Stochastic Stability of Simple Climate Models

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Outline

- **Introduction**

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- **Stochastic Stability I: Multiple Equilibria**

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- **Stochastic Stability II: Lyapunov Exponents**

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- Introduction
- Stochastic Stability I: Multiple Equilibria
- Stochastic Stability II: Lyapunov Exponents
- Conclusions

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- Stability properties of climate states important for determining:

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  - climate variability

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  - climate predictability

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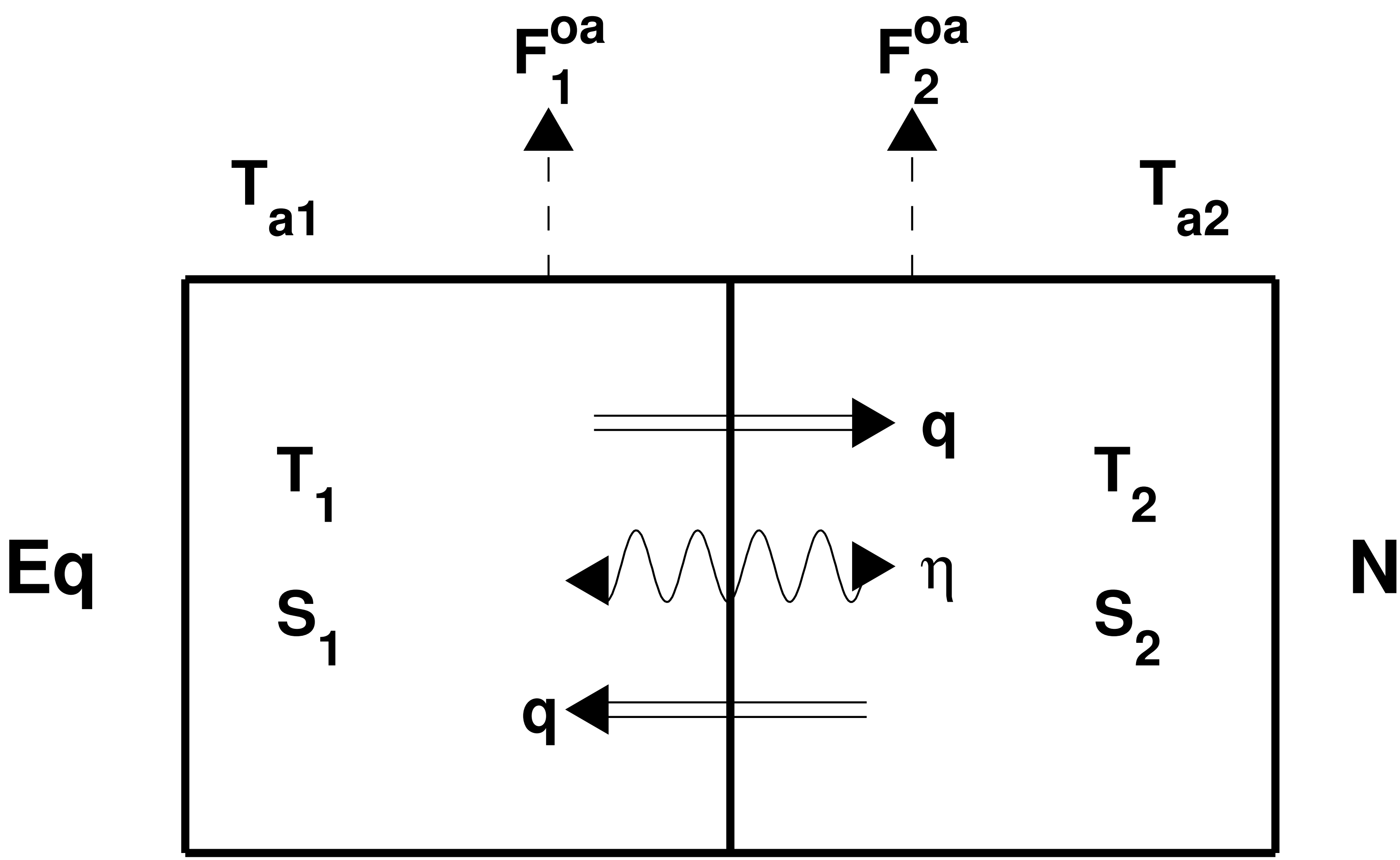
  - Two point: do initially nearby trajectories approach or diverge as time passes?
- Both of these are well-understood for deterministic systems; (ubiquitous) environmental fluctuations introduce new effects
- Will consider these in the context of simple climate models

Idealised Model: Stommel (61)

Idealised 2-box model of overturning circulation



Stommel Model: Equations

- Overturning strength $\propto$ density gradient

$$q = c(\alpha \Delta T - \beta \Delta S)$$

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$$q = c(\alpha\Delta T - \beta\Delta S)$$

- Dynamics of temperature and salinity gradients:

$$\begin{aligned} \frac{d}{dt}\Delta T &= -(|q|+\eta)\Delta T + \Gamma(\Delta T_a - \Delta T) \\ \frac{d}{dt}\Delta S &= -(|q|+\eta)\Delta S + \Delta F^{oa} \\ \frac{d}{dt}\eta &= -\frac{1}{\tau}\eta + \frac{\Sigma}{\tau}\dot{W}_1 \end{aligned}$$

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Stommel Model: Simplifications

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$$\begin{aligned} \dot{y} &= -|1-y|y - \eta y + \mu + \sigma_2 \dot{W}_2 \\ \dot{\eta} &= -\frac{1}{\tau}\eta + \frac{\sigma_1}{\tau}\dot{W}_1 \end{aligned}$$

$$\begin{aligned} y &= \text{salinity gradient} \\ \eta &= \text{mixing} \\ \mu &= \text{freshwater forcing} \\ \sigma_1, \sigma_2 &= \text{mixing, forcing fluctuation strength} \end{aligned}$$

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Stommel Model: Deterministic Dynamics

- Deterministic system:

$$\frac{d}{dt}\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} -|1-y|y - \eta y + \mu \\ -\eta/\tau \end{pmatrix}$$

has

$$\begin{array}{rl} \text{3 fixed points for} & 0 \leq \mu \leq 0.25 \\ \text{1 fixed point for} & 0 > \mu, \mu > 0.25 \end{array}$$

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- Fixed points meet in fold bifurcations
- System displays hysteresis behaviour

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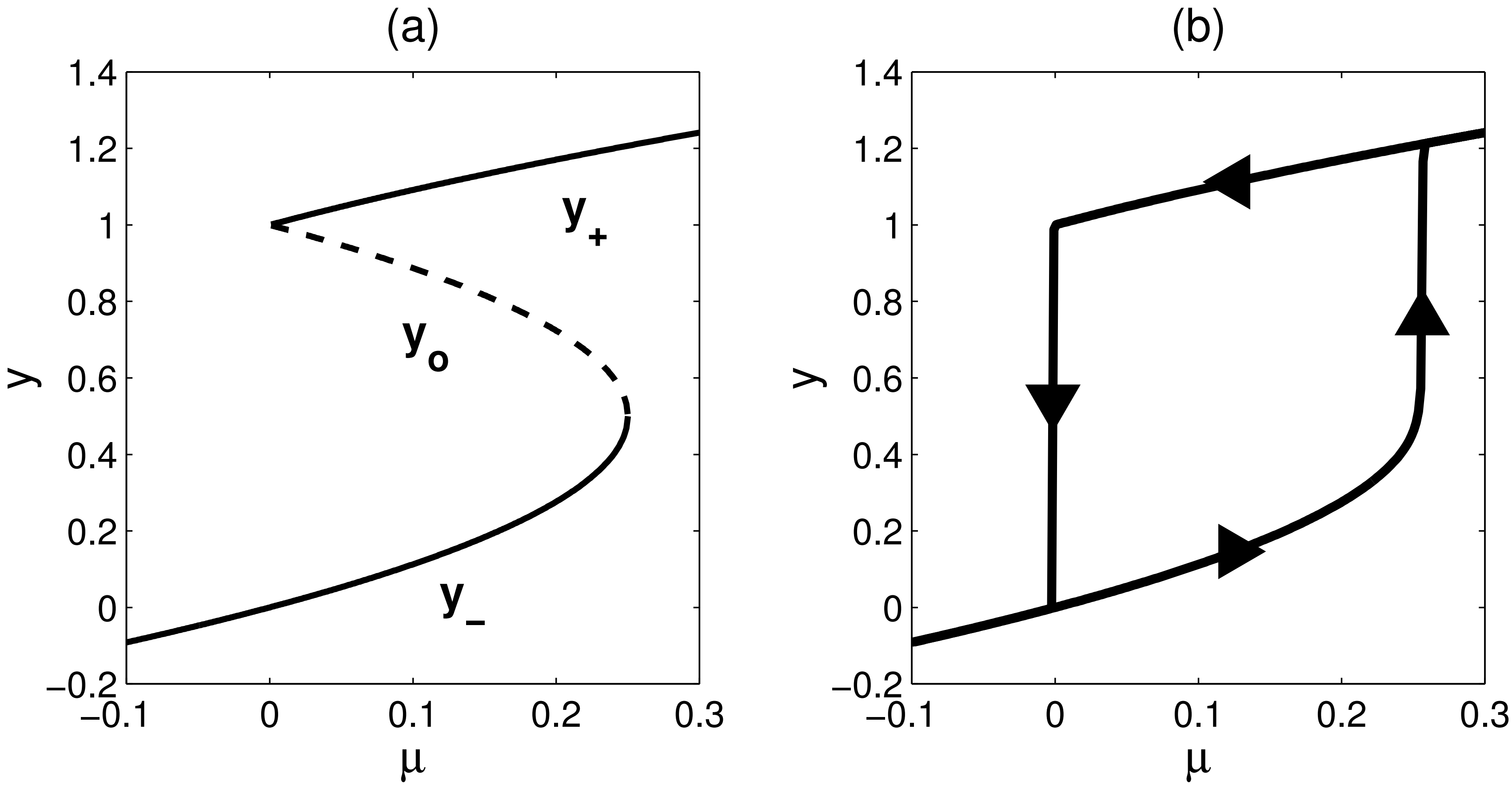
Stommel Model: Bifurcations & Hysteresis



