Stochastic Mode–Reduction in Large Deterministic Systems

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Related Publications:
http://www.math.uh.edu/~ilya
Plan

- Mode-Elimination as a Limit of Infinite Separation of Time-Scales
- Mode-Elimination for Conservative Systems
- Example - Truncated Burgers-Hopf Equation
  - Numerical Verification of the Limiting Behavior
  - Conservative Mode-Elimination
  - Comparison with Direct Numerical Simulations
  - Balance of Terms
Essence of Mode-Reduction

Dynamical Variables:

\[ \dot{Z} = f(Z) \]

Decomposition:

\[ Z = (\text{Essential, Non-Essential}) = (SLOW, FAST) \]

Goal: Eliminate Fast modes; Derive Closed-Form equation for Slow Dynamics

Motivation: Interested Only in Statistical Behavior of the Slow Dynamics

Asymptotic Approach:

\[ \text{Limit } \frac{\text{Time Scale}\{\text{FAST}\}}{\text{Time Scale}\{\text{SLOW}\}} \rightarrow \infty \]
Rewrite the Original System

\[ \dot{Z} = f(Z) \]

Decomposition:

\[ Z = (SLOW, FAST) \equiv (X, Y) \]

Quadratic System:

\[ \begin{align*}
\dot{X} &= \{X, X\} + \{Y, X\} + \{Y, Y\} \\
\dot{Y} &= \{X, X\} + \{Y, X\} + \{Y, Y\}
\end{align*} \]

Conservation of Energy:

\[ \frac{d}{dt}E = \frac{d}{dt}(X^2 + Y^2) = 2X\dot{X} + 2Y\dot{Y} = \]

\[ \{X, X, X\} + \{X, X, Y\} + \{Y, Y, X\} + \{Y, Y, Y\} = 0 \]
Modify the Original System

**Main Idea:** Introduce $\varepsilon$ in the Equations

Preserve Conservation of Energy

**Modified System**

\[
\begin{align*}
\dot{X} &= \{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon}\{Y, Y\} \\
\dot{Y} &= \frac{1}{\varepsilon}\{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon^2}\{Y, Y\}
\end{align*}
\]

**Conservation of Energy:**

\[
\dot{E} = \{X, X, X\} + \frac{1}{\varepsilon}\{X, X, Y\} + \frac{1}{\varepsilon}\{Y, Y, X\} + \frac{1}{\varepsilon^2}\{Y, Y, Y\} = 0
\]

- $\varepsilon = 1$ Corresponds to the Original System
- Conserves Energy
- $\dot{Y}$ $\sim \{Y, Y\}$
- Numerical & Analytical Approaches
Truncated Burgers-Hopf Model

Fourier-Galerkin Projection of

\[ u_t + uu_x = 0 \]

onto a finite number of Fourier modes

\[ u = \sum \hat{u}_k e^{ikx}, \quad 1 \leq |k| \leq \Lambda \]

2\(\Lambda\)-dimensional system of ODEs

\[ \frac{d}{dt} \hat{u}_k = -\frac{ik}{2} \sum_{p+q+k=0} \hat{u}_p^* \hat{u}_q^* \]

with Reality Condition \( \hat{u}_k^* = \hat{u}_{-k} \)

Main Features

- Conservation of Energy \( \sum |\hat{u}_k|^2 \); Equipartition
- Correlation scaling \( Corr.Time\{\hat{u}_k\} \sim k^{-1} \)
- Gaussian distribution in the limit \( \Lambda \to \infty \)
- Hamiltonian \( H = \frac{1}{6} \int u_\Lambda^3 dx \)
\( \hat{u}_1 \) is the Slow Mode

Consider:

\[
SLOW = \hat{u}_1, \quad FAST = \{ \hat{u}_2 \ldots \hat{u}_\Lambda \}
\]

Time-Scale Separation:

\[
\frac{\text{Corr. Time}\{SLOW\}}{\text{Corr. Time}\{FAST\}} = 2
\]

