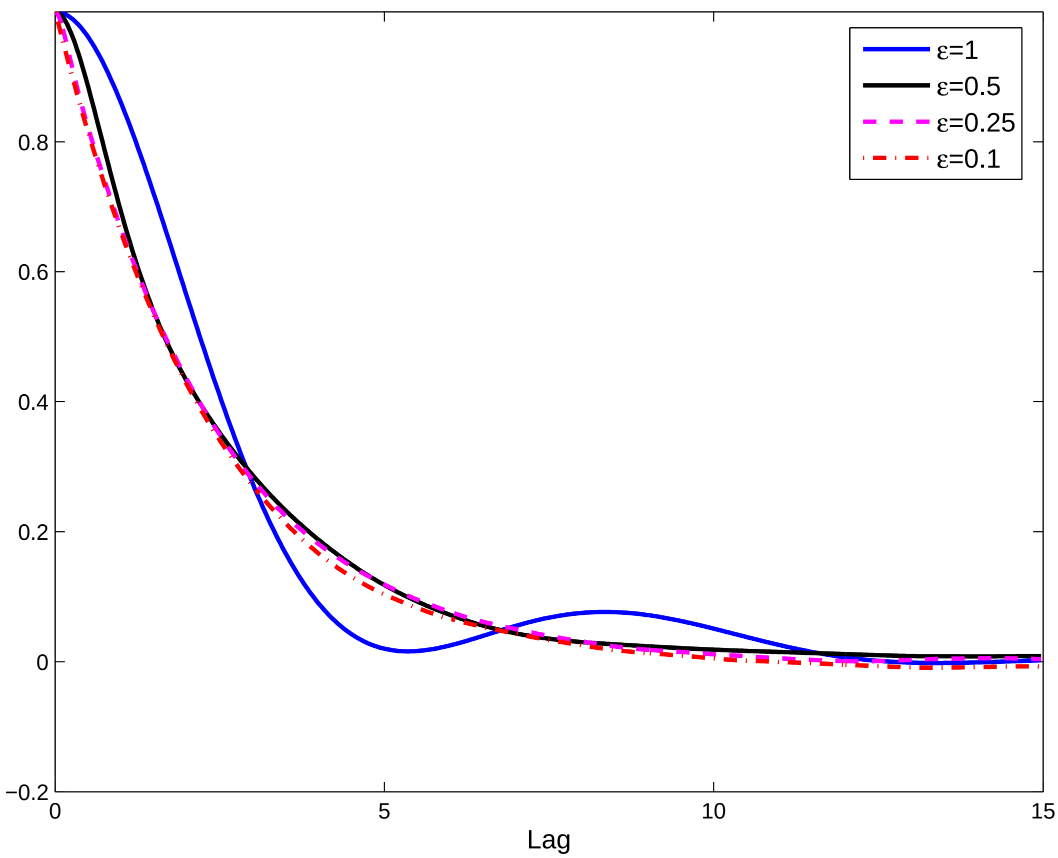
$$\begin{aligned}
\frac{d}{dt}\widehat{u}_1 &= -i\widehat{u}_2\widehat{u}_1^* - \frac{i}{2\varepsilon}\sum_{\substack{p+q+1=0\\ 3\le|p|,|q|\le\Lambda}} \widehat{u}_p^*\widehat{u}_q^*,\\
\frac{d}{dt}\widehat{u}_2 &= -i\widehat{u}_1^2 - \frac{i}{\varepsilon}\sum_{\substack{p+q+2=0\\ 3\le|p|,|q|\le\Lambda}} \widehat{u}_p^*\widehat{u}_q^*,\\
\frac{d}{dt}\widehat{u}_k &= -\frac{ik}{2\varepsilon}\left(\widehat{u}_{k+1}\widehat{u}_1^* + \widehat{u}_{k-1}\widehat{u}_1\right) - \frac{ik}{2\varepsilon}\left(\widehat{u}_{k+2}\widehat{u}_2^* + \widehat{u}_{k-2}\widehat{u}_2\right) -\\
&\quad -\frac{ik}{2\varepsilon^2}\sum_{\substack{p+q+k=0\\ 3\le|p|,|q|\le\Lambda}} \widehat{u}_p^*\widehat{u}_q^*
\end{aligned}$$

Numerical Approach

Question: Can we recover Bump Structure with More Modes?