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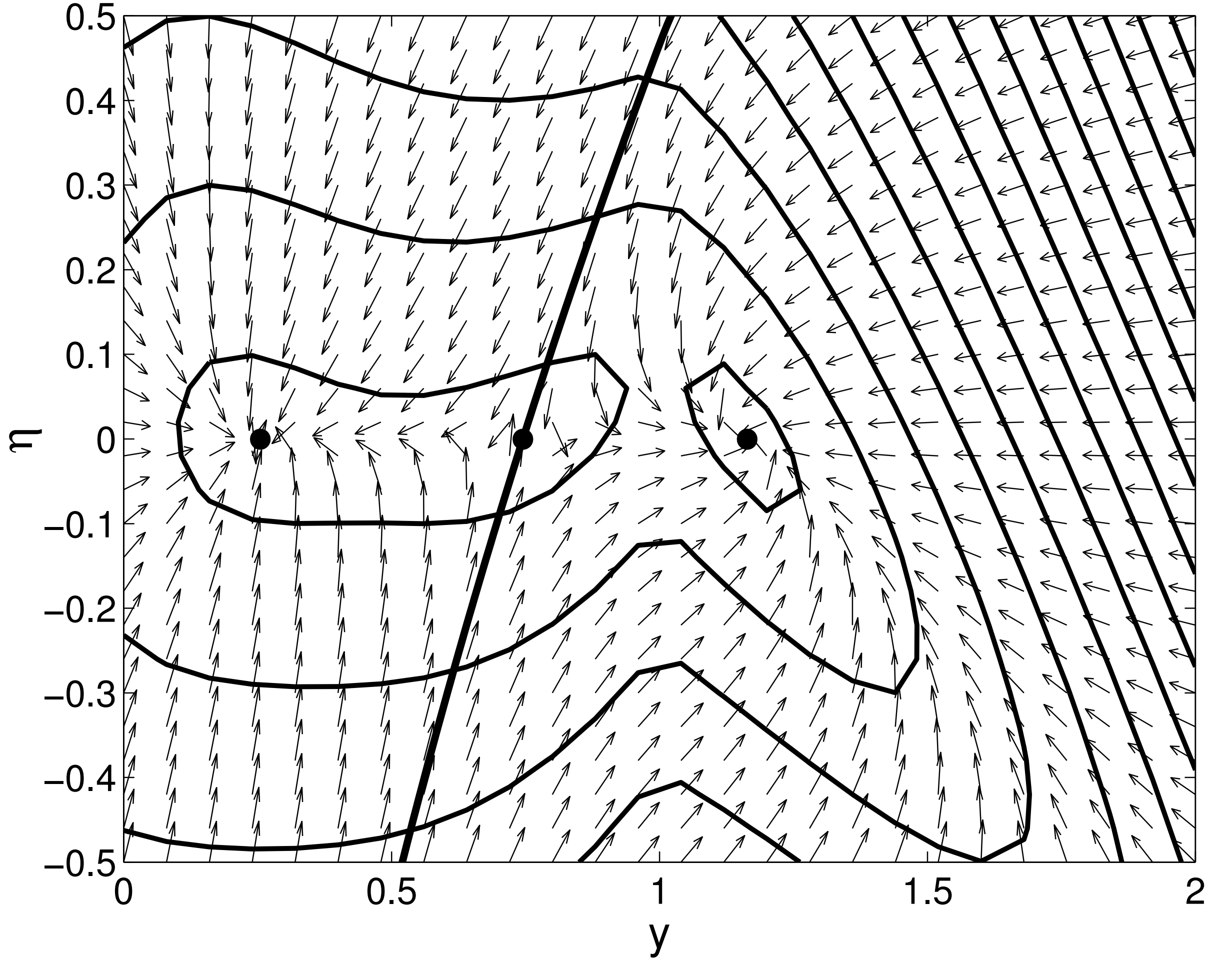
Stommel Model: Transient Dynamics

Vector field of deterministic dynamics



$$\mu = 0.19, \tau = 1$$

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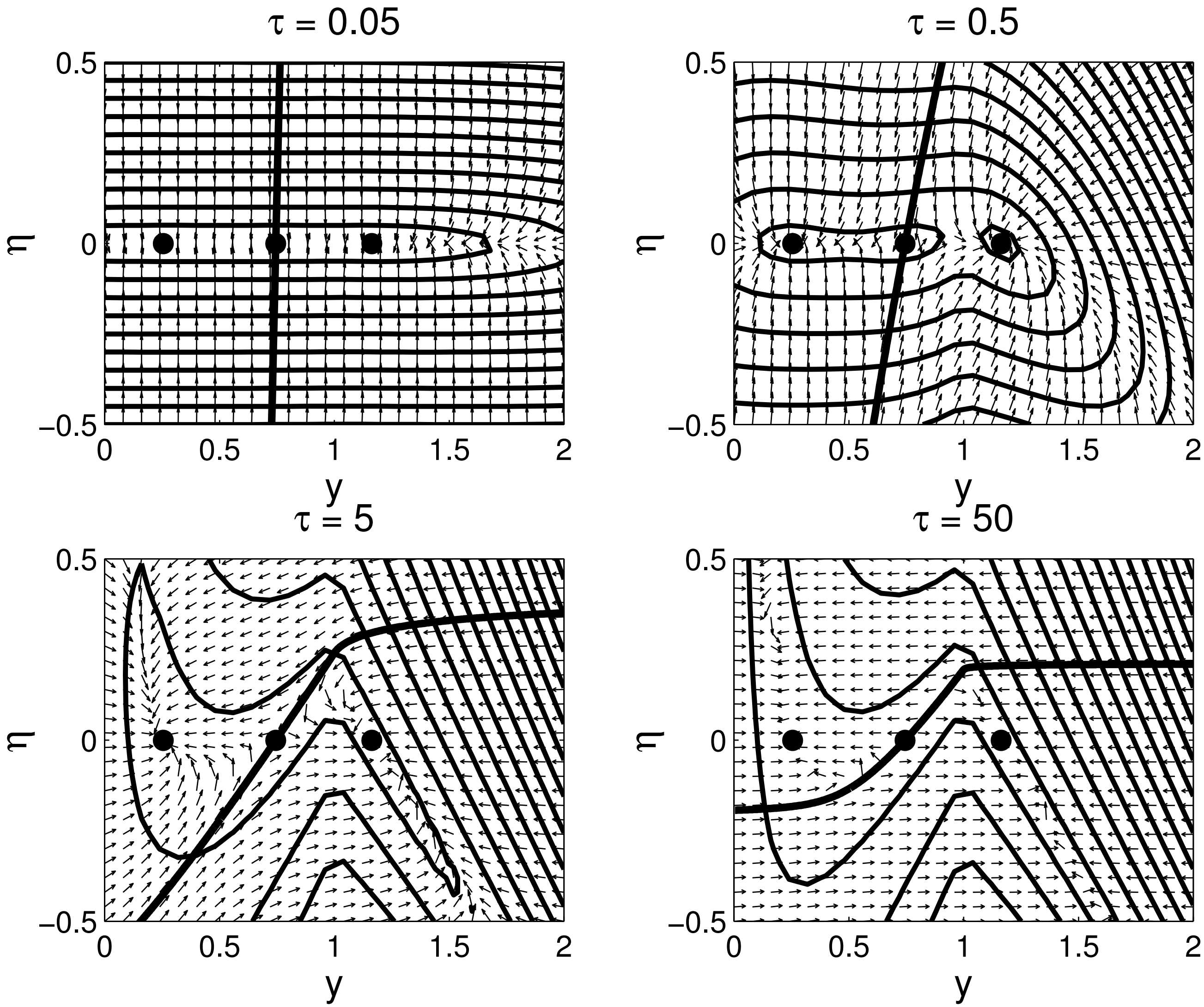
Stommel Model: Transient Dynamics



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Stommel Model: White Noise Limit

- As $\tau \to 0$, system approaches 1D SDE:

$$\dot{y} = -|1-y|y - \sigma_1 y \circ \dot{W}_1 + \sigma_2 \dot{W}_2$$

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- Instead of multiple steady states, have multimodal pdf