Modified System:

\[
\begin{align*}
\frac{d}{dt} \hat{u}_1 &= -\frac{i}{2\varepsilon} \sum_{p+q+1=0, \ 2 \leq |p|, |q| \leq \Lambda} \hat{u}_p^* \hat{u}_q^* \\
\frac{d}{dt} \hat{u}_k &= -\frac{ik}{2\varepsilon} [\hat{u}_{k+1}^* \hat{u}_1^* + \hat{u}_{k-1} \hat{u}_1] - \frac{ik}{2\varepsilon^2} \sum_{k+p+q=0, \ 2 \leq |p|, |q| \leq \Lambda} \hat{u}_p^* \hat{u}_q^*
\end{align*}
\]
Numerical Approach

**Goal:** Verify Existence of the Reduced Dynamics

**Also:** Understand the “shape” of the Limit

**Approach:** Simulate Modified System with

\[ \varepsilon = 1, 0.5, 0.25, 0.1 \]
Numerical Approach

More Severe Test: Behavior of the Two-Point Statistics

$$\langle Re \hat{u}_1(t) Re \hat{u}_1(t + s) \rangle$$

$$\varepsilon = 1, 0.5, 0.25, 0.1$$

Corr. Function of $Re \hat{u}_1$

Corr. Time_{\varepsilon=1} = 2.64, \hspace{0.5cm} Corr. Time_{\varepsilon=0.1} = 2.4
Numerical Approach

**Question:** How fast the Bump Disappears

**Approach:** Simulate Modified System with

\[ \epsilon = 1, \; 0.9, \; 0.8, \; 0.6 \]
Analytical Approach

Equation:

\[ \dot{X} = \frac{1}{\varepsilon} f(X, Y) \]
\[ \dot{Y} = \frac{1}{\varepsilon^2} g(X, Y) \]

Effective Equation:

\[ dX = \tilde{a}(X) dt + \tilde{b}(X) dW \]

\[ \tilde{a} = \int_0^\infty dt \int d\mu(Y) f(X, Y(0)) \partial_X f(X, Y(t)) \]

\[ \tilde{b}^2 = \int_0^\infty dt \int d\mu(Y) f(X, Y(0)) f(X, Y(t)) \]
Overview of the Approach

Backward Equation

\[- \frac{\partial p^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2} L_1 p^\varepsilon + \frac{1}{\varepsilon} L_2 p^\varepsilon\]

Represent formally as Power Series

\[p^\varepsilon = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots\]

Collect terms and impose solvability condition

\[\frac{\partial p_0}{\partial s} = \mathbb{P} L_2 L_1^{-1} L_2 \mathbb{P} p_0\]

\[\mathbb{P} L_2 \mathbb{P} = 0\]
Derivation of the Reduced Model

Consider a general quadratic system of equations for the variables $x = \{x_i\}$ and $y = \{y_i\}$

\[
\begin{align*}
\dot{x}_i &= \varepsilon^{-1} \sum_{j,k} m_{ijk}^{xxy} x_j y_k + \varepsilon^{-1} \sum_{j,k} m_{ijk}^{xyy} y_j y_k \\
\dot{y}_i &= \varepsilon^{-1} \sum_{j,k} m_{ijk}^{yyx} x_j x_k + \varepsilon^{-1} \sum_{j,k} m_{ijk}^{yxy} y_j y_k + \varepsilon^{-2} \sum_{j,k} m_{ijk}^{yyy} y_j y_k
\end{align*}
\]

Volume Preserving $\Rightarrow$ Consider Liouville Equation

\[-\frac{\partial p^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2} L_1 p^\varepsilon + \frac{1}{\varepsilon} L_2 p^\varepsilon\]

\[
L_1 = \sum_{i,j,k} m_{ijk}^{yyy} y_j y_k \frac{\partial}{\partial y_i}
\]

\[
L_2 = \left[ \sum_{i,j,k} m_{ijk}^{xxy} x_j y_k + \sum_{i,j,k} m_{ijk}^{xyy} y_j y_k \right] \frac{\partial}{\partial x_i} + \\
\left[ \sum_{i,j,k} m_{ijk}^{yyx} x_j x_k + \sum_{i,j,k} m_{ijk}^{yxy} y_j y_k \right] \frac{\partial}{\partial y_i}
\]
Derivation of the Reduced Model

