Reduced modeling with optimal bases

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Optimal bases

Idea: expand model PDE on a basis that is more efficient than a spectral (or gridpoint) basis.

Spectral model can describe many flow configurations that are never realised

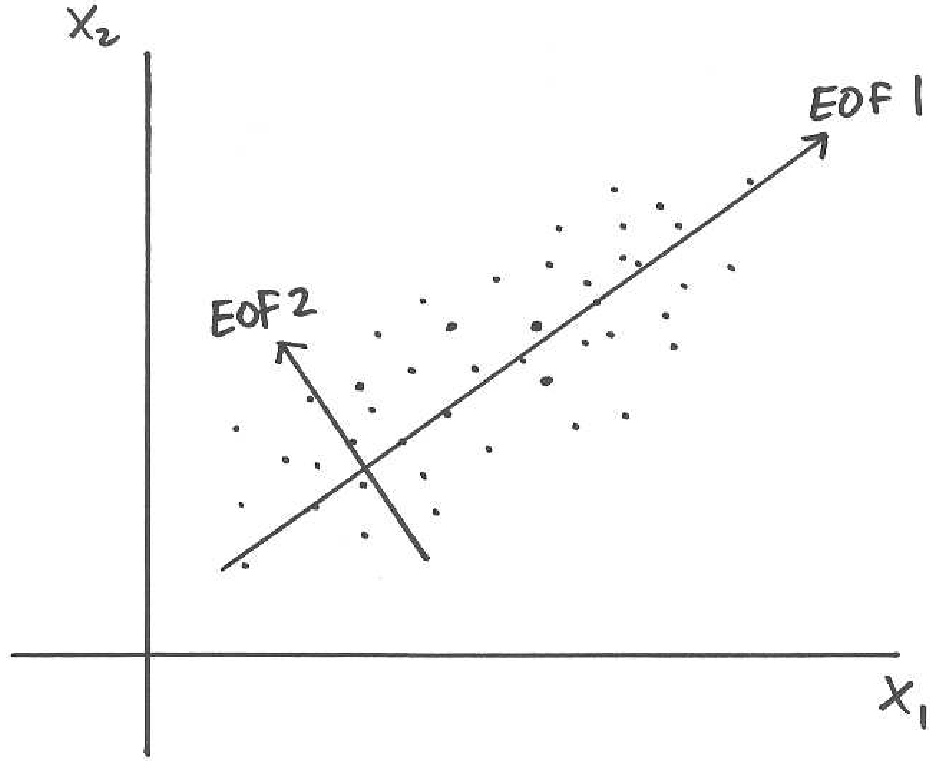
PDE may have low-dimensional attractor

EOFs

Best known and most used example of optimal bases:
Empirical Orthogonal Functions (EOFs),
(a.k.a. Proper Orthogonal Decomposition (POD)
or Karhunen-Loève expansion)

Atmospheric science: Lorenz 1956, Rinne and Karhilla 1975, Schubert 1985,1986, Selten 1993, 1995, 1997, Achatz and Branstator 1999, Achatz and Opsteegh 2003a,b, Franzke et al. 2005, 2006, ...

Fluid dynamics: Lumley 1967,1981, Aubry et al. 1988, Sirovich 1989, Cazemier et al. 1998, ...



Given a dataset, EOF 1 is the direction of maximal variance in phase space

Variance: **metric dependent**

Calculation of EOFs

Covariance matrix of dataset $\{\mathbf{x}(t)\}$: $C_{ij} = \overline{(x_i - \bar{x}_i)(x_j - \bar{x}_j)}$

