Large Scale Coherent Structure under Random Small Scale Bombardment

Emergence of Large Structure

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joint work with Andrew J. Majda

Overview
Overview

Introduction
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- Introduction
- Heuristics
Overview

- Introduction
- Heuristics
- Rigorous result
Overview

- Introduction
- Heuristics
- Rigorous result
- Summary and comments
Great Red Spot
Goal
Goal

- Understand the emergence and persistence of such large scale coherent structure
Goal

- Understand the emergence and persistence of such large scale coherent structure
- Prediction
Mathematical model
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One layer model (Two dimensional fluid system for potential vorticity)
Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

\[
\frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = D(-\Delta)\psi + F, \\
q = \Delta \psi + \beta y - F\psi + h \\
D(-\Delta)\psi = \sum_{j \geq 1} d_j (-\Delta)^j \psi
\]
Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

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\frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = \mathcal{D}(-\Delta)\psi + \mathcal{F},
\]

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\mathcal{D}(-\Delta)\psi = \sum_{j \geq 1} d_j (-\Delta)^j \psi
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\[d_1: \text{Ekman damping, } d_2: \text{Newtonian viscosity, } d_j, j \geq 3: \text{hyper-viscosity}\]
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Rationale: 1 fast rotation, (Charney, Bourgeois-Beale, Embid-Majda, Lions-Temam-Wang ...), 2. relative thinness (Raugel-Sell, Temam-Ziane, ...)
Equilibrium/empirical statistical mechanical
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- undamped/unforced setting customary
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- Information theoretical approach: Maximize Shannon entropy with given information
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- Mean field equation

$$\bar{q} = G(\bar{\psi})$$
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- Majda and W., Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows, CUP, 2006
Dynamical approach
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- Forcing on the largest scale (Yudovitch, Marchioro, Constantin-Foias-Temam)
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- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)
Dynamical approach

- Forcing on the largest scale (Yudovich, Marchioro, Constantin-Foias-Temam)
- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)
- Large scale structure: ground energy shell
Damped driven environment
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- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans’ mixing layer)
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- random small scale forcing (in Jupiter’s case: predominantly positive)
Damped driven environment

- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans’ mixing layer)
- random small scale forcing (in Jupiter’s case: predominantly positive)
- Newtonian viscosity needed
Two dimensional Navier-Stokes equation (vorticity-stream function)

\[
\frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = \nu \Delta q + F, \\
\Delta \psi = q, \\
q|_{t=0} = q_0(\geq 0) \\
\psi = q = 0, \text{on } \partial Q(Q = [0, \pi] \times [0, \pi])
\]
Impulse(kick) random small scale forcing
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\[ \mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t) A\omega_r(\vec{x} - \vec{x}_j) \]
Impulse(kick) random small scale forcing

\[ F = \sum_{j=1}^{\infty} \delta(t - j\Delta t) A\omega_r(\vec{x} - \vec{x}_j) \]

\[ \omega_r(\vec{x}) = \begin{cases} 
(1 - |\vec{x} - \vec{x}_j|^2/r^2)^2, & |\vec{x} - \vec{x}_j|^2 \leq r^2 \\
0, & |\vec{x} - \vec{x}_j|^2 > r^2 
\end{cases} \]
Impulse(kick) random small scale forcing

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\end{cases} \]

\[ x_j: \text{uniform distribution on } Q_{r_0} = [r_0, \pi - r_0] \times [r_0, \pi - r_0] \]
Forcing figure
Prediction via statistical theory (Grote-Majda)
EEST leads to the ground state $\sin x \sin y$
• EEST leads to the ground state $\sin x \sin y$
• PVST or ESTP leads to sinh-Poisson
EEST leads to the ground state $\sin x \sin y$

PVST or ESTP leads to sinh-Poisson

crude closure (tracking energy and circulation only) works very well
Numerical results (contours)
Numerical results (vorticity)
Numerical results (correlation, D quotient, energy)
Stochastic approach
Stochastic approach