Compute
\[ \bar{L} = P L_2 L_1^{-1} L_2 P \]

\( P \) - IM of the fast subsystem on \( E^{fast} = E - |x|^2 \)

\( L_1^{-1} \) - Shift fast variables in time; Integrate for all times

\[ \bar{L} = \sum_i B_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j} \frac{\partial}{\partial x_i} D_{ij}(x) \frac{\partial}{\partial x_j} \]

Where

\[ D_{ij}(x) = \mathcal{E}^{1/2}(x) \int_0^\infty dt \int d\mu_N(y) P_i(y(0)) P_j(y(t)) \]

\[ P_i(y) = \sum_{j,k} m_{ijk} x_j y_k + \mathcal{E}^{1/2}(x) \sum_{j,k} m_{ijk} x_j y_k \]

\[ \mathcal{E}(x) := (E - |x|^2)/N \]

\[ B_i(x) = -(1 - 2N^{-1}) \mathcal{E}^{-1}(x) \sum_j D_{ij}(x) x_j \]
Reduced Model for $\hat{u}_1$

$$d\hat{u}_1 = B(|\hat{u}_1|)\hat{u}_1 dt + \sigma(|\hat{u}_1|)dW(t)$$

**Additional Assumptions:** Ergodicity, Structure of the Correlation Matrix, etc.

$$B(|\hat{u}_1|^2) = 2\sqrt{\mathcal{E}} I_2 - \frac{I_2}{\sqrt{\mathcal{E}}} |\hat{u}_1|^2 - \left[ 1 + \frac{2}{N} \right] \sqrt{\mathcal{E}} I_f$$

$$\sigma^2(|\hat{u}_1|^2) = 2\sqrt{\mathcal{E}} |\hat{u}_1|^2 I_2 + 2 \left( \sqrt{\mathcal{E}} \right)^3 I_f$$

$$\mathcal{E} = \mathcal{E}(|\hat{u}_1|^2) = (E - |\hat{u}_1|^2)/N$$

where

$E$ - Total Energy of the System
$N$ - Number of Fast Modes
$I_2$ - Corr.Time($\hat{u}_2$)
$I_f$ - Corr.Time(RHS of $\hat{u}_1$ projected onto Fast Modes)
Reduced Model for $u_1$

**Need to Know:** Moments of Fast Modes

**Simplification:** Can be Recast as CF of RHS $\hat{u}_1$

**Approach:** Compute Correlations from a single micro-canonical realization of the fast subsystem $\{y_k\}$

\[
d\hat{u}_1 = B(|\hat{u}_1|)\hat{u}_1 dt + \sigma(|\hat{u}_1|)dW(t)
\]

\[
B = 2\sqrt{\varepsilon} I_2 - \frac{I_2}{\sqrt{\varepsilon}}|\hat{u}_1|^2 - \left[1 + \frac{2}{N}\right]\sqrt{\varepsilon} I_f
\]

Drift Term $B(|\hat{u}_1|)$; Total Energy $E = 0.4$

$I_2 = 0.14$, $I_f = 4.3$
Reduced Model for $\hat{u}_1$

One–Point Statistics: Perfect Agreement

PDF of $\text{Re}\,\hat{u}_1$
Blue - DNS with 20 Modes
Magenta - Reduced Equation
Reduced Model for $\hat{u}_1$

**Two Point Statistics:** Cannot Reproduce DNS Exactly

**Analytical vs Numerical:** Should be Identical; It’s the same Limit $\varepsilon \to 0$
\( \hat{u}_1 \) and \( \hat{u}_2 \) are the Slow Modes