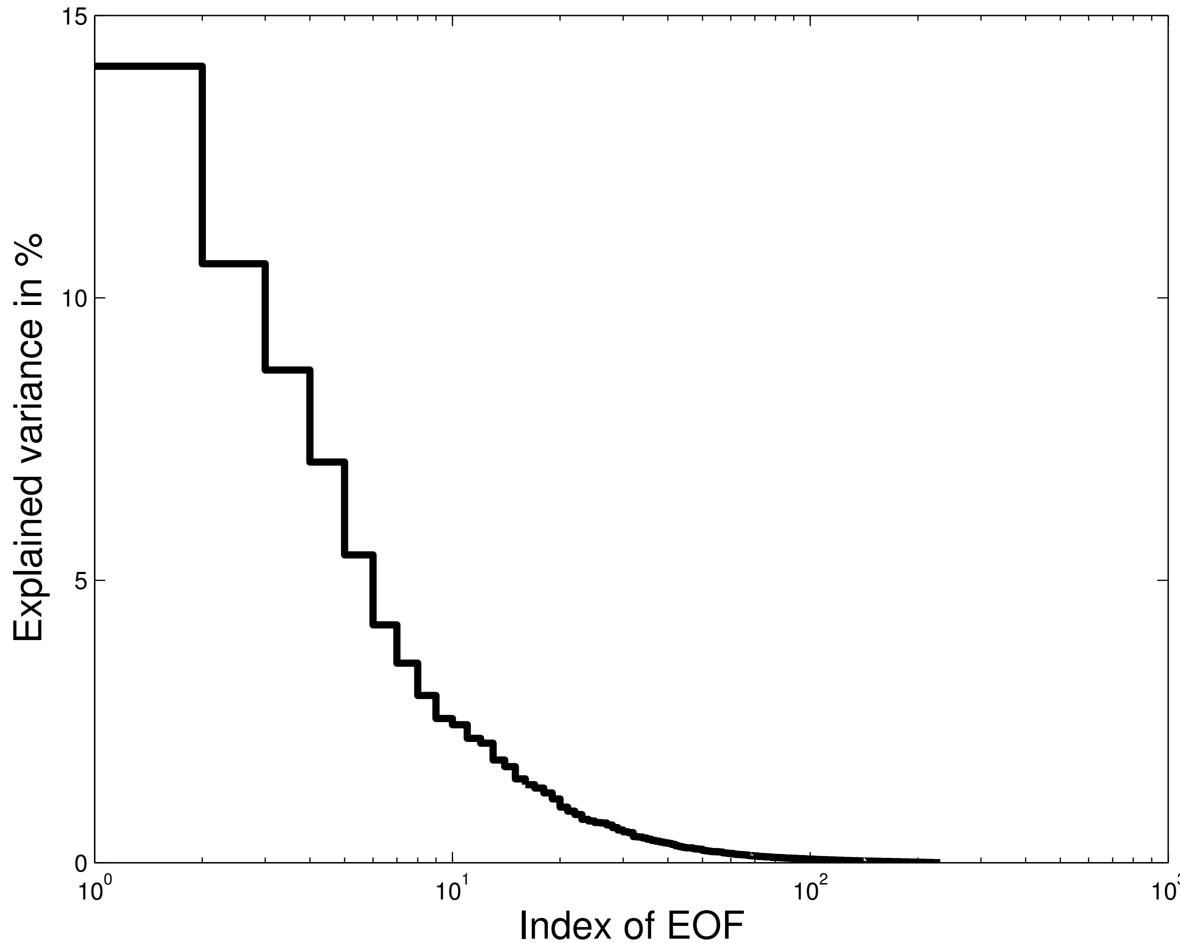
Given metric M, solve eigenvalue problem $\boxed{C\,M\,\mathbf{p}^i = \lambda_i\,\mathbf{p}^i}$

$\mathbf{p}^i$ is i-th EOF with associated variance λ_i, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$

Total variance: $\sum_{i=1}^{n} \overline{x_i^2} = \sum_{i=1}^{n} \lambda_i$, fractional variance: $\dfrac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$

Maximum variance captured in minimum number of patterns.

Example of variance spectrum (from Franzke et al. 2005)



EOF expansion: $\mathbf{x}(t) = \bar{\mathbf{x}} + \sum_{i=1}^{n} a_i(t)\mathbf{p}^i$

Project model $\dot{\mathbf{x}} = F(\mathbf{x})$ onto EOFs to obtain EOF model $\dot{\mathbf{a}} = G(\mathbf{a})$

Truncate: $\mathbf{a} = (\mathbf{b}, \mathbf{c}) \rightarrow \begin{cases} \dot{\mathbf{b}} = G^1(\mathbf{b}) + G^2(\mathbf{b}, \mathbf{c}) \\ \dot{\mathbf{c}} = G^3(\mathbf{b}, \mathbf{c}) \end{cases}$

Closure scheme: $\dot{\mathbf{b}} = G^1(\mathbf{b}) + G^4(\mathbf{b})$
(additional damping, regression fitting, stochastic, ...)