- Decomposition of the kick as mean plus fluctuation

\[ \omega_r = \bar{\omega}_r + \omega'_r, \quad \bar{\omega}_r = \mathbb{E}\omega_r \]
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  \[ \omega_r = \bar{\omega}_r + \omega'_r, \quad \bar{\omega}_r = \mathbb{E}\omega_r \]

- Cumulative forcing effect (deterministic part)
  \[ \left[ \frac{t}{\Delta t} \right] A\bar{\omega}_r \]
Stochastic approach

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- Cumulative forcing effect (deterministic part)
  \[ \left[ \frac{t}{\Delta t} \right] A\bar{\omega}_r \]

- Deterministic part remain order one requires
  \[ A \approx \Delta t, \text{ or } A = c_r \Delta t \]
stochastic forcing (fluctuation part)
Stochastic forcing (fluctuation part)

Cumulative forcing effect (fluctuation part)

\[
\int_0^t \mathcal{F}' = A \frac{\omega'_r(1) + \cdots + \omega'_r(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \sqrt{\frac{1}{\Delta t}}
\]
stochastic forcing (fluctuation part)

- cumulative forcing effect (fluctuation part)

\[
\int_0^t \mathcal{F}' = A \omega_r'(1) + \cdots + \omega_r'([\frac{t}{\Delta t}]) \sqrt{\frac{1}{\Delta t}} \cdot \sqrt{\frac{1}{\Delta t}}
\]

- Donsker’s invariance principle

\[
\int_0^t \mathcal{F}' \approx \frac{A}{\sqrt{\Delta t}} G(t) = c_r \epsilon G(t)
\]

\[
\epsilon = \sqrt{\Delta t}
\]
Stochastic continuous version
The continuous equation

\[ \frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}, \]

\[ q = \Delta \psi \]
The continuous equation

\[ \frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = \nu \Delta q + c_r \tilde{\omega}_r + c_r \epsilon \frac{dG}{dt}, \]

\[ q = \Delta \psi \]

existence and uniqueness of solutions well known, existence of invariant measure, random dynamical system, existence of random attractor well-known (Benssouson-Temam, Vishik-Fursikov, Schmalfuss, Crauel-Debussche-Flandoli...)

Stochastic continuous version
Heuristic limit
Heuristic limit

Heuristic limit as $\epsilon \to 0$

\[
\frac{\partial q^0}{\partial t} + \nabla^\bot \psi^0 \cdot \nabla q^0 = \nu \Delta q^0 + c_r \bar{\omega}_r,
\]

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q^0 = \Delta \psi^0
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Heuristic limit

heuristic limit as $\epsilon \to 0$

$$\frac{\partial q^0}{\partial t} + \nabla \psi^0 \cdot \nabla q^0 = \nu \Delta q^0 + c_r \bar{\omega}_r,$$

$$q^0 = \Delta \psi^0$$

limiting behavior in time for relatively small $c_r \bar{\omega}_r$

$$\nabla \psi^0 \cdot \nabla q^0 = \nu \Delta^2 \psi^0 + c_r \bar{\omega}_r$$
Heuristic limit

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$$\nabla^\perp \psi^0 \cdot \nabla q^0 = \nu \Delta^2 \psi^0 + c_r \bar{\omega}_r$$

limiting behavior as $c_r \to 0$

$$q^0 \approx \frac{c_r}{\nu} (-\Delta)^{-1} (\bar{\omega}_r)$$
Heuristic limit (approximation)
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\[ \bar{\omega}_r \approx r^2 \]
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Heuristic limit (approximation)

\[ \omega_r \approx r^2 \]

\[ q^0 \approx \frac{r^2 c_r}{\nu} (-\Delta)^{-1}(1) \]

\[ (-\Delta)^{-1}(1) = \sum_{k_j \text{pos.odd}, j=1,2} \frac{16}{\pi^2 k_1 k_2 |\vec{k}|^2} \sin(k_1 x) \sin(k_2 y) \]
Heuristic limit (approximation)

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\[ (-\Delta)^{-1}(1) = \sum_{k_j \text{pos. odd}, j=1,2} \frac{16}{\pi^2 k_1 k_2 |\vec{k}|^2} \sin(k_1 x) \sin(k_2 y) \]

\[ \text{corr}(\sin x \sin y, (-\Delta)^{-1}(1)) \approx 0.99 \]
Numerical results and prediction
Pathwise convergence (Majda-W.)
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Theorem

\[ \| q - q^0 \|_{L^\infty(0,T;L^2(\Omega))} \to 0, \text{a.s.} \]
Pathwise convergence (Majda-W.)