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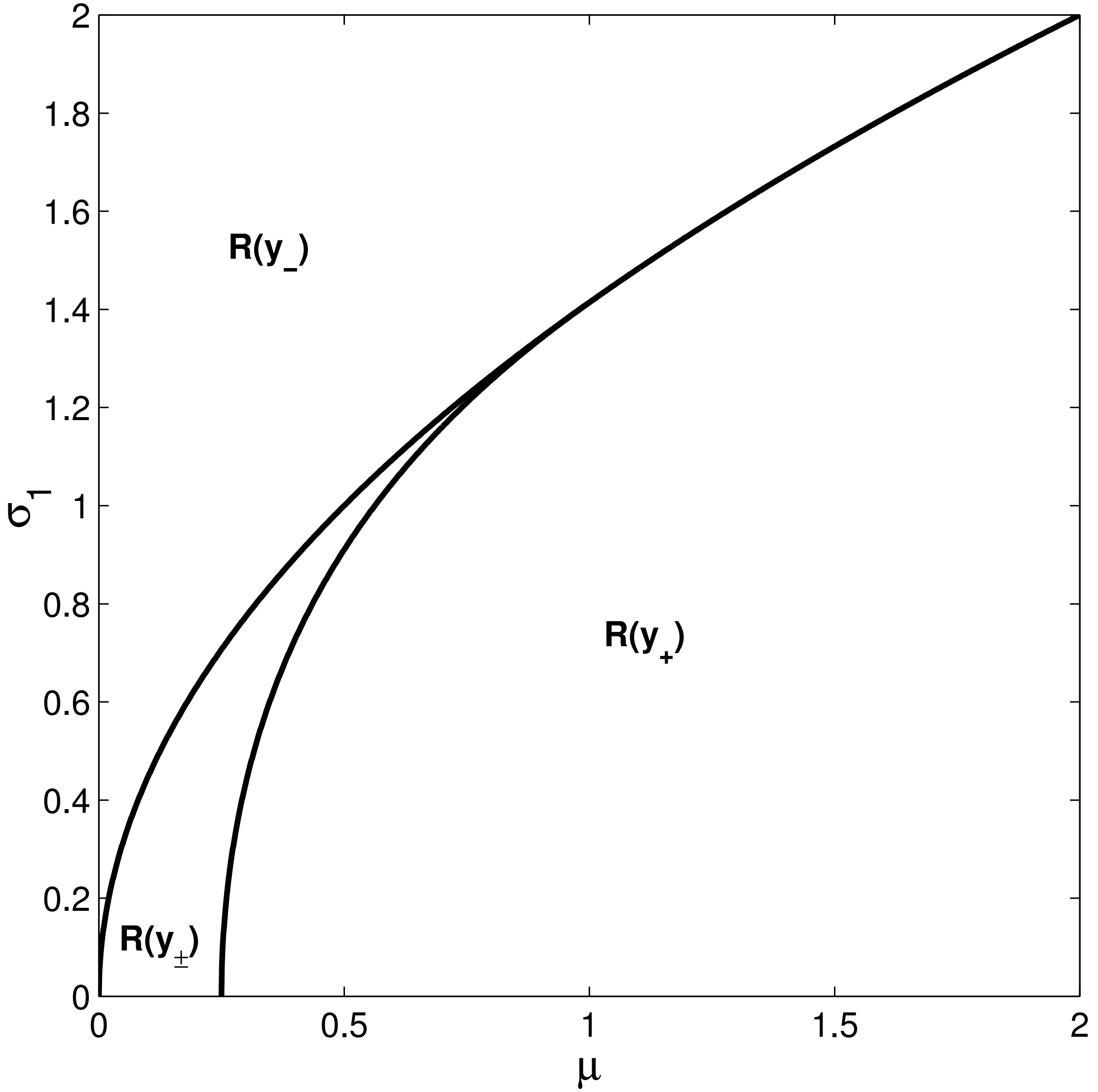
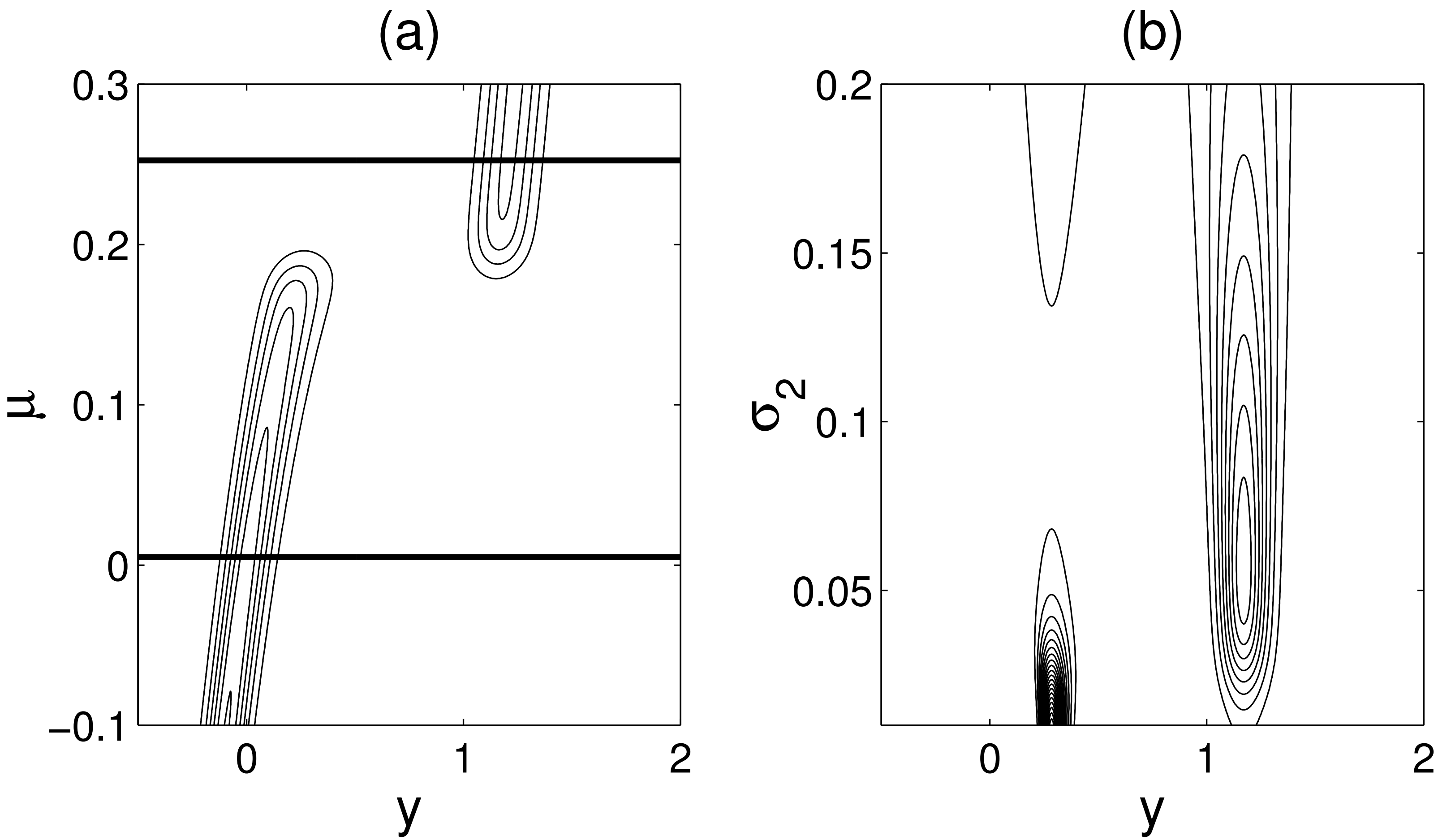
Stommel Model: Phase Diagram



Stommel Model: pdfs



(a) $\sigma_1 = \sigma_2 = 0.1$ (b) $\mu = 0.205$ and $\sigma_1 = 0.1$

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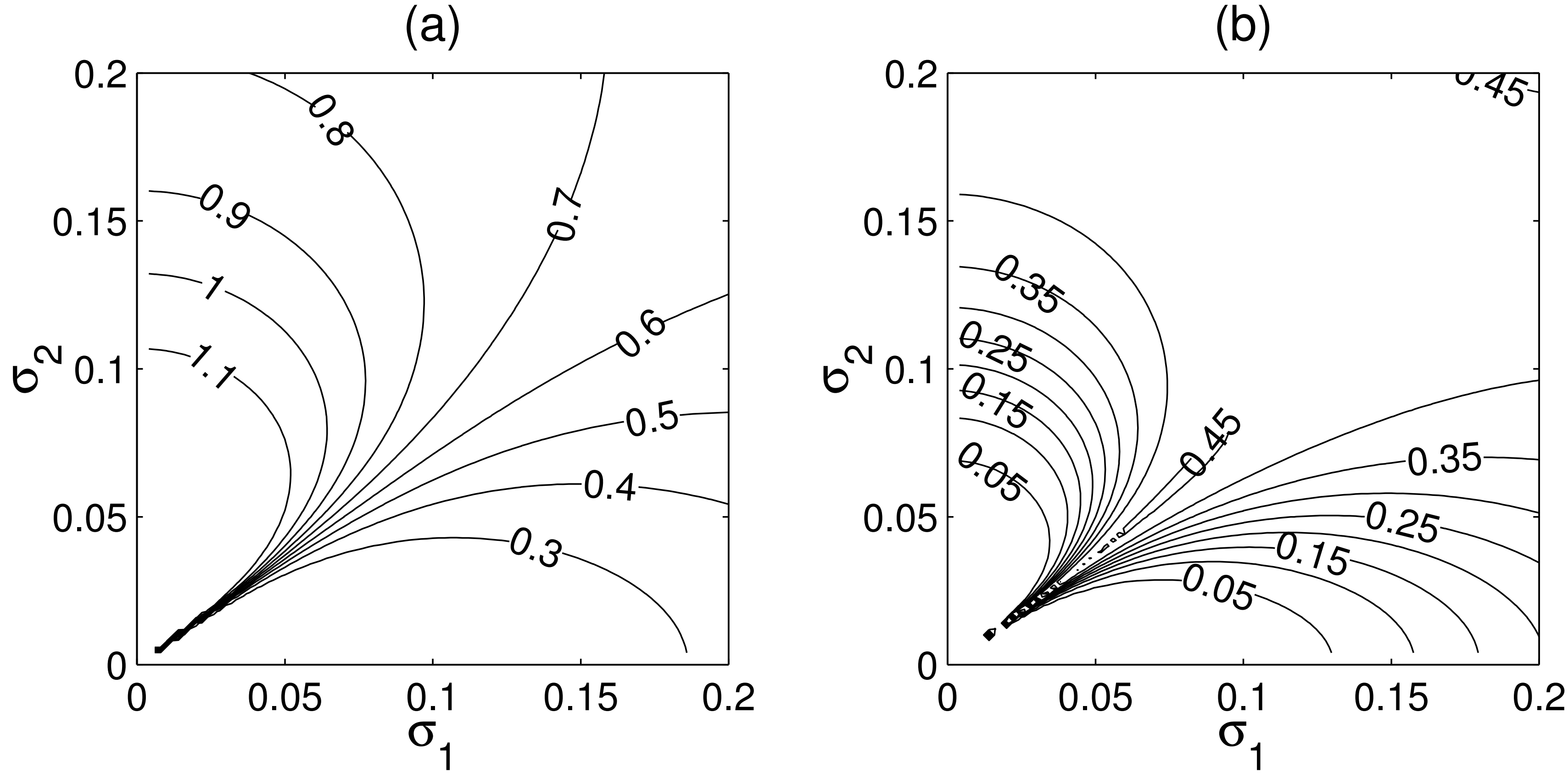
Stommel Model: Moments

$\text{mean}(y)$ $\text{std}(y)$



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- Define $\mu_{0.5}$ as value of μ for which both regimes equally populated

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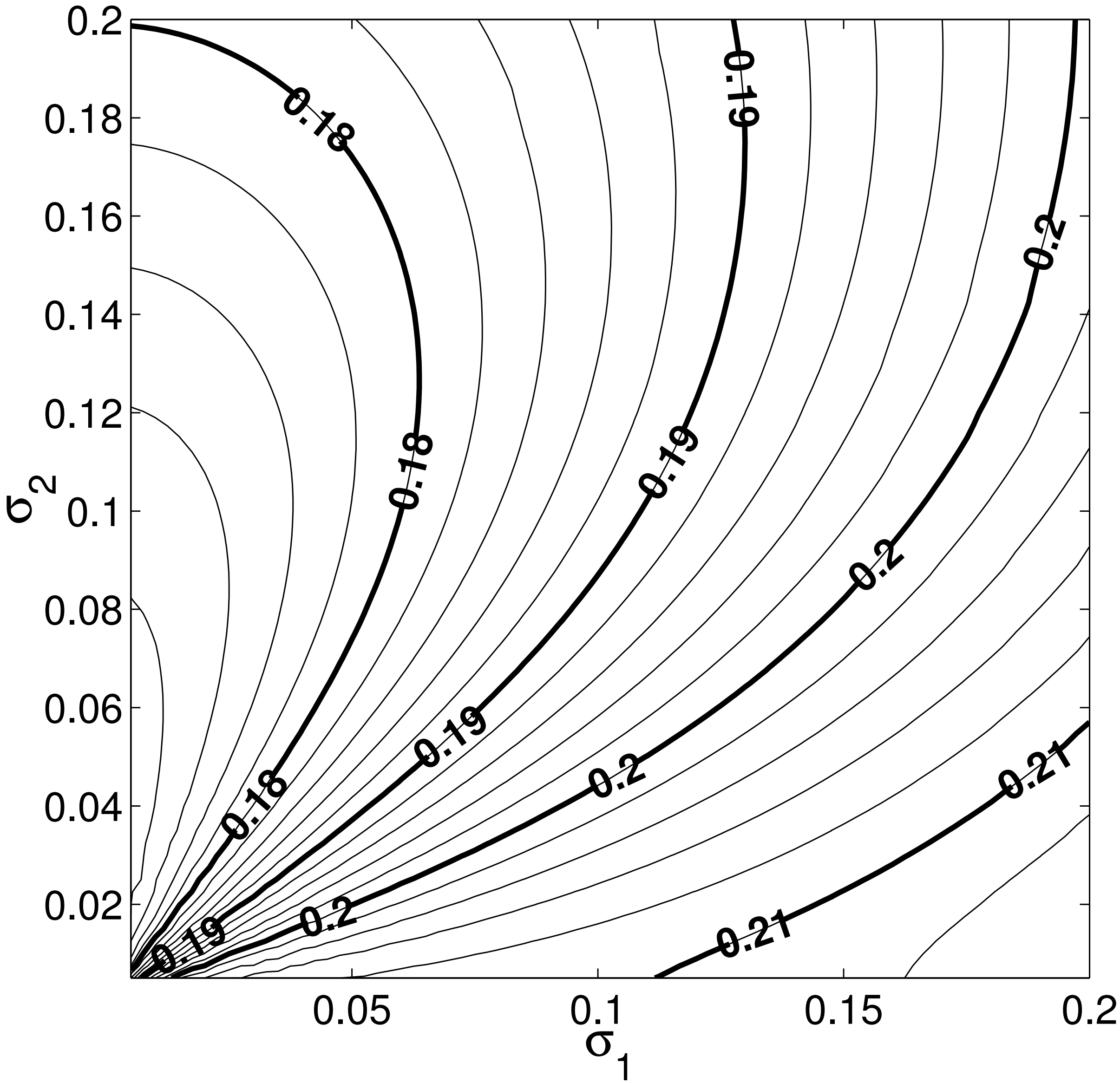
Stommel Model: $\mu_{0.5}$