EOF models have difficulties reproducing bursting-type behaviour

Aubry et al. (1993):
6 EOFs representing 99.9995 % of the variance could **not** reproduce the right behaviour (Kuramoto-Sivashinsky eqn.)

Comparing different bases

Model: De Swart (1989), equivalent to Charney and DeVore (1979)

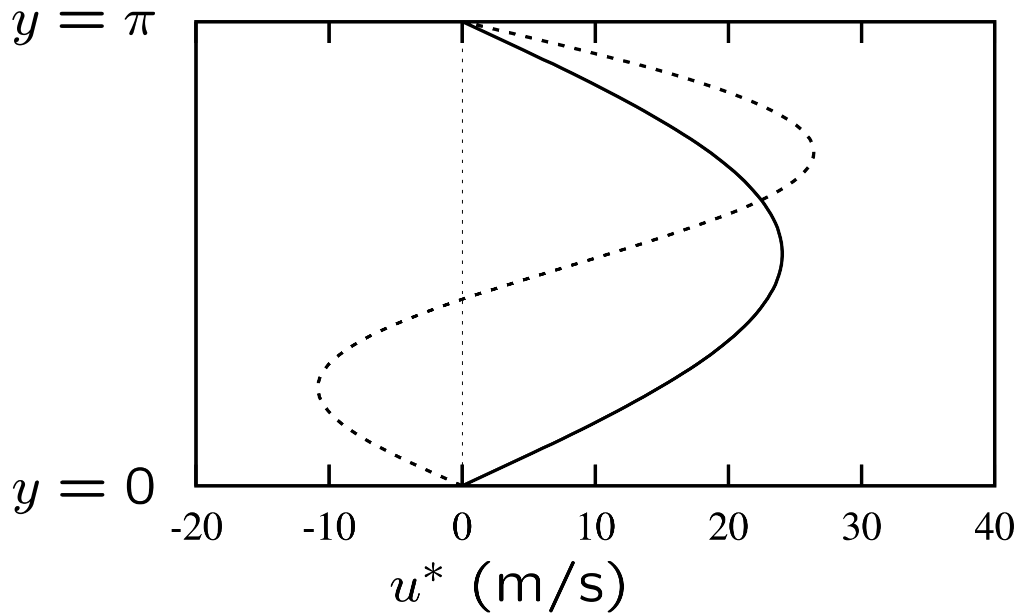
A channel model for midlatitude barotropic flow over topography, expanded on a 6-dim. Fourier basis. It includes:

β-effect
topography
Ekman friction
constant zonal forcing

Variables:
x_1, x_4 zonal modes [(0,1) and (0,2)]
x_2, x_3, x_5, x_6 wave modes [(1,1) and (1,2)]

Important parameters: strength (x_1^*) and shape (r) of the forcing.

Zonal wind speed forcing profile $u^*(y) \sim x_1^* \, (\sin(y) + 2r \sin(2y))$.



Other parameters:

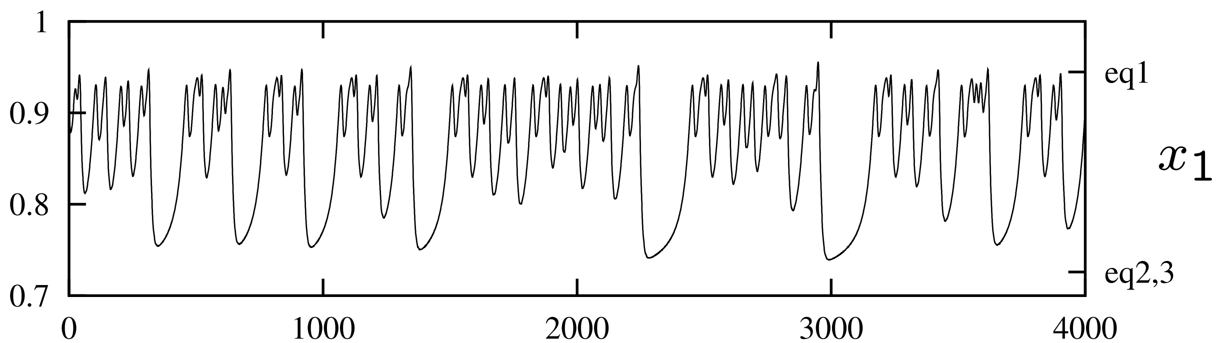
Ekman damping timescale: 10 days

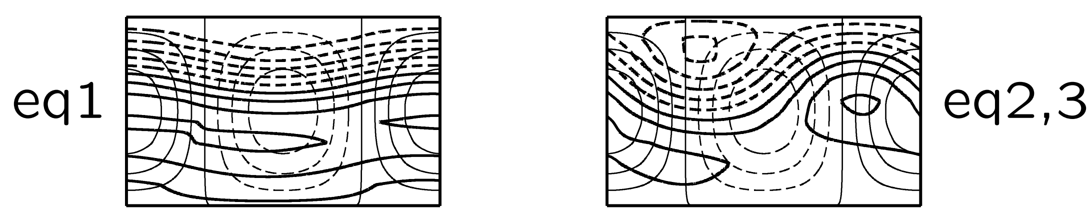
Topographic height: 0.2 km

Length-width ratio of the channel: 4

Central latitude of β-plane: 45°

For right choice of parameters x_1^* and r:





(Crommelin et al. 2003)

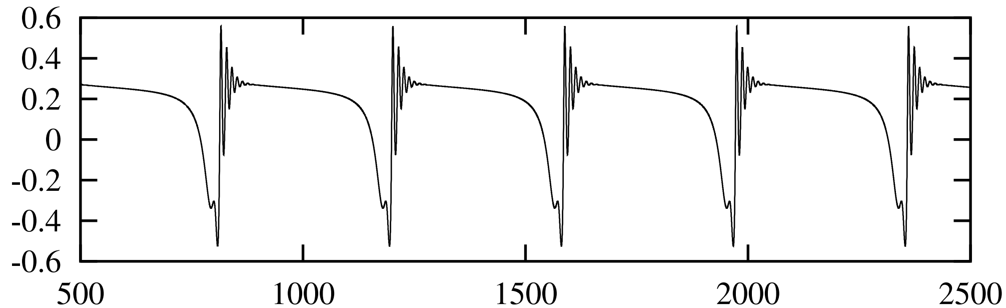
EOF variance spectrum:

No. of EOF	Variance	Cumulative variance
1	0.660	0.660
2	0.287	0.947
3	0.041	0.988
4	0.006	0.994
5	0.004	0.998
6	0.002	1.000

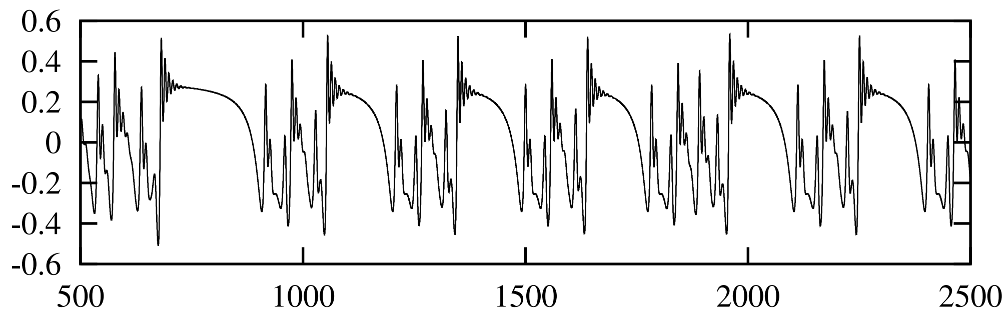
(kinetic energy metric)