Consider:

\[
SLOW = \{ \hat{u}_1, \hat{u}_2 \}, \quad FAST = \{ \hat{u}_3 \ldots \hat{u}_\Lambda \}
\]

Time-Scale Separation:

\[
\frac{Corr.\, Time\{SLOW\}}{Corr.\, Time\{FAST\}} = \frac{3}{2}
\]

Modified System:

\[
\begin{aligned}
\frac{d}{dt} \hat{u}_1 &= -i\hat{u}_2 \hat{u}_1^* - \frac{i}{2\varepsilon} \sum_{\substack{p+q+1=0 \\ 3\leq|p|,|q|\leq\Lambda}} \hat{u}_p^* \hat{u}_q^*, \\
\frac{d}{dt} \hat{u}_2 &= -i\hat{u}_2 - \frac{i}{\varepsilon} \sum_{\substack{p+q+2=0 \\ 3\leq|p|,|q|\leq\Lambda}} \hat{u}_p^* \hat{u}_q^*, \\
\frac{d}{dt} \hat{u}_k &= -\frac{ik}{2\varepsilon} (\hat{u}_{k+1} \hat{u}_1^* + \hat{u}_{k-1} \hat{u}_1) - \frac{ik}{2\varepsilon} (\hat{u}_{k+2} \hat{u}_2^* + \hat{u}_{k-2} \hat{u}_2) - \frac{ik}{2\varepsilon^2} \sum_{\substack{p+q+k=0 \\ 3\leq|p|,|q|\leq\Lambda}} \hat{u}_p^* \hat{u}_q^*
\end{aligned}
\]
Numerical Approach

**Question:** Can we recover Bump Structure with More Modes?

**Approach:** Simulate Modified System with

\[ \varepsilon = 1, 0.5, 0.25, 0.1 \]

Correlation Function of \( Re \hat{u}_1 \)
Reduced Model for $\hat{u}_1$ and $\hat{u}_2$

**Analytical vs Numerical:** Should be Identical; It’s the same Limit $\varepsilon \to 0$

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The diagram shows the correlation function of $\hat{u}_1$. The lines are labeled as follows:
- **Blue** - DNS with 20 Modes
- **Magenta** - Reduced Equation
- **Black** - Modified Equation with $\varepsilon = 0.1$
Reduced Model for $\hat{u}_1$ and $\hat{u}_2$

Include More Modes: Discrepancies are Larger for $\hat{u}_2$

Correlation Function of $\hat{u}_2$
- Blue - DNS with 20 Modes
- Magenta - Reduced Equation
- Black - Modified Equation with $\varepsilon = 0.1$
Multiplicative Noise vs Additive Noise

\[
\begin{align*}
\dot{X} &= \{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon}\{Y, Y\} \\
\dot{Y} &= \frac{1}{\varepsilon}\{X, X\} + \frac{1}{\varepsilon}\{Y, X\} + \frac{1}{\varepsilon^2}\{Y, Y\}
\end{align*}
\]

Intuition: \( Y \sim \text{Noise} \)

\[\{Y, X\} \rightarrow \text{Mult Noise } + \text{ Cubic Damping}\]

\[\{Y, Y\} \rightarrow \text{Additive Noise } + \text{ Linear Damping}\]
Reduced Model for $\tilde{u}_1$ and $\tilde{u}_2$

**Additional Advantage:** Can Analyze Balance of Various Terms

**Mult. Noise:** Contribution $\approx 6\%$

Correlation Function of $\tilde{u}_1$
- Blue - DNS with 20 Modes
- Magenta - Reduced Equation
- Red - Reduced Equation without Mult. Noise Terms
Conclusions

- Effectively computable theory for deriving closed systems of reduced equations for slow variables

- No ad-hoc assumptions about the fast modes are necessary

- Parameters are Estimated from a single micro-canonical simulation

- Necessary Information can be recast in terms of correlation functions of the RHS

- Numerical and Analytical Approaches
  - Analyze the source of discrepancies
  - Better understanding of the Limit
  - Analyze Balance of Various Terms