Approach: Simulate Modified System with

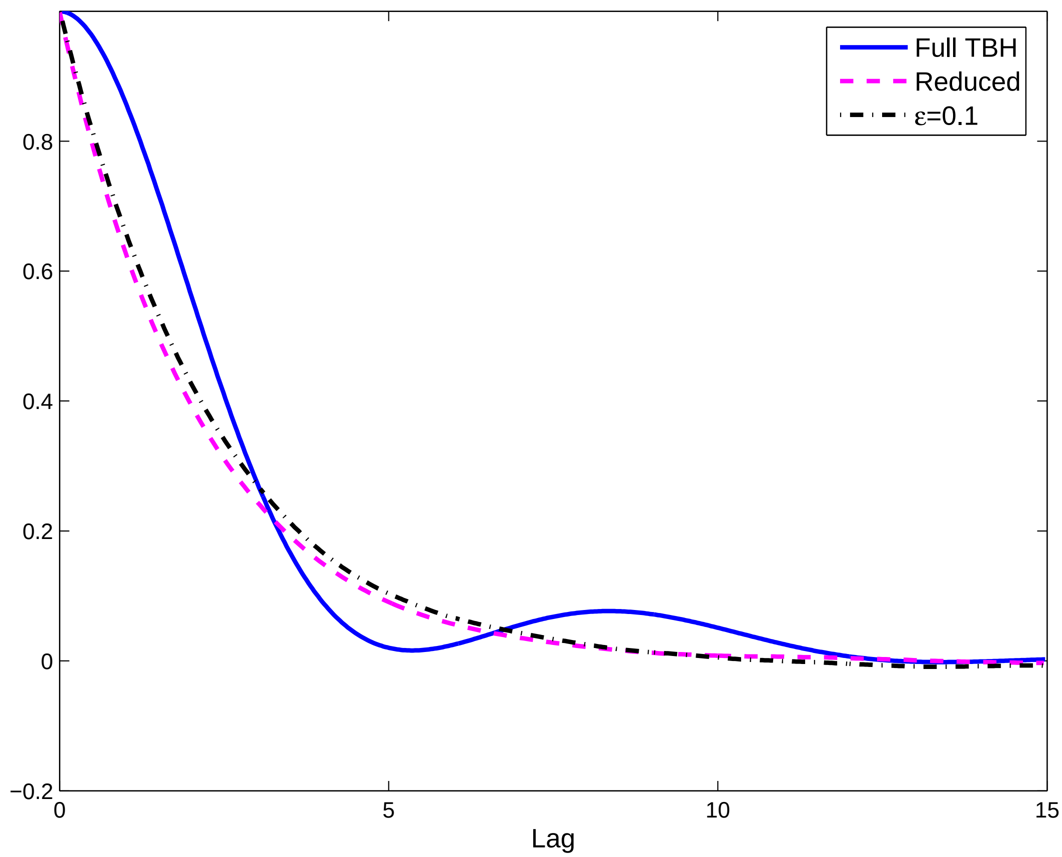
$$\varepsilon = 1,\ 0.5,\ 0.25,\ 0.1$$



Correlation Function of $Re\ \widehat{u}_1$

Reduced Model for $\hat{u}_1$ and $\hat{u}_2$

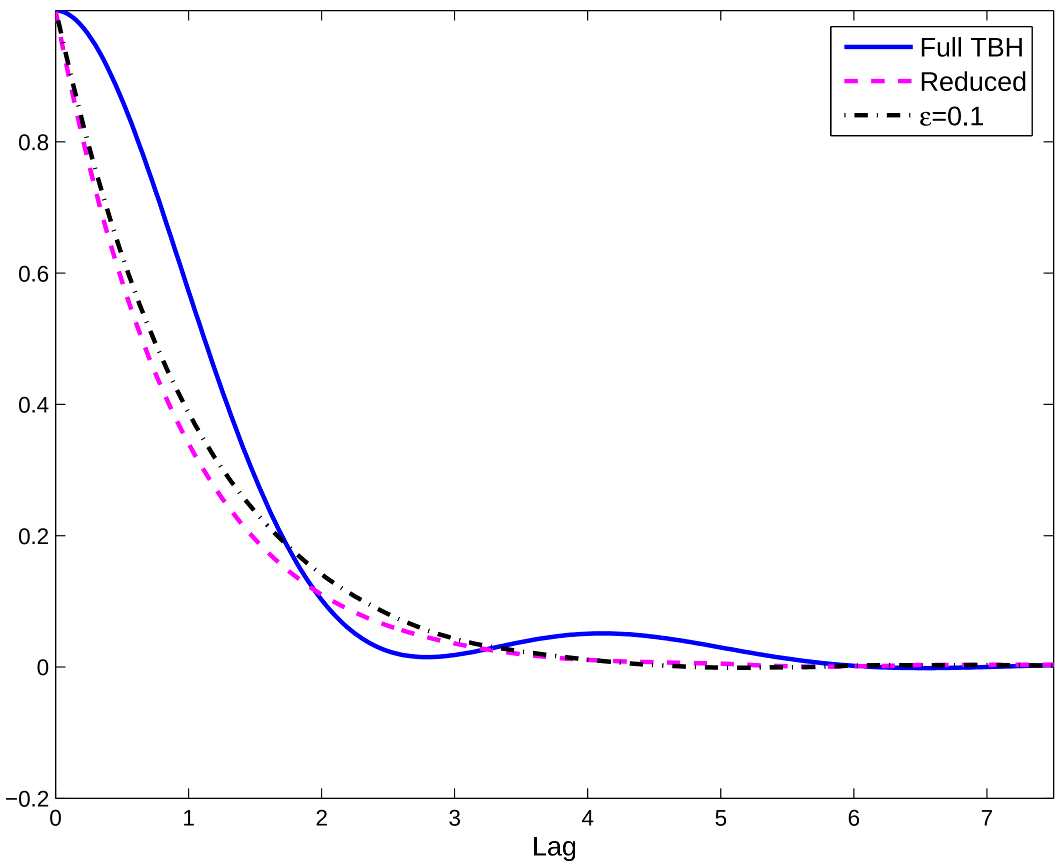
Analytical vs Numerical: Should be Identical; It's the same Limit $\varepsilon \longrightarrow 0$