All EOF models fail

4 EOF model version (99.4 % variance):



CDV model:



Optimal Persistence Patterns (OPPs)

OPPs are patterns that maximize **persistence** (DelSole 2001)

Take autocorrelation function $\rho(\tau)$ of a pattern.
Define persistence measures:

$$T_1 = \int_0^\infty \rho(\tau)\, \mathrm{d}\tau \quad \text{and} \quad T_2 = 2 \int_0^\infty \rho^2(\tau)\, \mathrm{d}\tau$$

Find (orthogonal) patterns that maximize T_1 (or T_2)

$T_1 \longrightarrow$ eigenvalue problem
$T_2 \longrightarrow$ optimization problem

OPP model versions of CDV model.

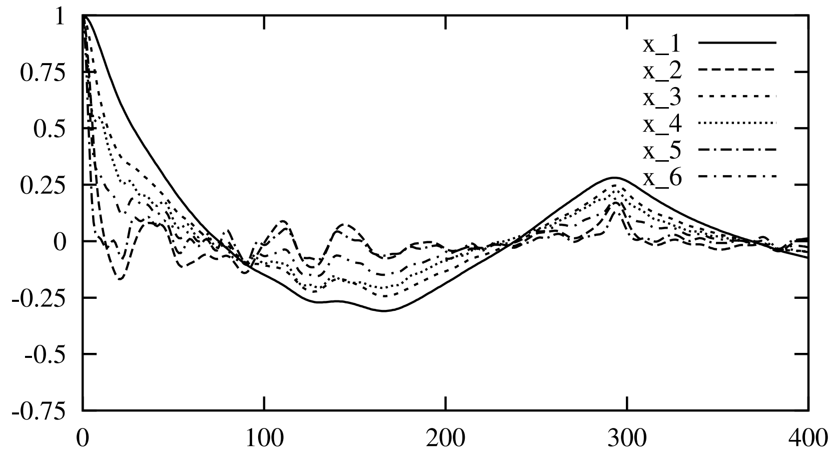
Using T_2, patterns orthogonal in time:

No. of OPP	T_2
1	92.8
2	15.3
3	13.8
4	7.5
5	6.3
6	5.7

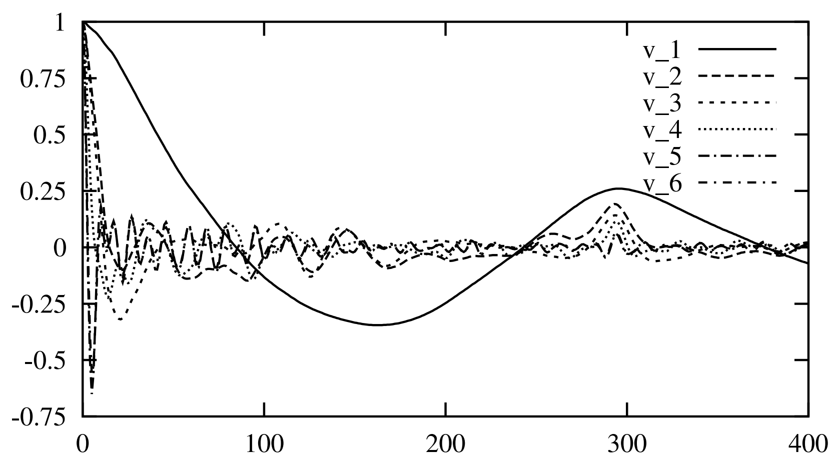
Behaviour of OPP models: unbounded (2, 4), fixed point (3), fixed point / periodic (5).