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$\Rightarrow$ stochastically perturbed hysteresis loops "shrink"

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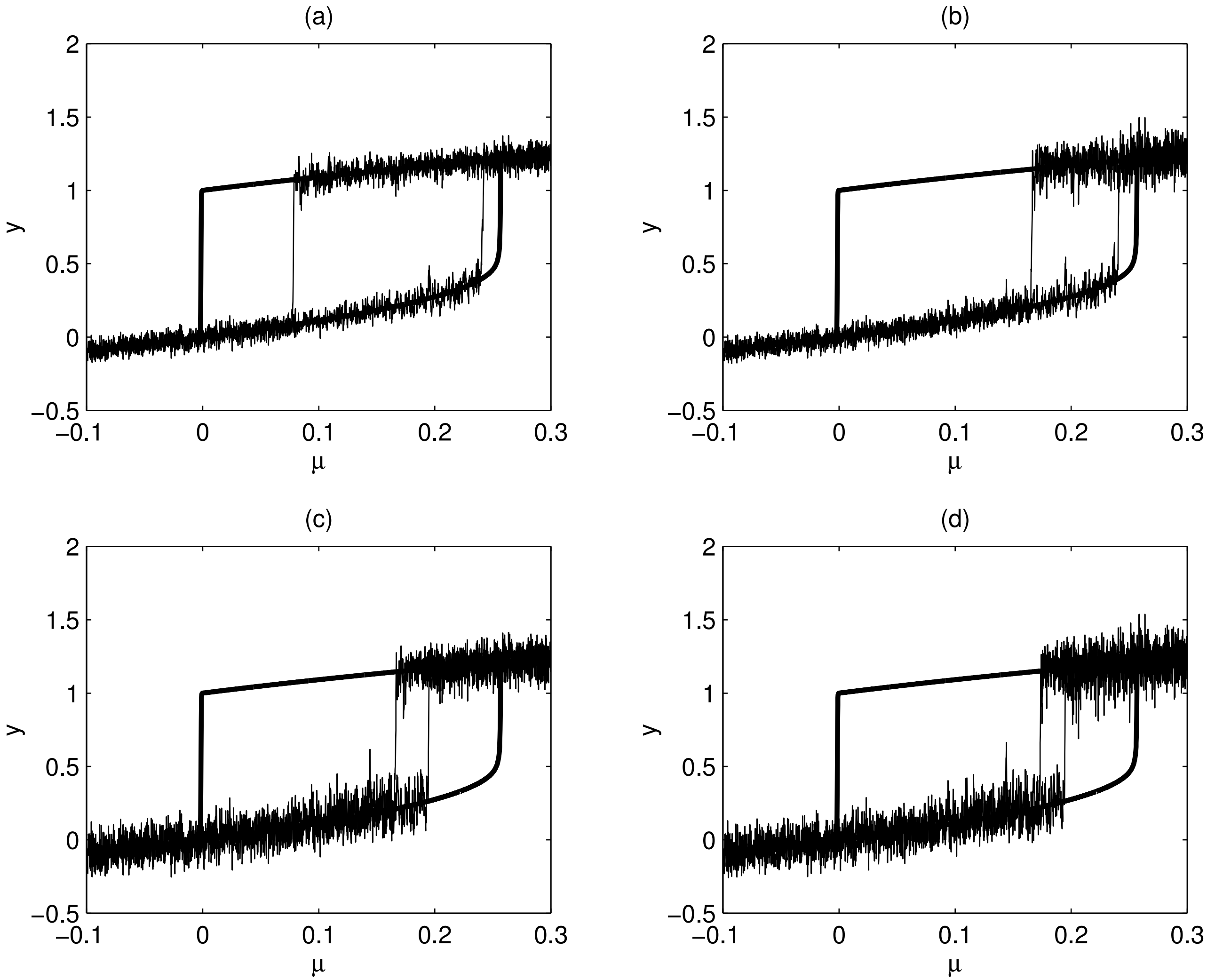
Stommel Model: Stochastic Hysteresis Loops



$\{\sigma_1, \sigma_2\}$: (a) $\{0.05,0.05\}$, (b) $\{0.1,0.05\}$, (c) $\{0.05,0.1\}$, (d) $\{0.1,0.1\}$

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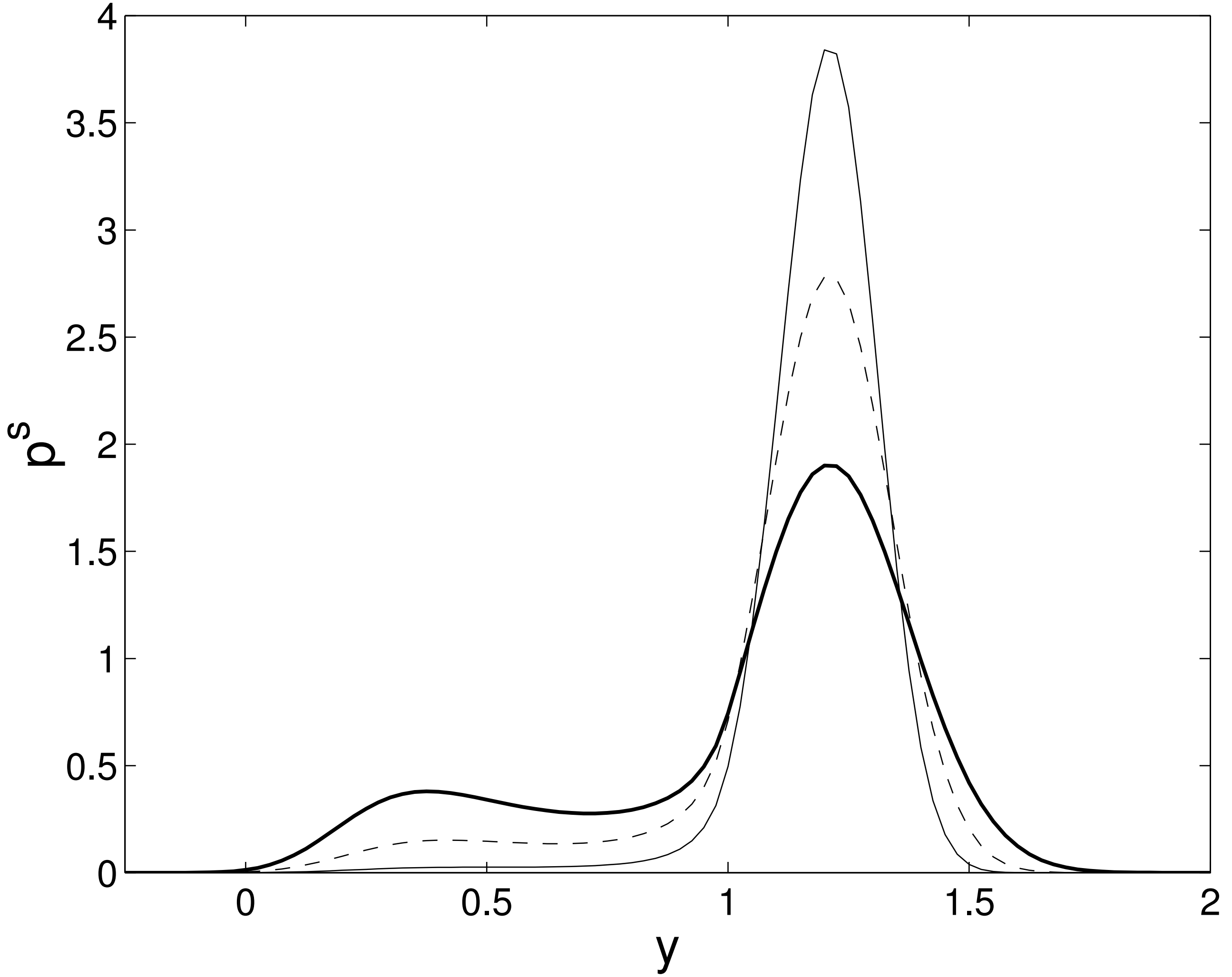

Thin: $\sigma_1 = 0.1$; Dashed: $\sigma_1 = 0.2$; Thick: $\sigma_3 = 0.3$

$$\mu = 0.255$$

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Stommel Model: Red Noise Mixing

- For $\tau \neq 0$, no analytic expression for pdf; resort to numerical simulation
- Increasing τ:
  - regime peaks become narrower
  - peaks in pdf can occur where there is no deterministic fixed point
- Last effect can be understood by considering diffusion in deterministic vector field; probability mass can accumulate where deterministic tendency minimised