Correlation Function of $\hat{u}_1$
Blue - DNS with 20 Modes
Magenta - Reduced Equation
Black - Modified Equation with $\varepsilon = 0.1$

Reduced Model for $\widehat{u}_1$ and $\widehat{u}_2$

Include More Modes: Discrepancies are Larger for $\widehat{u}_2$



Correlation Function of $\widehat{u}_2$

Blue - DNS with 20 Modes

Magenta - Reduced Equation

Black - Modified Equation with $\varepsilon = 0.1$

Multiplicative Noise vs Additive Noise

$$\dot{X} = \{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon}\{Y, Y\}$$

$$\dot{Y} = \frac{1}{\varepsilon}\{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon^2}\{Y, Y\}$$

Intuition: $Y \sim$ Noise

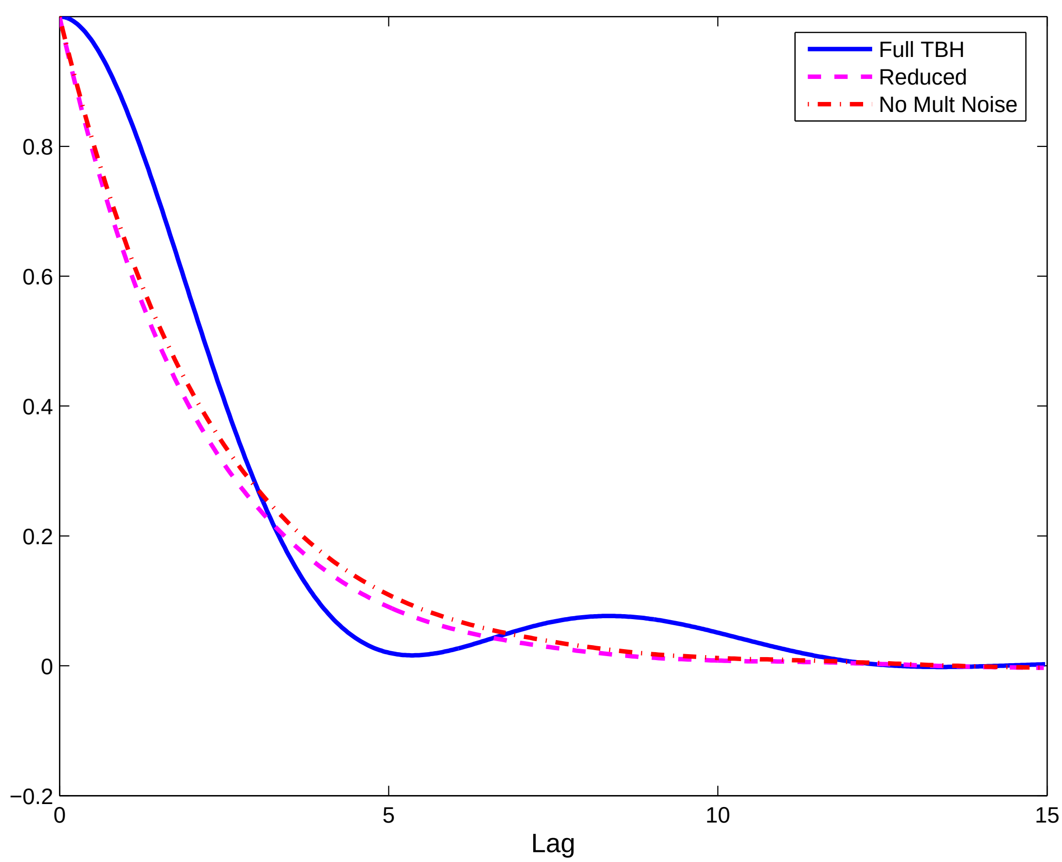
$\{Y, X\} \rightarrow$ Mult Noise $+$ Cubic Damping

$\{Y, Y\} \rightarrow$ Additive Noise $+$ Linear Damping

Reduced Model for $\widehat{u}_1$ and $\widehat{u}_2$

Additional Advantage: Can Analyze Balance of Various Terms

Mult. Noise: Contribution $\approx 6\%$



Correlation Function of $\widehat{u}_1$

Blue - DNS with 20 Modes

Magenta - Reduced Equation

Red - Reduced Equation without Mult. Noise Terms

Conclusions

- Effectively computable theory for deriving closed systems of reduced equations for slow variables
- No ad-hoc assumtions about the fast modes are necessary
- Parameters are Estimated from a single micro-canonical simulation
- Necessary Information can be recast in terms of correlation functions of the RHS
- Numerical and Analytical Approaches
  - Analyze the source of discrepancies
  - Better understanding of the Limit
  - Analyze Balance of Various Terms