Autocorrelation functions:

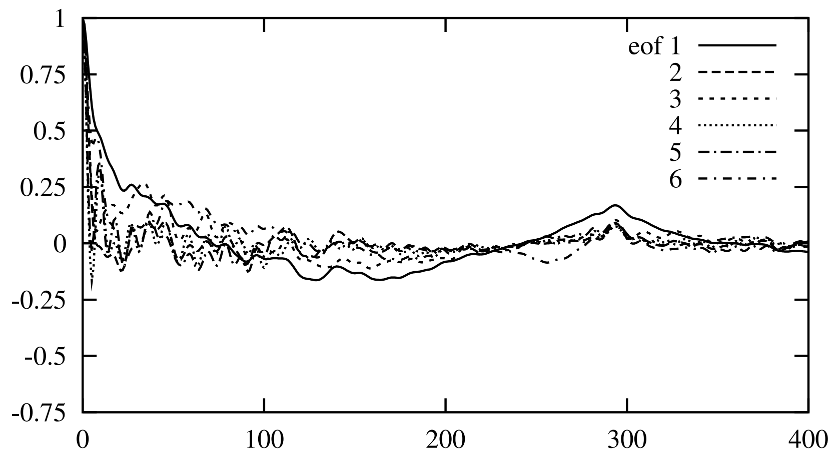
CDV model variables:



OPPs:



EOFs:



Principal Interaction Patterns (PIPs)

PIPs minimize average error in (brief) model **trajectories**

Proposed by Hasselmann 1988, refined by Kwasniok 1996, 1997, ...

Original model: $\dot{x} = F(x)$
Reduced model: $\dot{x}_\mathsf{p} = F_\mathsf{p}(x_\mathsf{p})$ (given a set of patterns P)

Integration gives model trajectories x^t and x^t_p, with $t \in [0, t_\mathsf{max}]$
and $x^0_\mathsf{p} = \mathsf{Proj}(x^0, P)$

Difference: $d^t = x^t_\mathsf{p} - x^t$

Define $Q = \dfrac{1}{\text{var } t_{\text{max}}} \int_0^{t_{\text{max}}} \|d^t\|^2 \, \text{d}t$

Ensemble average: $\chi(t_{\text{max}}, P) = \langle Q(x^0, t_{\text{max}}, P) \rangle$

Finding PIPs: minimize χ under variation of the set of patterns P

t_{max} remains a free parameter

PIPs and EOFs

Projected error: $\varepsilon_i^t = [\mathbf{p}_i^*, d^\tau]$

Two contributions: $\boxed{\chi(t_{\mathsf{max}}, P) = \chi^{\mathsf{0}}(P) + \chi^{\mathsf{dyn}}(t_{\mathsf{max}}, P)}$

with
$$\begin{cases} \chi^{\mathsf{0}}(P) = 1 - \dfrac{1}{\mathsf{var}} \displaystyle\sum_i \lambda_i^{\mathsf{PIP}} & \text{(projection error)} \\ \\ \chi^{\mathsf{dyn}}(t_{\mathsf{max}}, P) = \dfrac{1}{\mathsf{var}\, t_{\mathsf{max}}} \displaystyle\int_0^{t_{\mathsf{max}}} \sum_i \langle (\varepsilon_i^t)^2 \rangle \mathsf{d}t & \text{(dynamical error)} \end{cases}$$

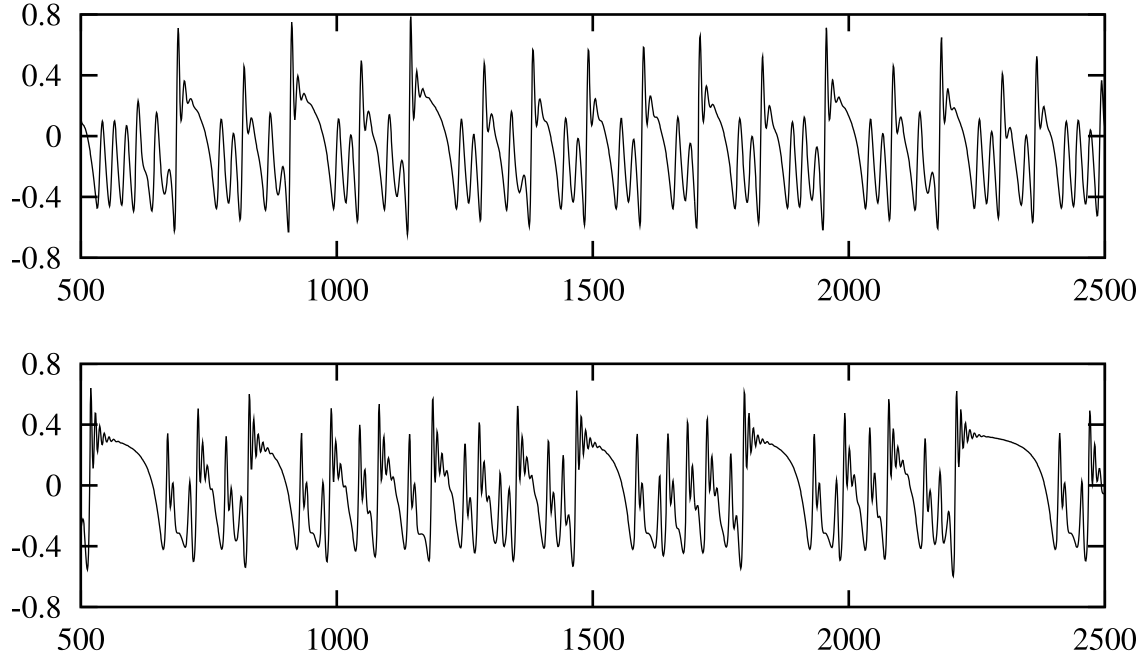
if $\quad t_{\mathsf{max}} = 0: \quad \chi = \chi^{\mathsf{0}}$, PIPs = EOFs