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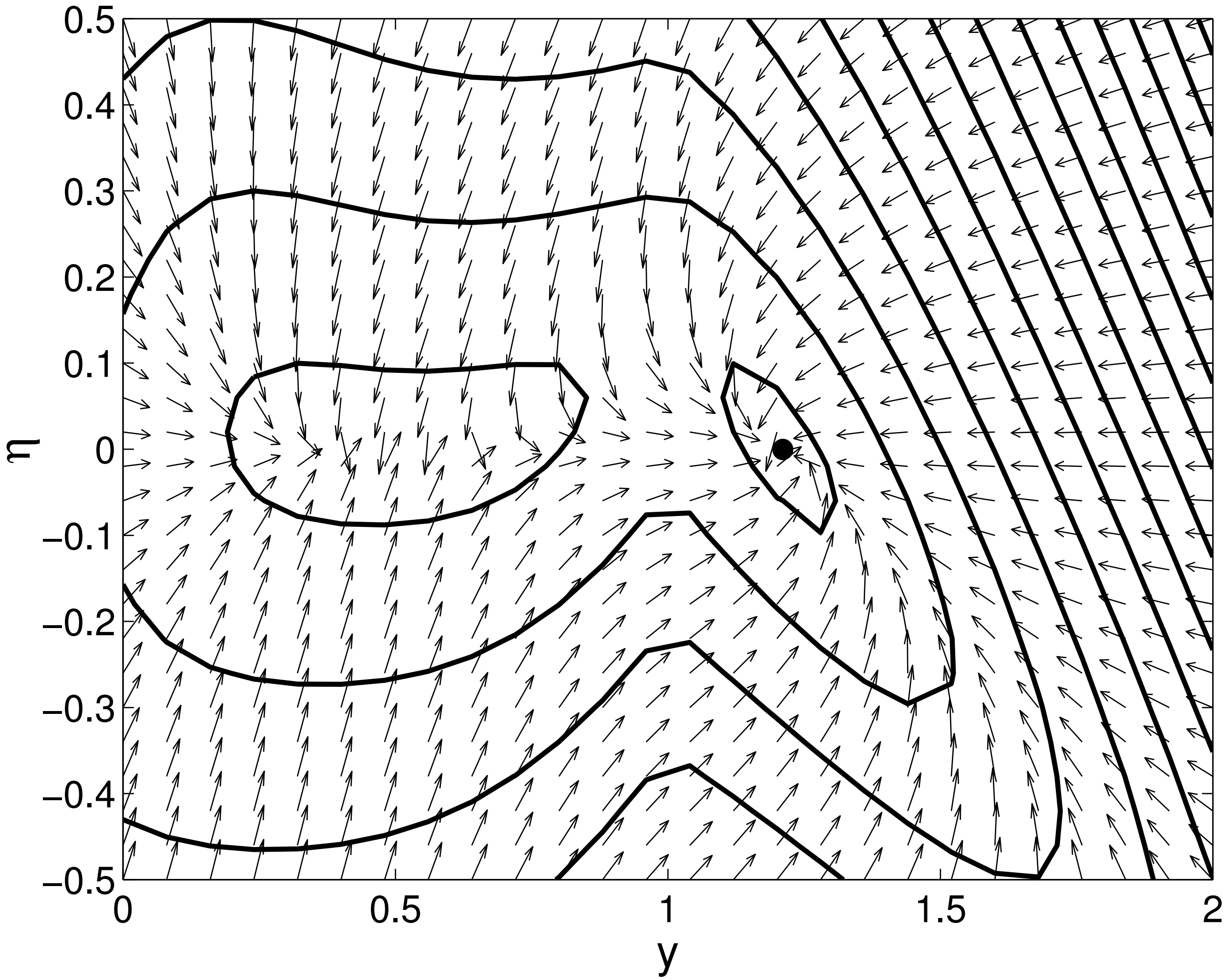
Stommel Model: Deterministic Flow



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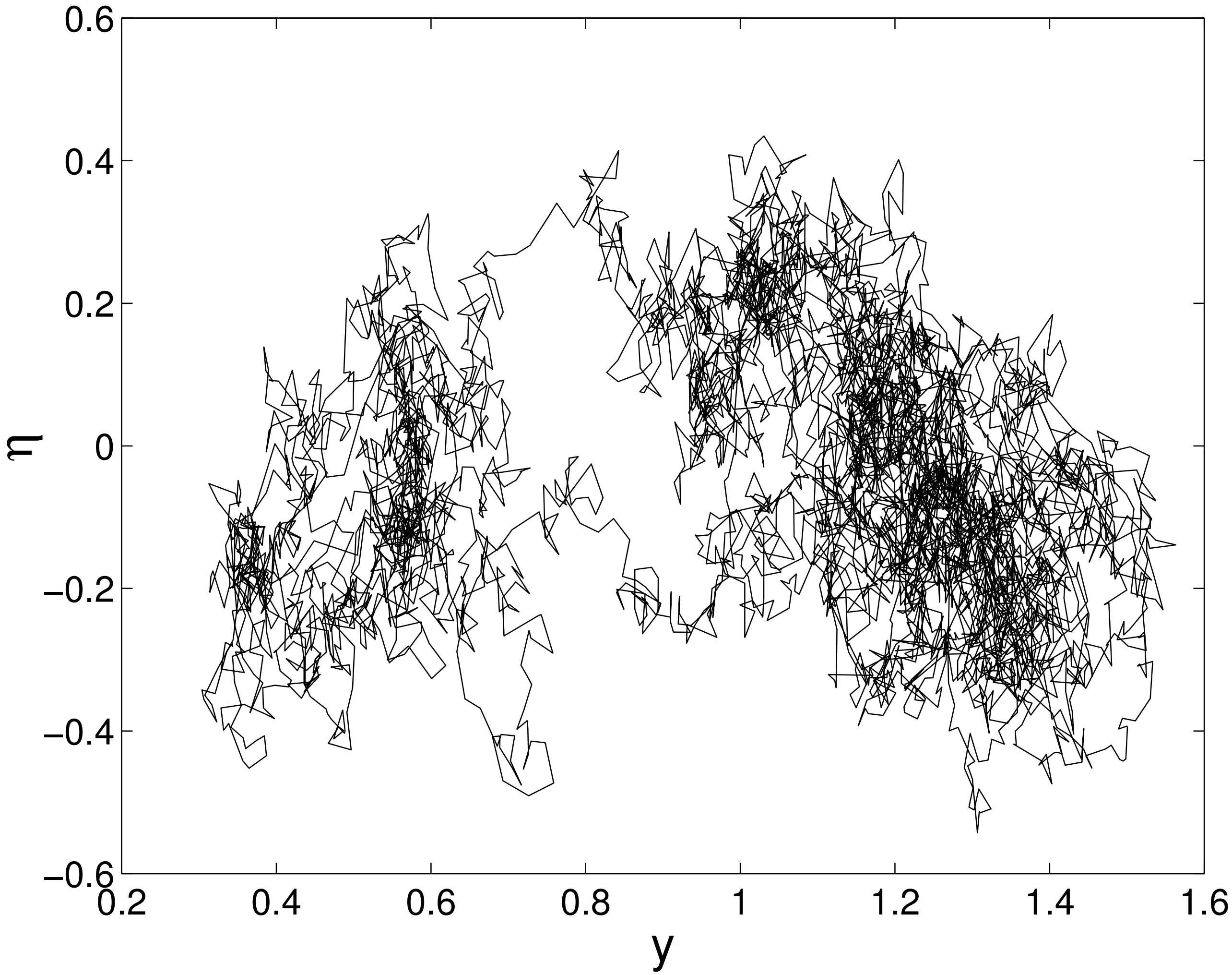
Stommel Model: Sample Trajectory



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Conclusions: Part I

- Stochastic and deterministic systems have very different stability properties

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- Stochastic and deterministic systems have very different stability properties
- In presence of fluctuations, one regime typically preferred over other when both are deterministically stable $\Rightarrow$ **stabilisation by noise**
- As control parameter varies, transitions between regimes typically occur well before deterministic bifurcations
- Peaks of pdf do not necessarily coincide with deterministic fixed points

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Lyapunov Exponents of a Stochastic System

- Consider the SDE

$$\frac{d}{dt}\mathbf{x} = \mathbf{a}(\mathbf{x}) + \sum_{p=1}^{P} \mathbf{b}^{(p)}(\mathbf{x}) \circ \dot{W}_p(t)$$

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- Linearised dynamics around trajectory $\mathbf{X}(t)$ for perturbation $\mathbf{z}(t)$:

$$\frac{d}{dt}\mathbf{z} = A(\mathbf{X}(t))\mathbf{z} + \sum_{p=1}^{P} B^{(p)}(\mathbf{X}(t))\mathbf{z} \circ \dot{W}_p(t)$$

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  - $A_{ij} = \partial_i a_j$, $B_{ij}^{(p)} = \partial_i b_j^{(p)}$,

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  - $A_{ij} = \partial_i a_j$, $B_{ij}^{(p)} = \partial_i b_j^{(p)}$,
  - same realisation $W_p(t)$ as used to get $\mathbf{X}(t)$

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- Leading Lyapunov exponent given by

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- Convenient formula for λ using spherical coordinates, $\mathbf{S} = \mathbf{z}/||\mathbf{z}||$ (Furstenberg-Khasminskii):

$$\lambda = E\{q(\mathbf{S})\} = \lim_{t\to\infty} \frac{1}{t} \int_0^t q(\mathbf{S}_u)du$$

where

$$q(\mathbf{S}) = \mathbf{S}^T A(t)\mathbf{S} + \sum_{p=1}^{P} \left(\frac{1}{2}\mathbf{S}^T [B^{(p)}(t) + B^{(p)}(t)^T] B^{(p)}(t)\mathbf{S} - (\mathbf{S}^T B^{(p)}(t)\mathbf{S})^2 \right)$$

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Lyapunov Exponents: Computational Strategy

For time T long enough to "ensure" ergodicity:

1. Generate Wiener processes

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4. Compute λ using Furstenberg-Khasminskii

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Lyapunov Exponents: Discussion

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Additive noise enters calculation of λ through invariant measure of $\mathbf{X}(t)$

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$$\rho = x\overline{\rho}_x + y\overline{\rho}_y + z\overline{\rho}_z = (x, y, z) \cdot \nabla\overline{\rho}$$

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  in a rectangular basin on an f-plane
- Incorporates both surface mechanical and buoyancy forcing
- Assumes single component fluid, planar isopycnal surfaces

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Maas Model: Dynamics

- Nondimensionalised equations:

$$\gamma \frac{d}{dt}\mathbf{L} + \frac{1}{2}\mathbf{k} \times \mathbf{L} = -\overline{\rho}_y \mathbf{i} + \overline{\rho}_x \mathbf{j} - \epsilon(L_1 \mathbf{i} + L_2 \mathbf{j} + r L_3 \mathbf{k}) - \hat{T}\mathbf{k}$$

$$\frac{d}{dt}\nabla\overline{\rho} + \frac{1}{2}\nabla\overline{\rho} \times \mathbf{L} = -(\overline{\rho}_x \mathbf{i} + \overline{\rho}_y \mathbf{j} + \mu \overline{\rho}_z \mathbf{k}) + B_2 \mathbf{j},$$

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\frac{d}{dt}\nabla\overline{\rho} + \frac{1}{2}\nabla\overline{\rho} \times \mathbf{L} &= -(\overline{\rho}_x \mathbf{i} + \overline{\rho}_y \mathbf{j} + \mu \overline{\rho}_z \mathbf{k}) + B_2 \mathbf{j},
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where

- $\epsilon \sim$ friction

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Maas Model: Dynamics

- Nondimensionalised equations:

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Maas Model: Simplifications

- Small $\gamma \Rightarrow$ assume balanced $\mathbf{L}$

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\end{aligned}$$

- Includes friction, buoyancy forcing, and interaction with wind-driven circulation

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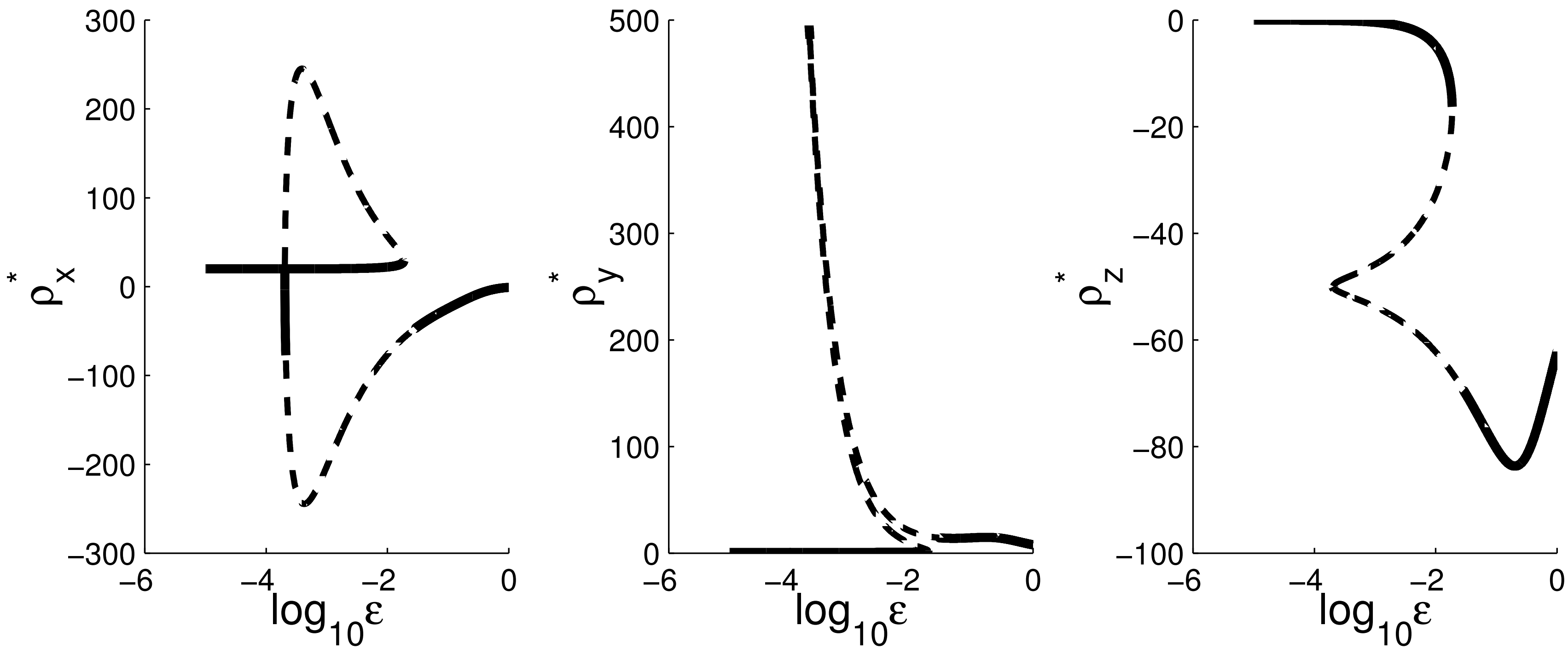
Maas Model: Bifurcations



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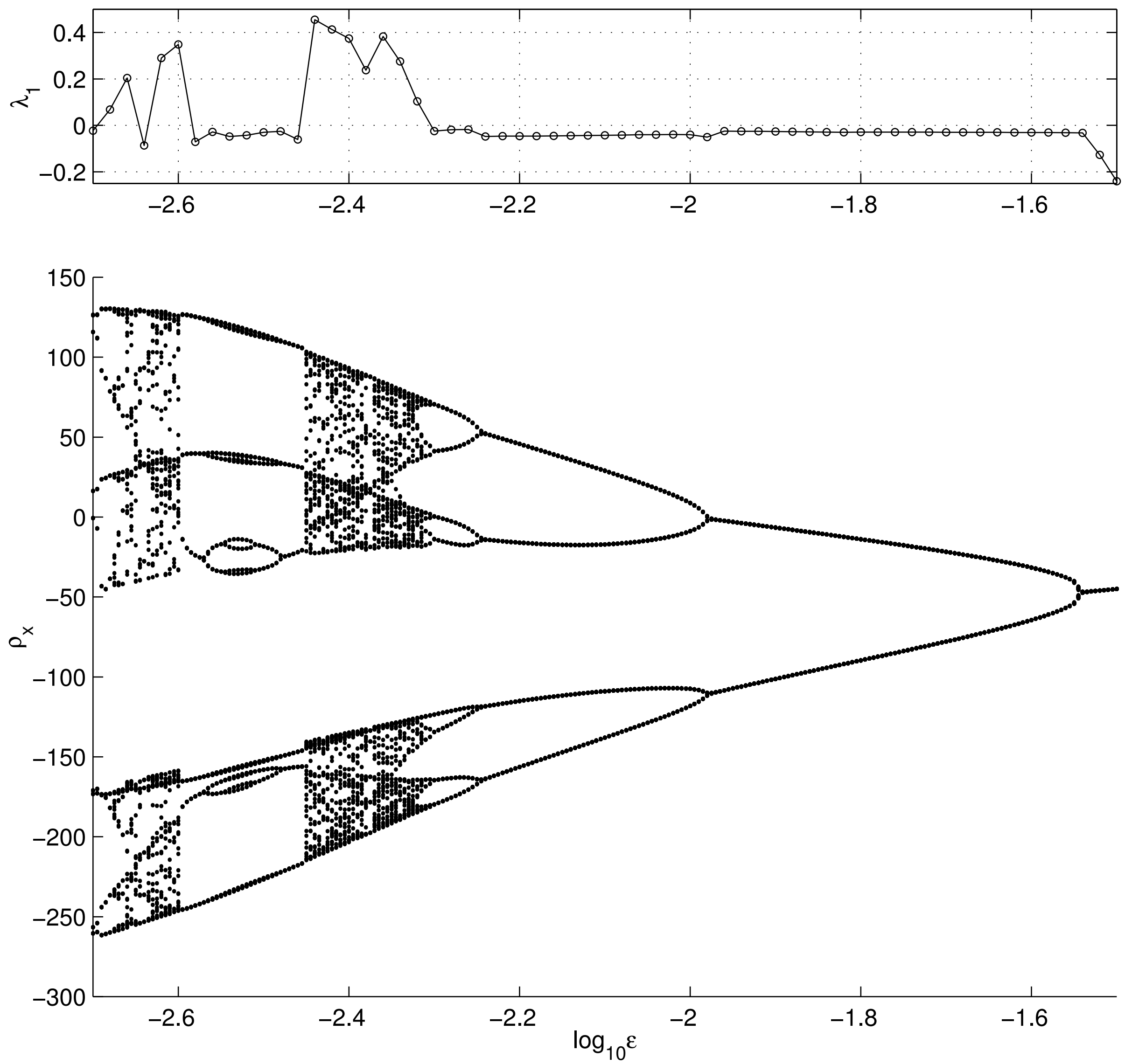

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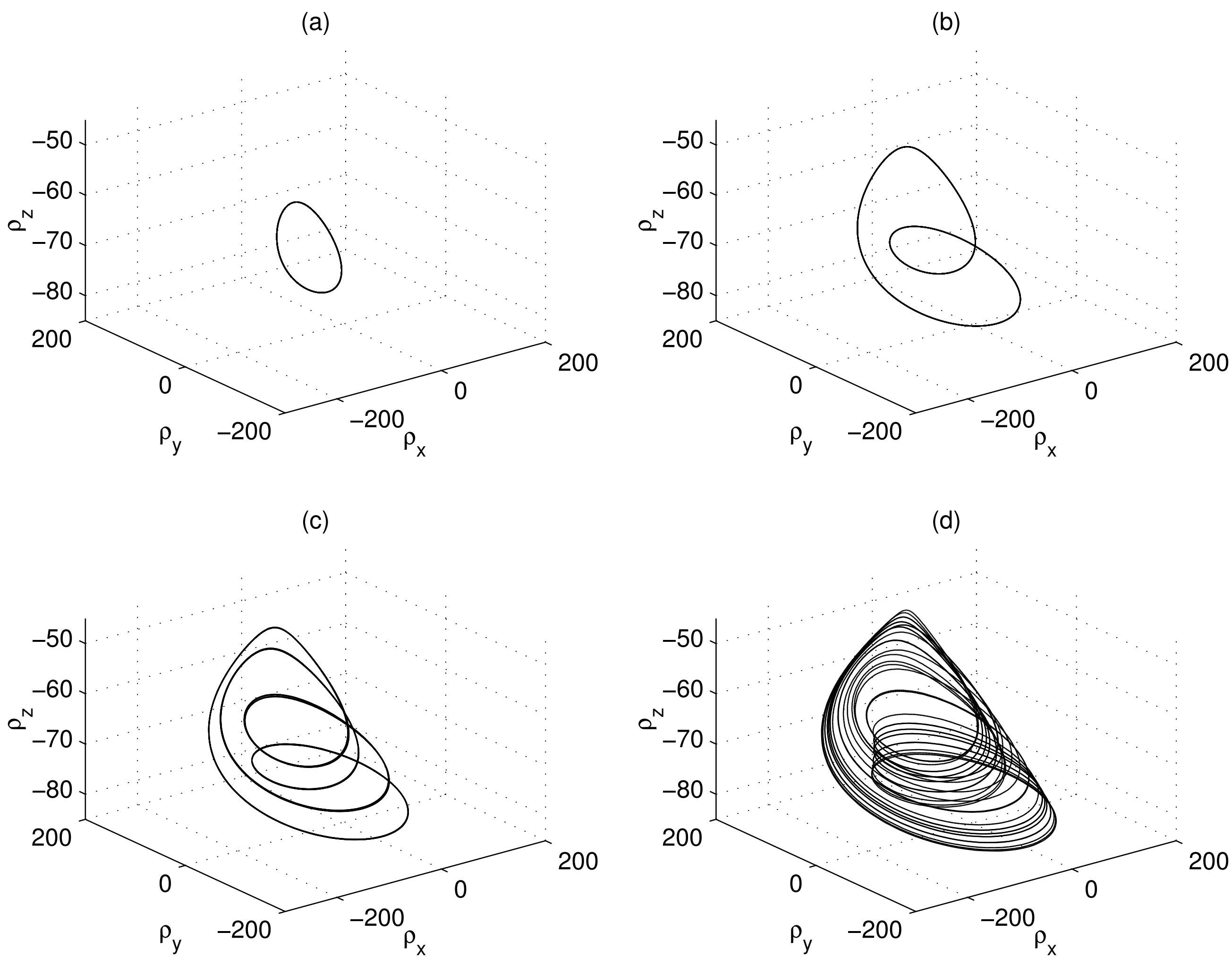
Maas Model: Attractors