Chaotic regime behaviour of CDV model can be reproduced with models using 5, 4 and 3 PIPs (Crommelin and Majda, 2003)

Sensitive dependence on t_{max} and number of integrations.

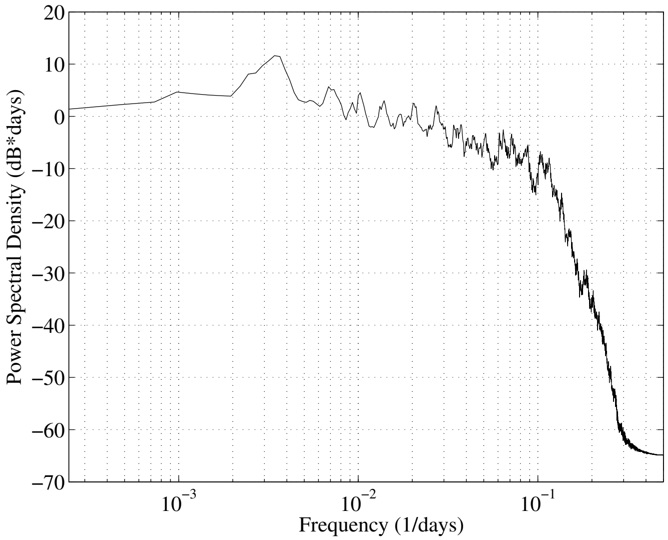
PIPs superior to EOFs and OPPs

3 PIP model

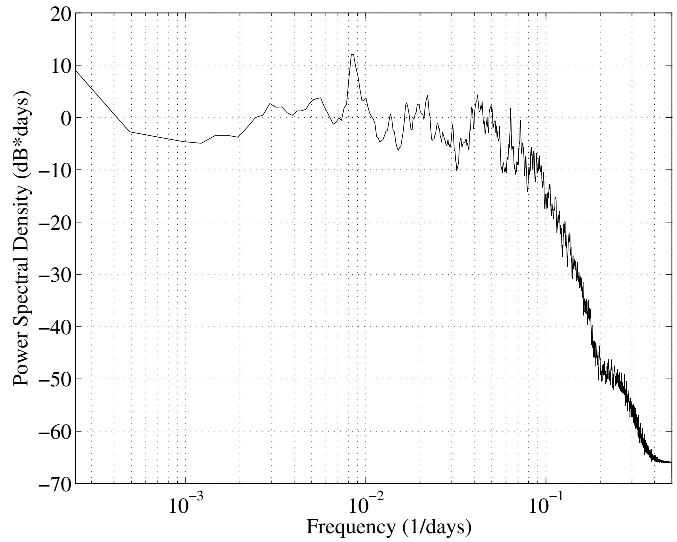


CDV model

Power spectrum of leading pattern:



CDV model



3 PIP model