$\log_{10} \epsilon = (a) - 1.8, (b) - 2.2, (c) - 2.3, (d) - 2.4$

Maas Model: Fluctuating Forcing

- Introducing fluctuations in mechanical & buoyancy forcing:

$$\begin{aligned} L_3 &\rightarrow L_3 + \sigma_1 \dot{W}_1 \\ B_2 &\rightarrow B_2 + \sigma_2 \dot{W}_2 \end{aligned}$$

(respectively multiplicative & additive noises) gives:

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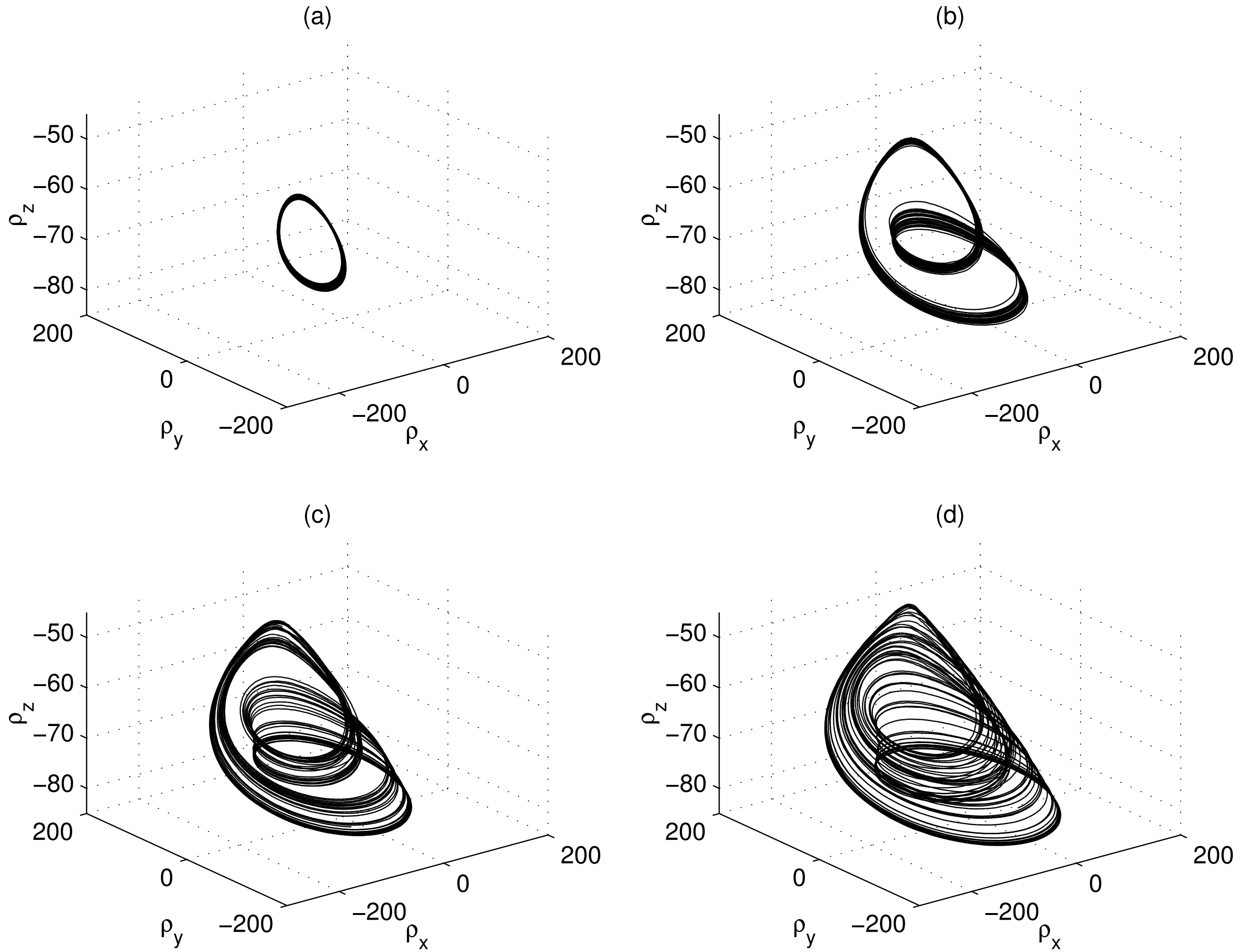
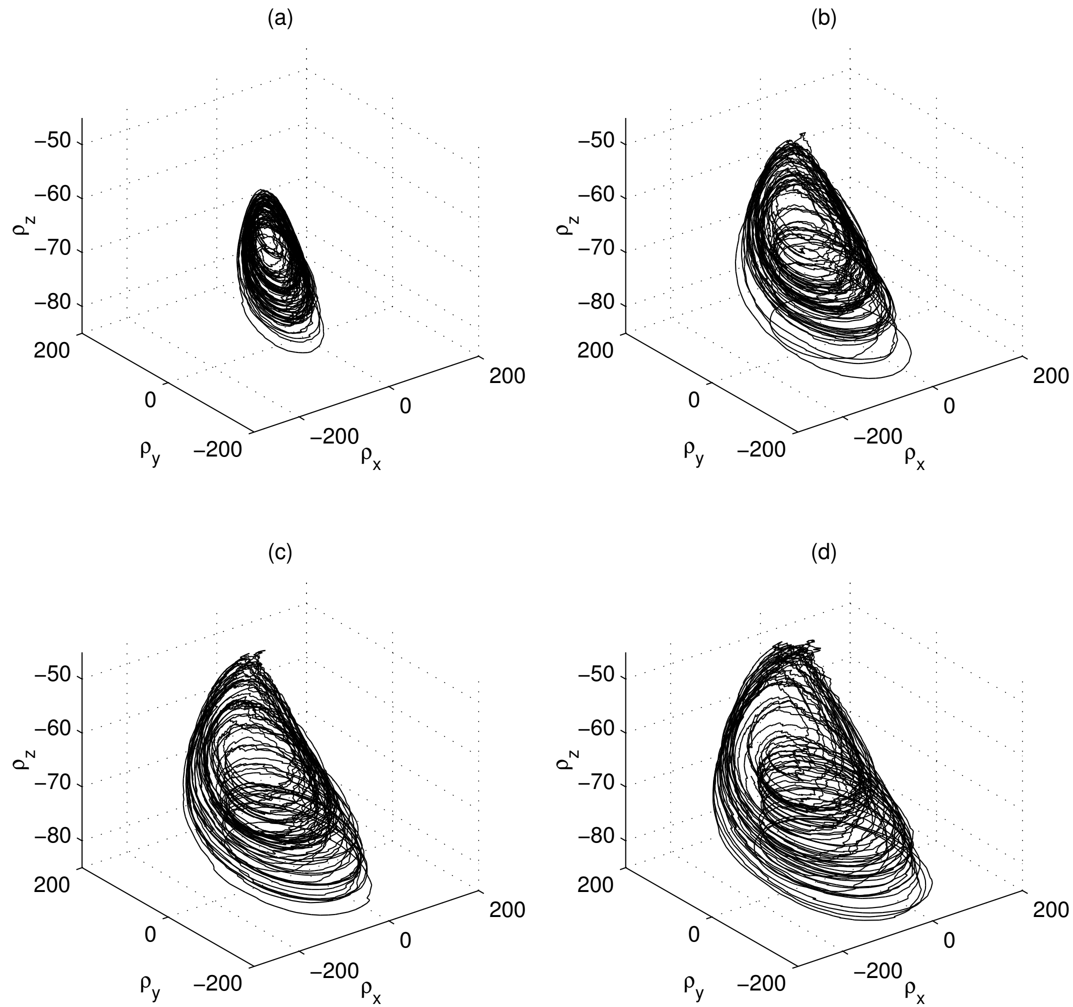
Stochastic Maas Model: Trajectories



$$\log_{10} \epsilon = (a) - 1.8, (b) - 2.2, (c) - 2.3, (d) - 2.4$$

$$(\sigma_1, \sigma_2) = (0.1, 0)$$

Stochastic Maas Model: Trajectories



$$\log_{10}\epsilon = (a) - 1.8, (b) - 2.2, (c) - 2.3, (d) - 2.4$$
$$(\sigma_1, \sigma_2) = (1.0, 0)$$

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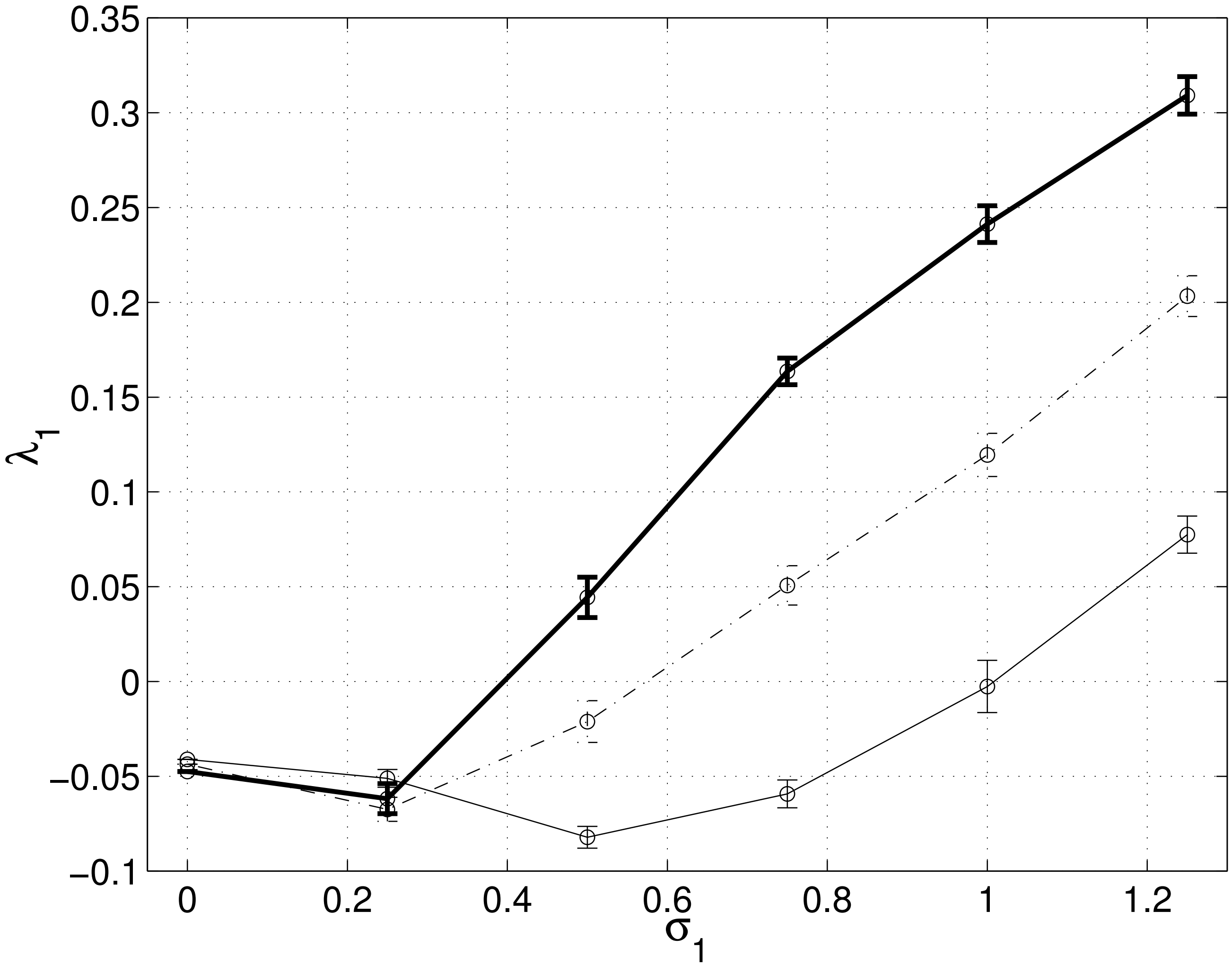
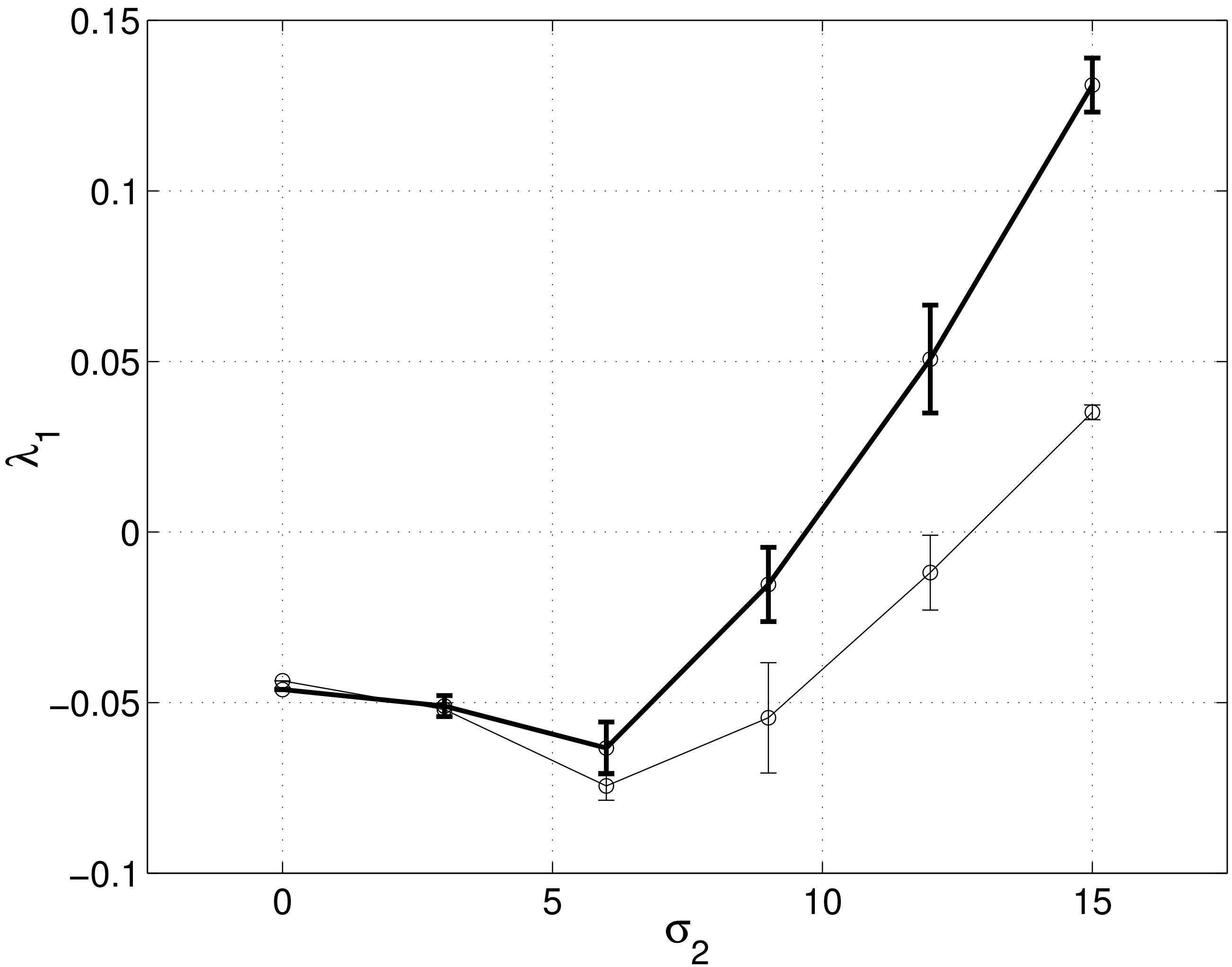
Stochastic Maas Model: Lyapunov Exponents



$\sigma_2 = 0$; $\log_{10} \epsilon = -2.0$ (thin), -2.1 (dash), -2.2 (thick)

Stochastic Maas Model: Lyapunov Exponents



$\sigma_1 = 0$; $\log_{10}\epsilon = -2.1$ (thin) and $\log_{10}\epsilon = -2.2$ (thick)

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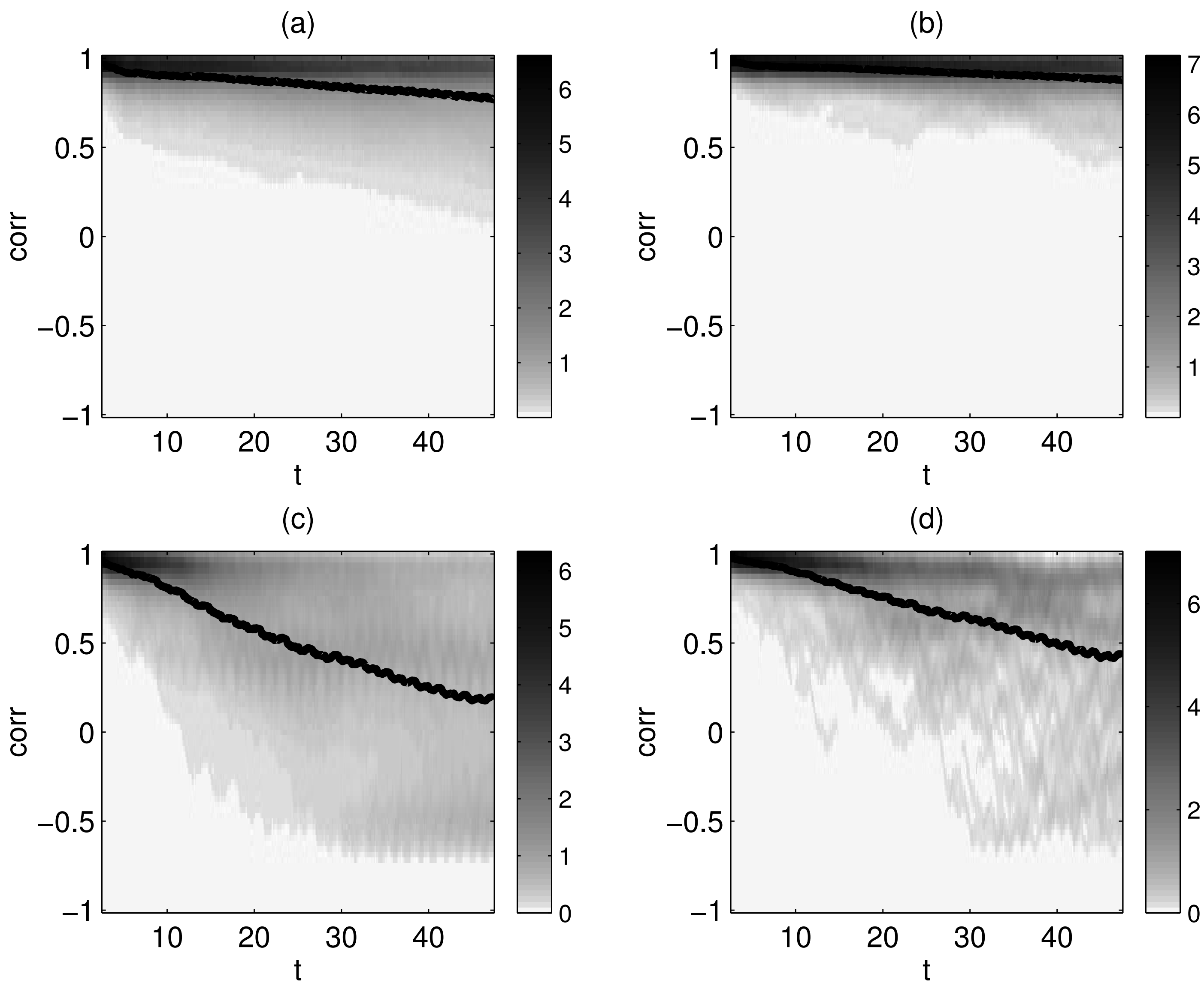
Stochastic Maas Model: Predictability



Conclusions: Part II

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  - random perturbations perturb trajectory

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- Second effect has been called “Noise-induced chaos”

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Overall Conclusions

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Overall Conclusions

- Can generalise deterministic stability concepts to stochastic systems, both for
  - one-point measures (e.g. fixed points)
  - two-point measures (e.g. Lyapunov exponents)
- Fluctuating forcing has a non-trivial impact on stability, particularly in nonlinear systems
- "Weather" variability always present, and should be accounted for in determination of climate stability

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References

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