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\[ \frac{\partial q}{\partial t} + \nabla \perp \psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt} \]
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For \( \tilde{q} = q - c_r \epsilon G \)

\[ \frac{\partial \tilde{q}}{\partial t} + \nabla \perp (\tilde{\psi} + c_r \epsilon \Delta^{-1} G) \cdot \nabla (\tilde{q} + c_r \epsilon G) = \nu \Delta \tilde{q} + c_r \bar{\omega}_r + \nu c_r \epsilon \Delta G \]
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\]

For \( q' = \tilde{q} - q^0 \)

\[
\frac{\partial q'}{\partial t} + \nabla \perp \psi \cdot \nabla q' + \nabla \perp (\psi' + c_r \epsilon \Delta^{-1} G) \cdot \nabla q^0 = \nu \Delta q' + \nu c_r \epsilon \Delta G
\]
Rate of convergence (Majda-W.)
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Theorem

\[ \mathbb{E}(\|q - q^0\|_{L^2}^2) \leq \kappa \epsilon^2 \]
Rate of convergence (Majda-W.)

**Theorem**

\[ E(\|q - q^0\|_{L^2}^2) \leq \kappa \varepsilon^2 \]

\[ q' = q - q^0 \]

\[ dq' + (-\nu \Delta q' + \nabla^\perp \psi \cdot \nabla q' + \nabla \psi' \cdot \nabla q^0) dt = c_r \varepsilon dG \]
Theorem

\[ \mathbb{E}(\| q - q^0 \|^2_{L^2}) \leq \kappa \epsilon^2 \]

\[ q' = q - q^0 \]

\[ dq' + (-\nu \Delta q' + \nabla^\perp \psi \cdot \nabla q' + \nabla \psi' \cdot \nabla q^0) dt = c_r \epsilon dG \]

Ito’s formula \[\Rightarrow\]

\[ \frac{d}{dt} \mathbb{E}(\| q' \|^2_{L^2}) \leq -(2\nu - \frac{c}{\nu^3} \| q^0 \|^8_{L^2}) \mathbb{E}(\| q' \|^2_{L^2}) + c_r^2 \epsilon^2 \sum b_k^2 \]

where

\[ G(\vec{x}, t) = \sum b_k \vec{e}_k(\vec{x}) \beta_k(t) \]

\[ \{ \vec{e}_k(\vec{x}) \} \text{ o.n.b., } \{ \beta_k(t) \} \text{ Brownians} \]
Convergence of attractors (Majda-W.)
Theorem

\[ \lim_{\epsilon \to 0} dist(A_\epsilon(\omega), A_0) = 0, \ a.s. \]
Convergence of attractors (Majda-W.)

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**Random dynamical system**

\[
\varphi : \mathbb{R}^+ \times \Omega \times H \to H, (t, \omega, u) \mapsto \varphi(t, \omega)u
\]

\[
\varphi(0, \omega) = id, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)
\]

\[
(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}),
\]

\[
\theta_t \text{ measure preserving, } \theta_0 = id, \quad \theta_{t+s} = \theta_t \theta_s
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**random attractor** \(\mathcal{A}(\omega)(\text{compact, measurable})\)
Convergence of attractors (Majda-W.)

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**random attractor** \(\mathcal{A}(\omega)\)(compact, measurable)

\[
\varphi(t, \omega) \mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)
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Convergence of attractors (Majda-W.)

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**random attractor** \(A(\omega)(\text{compact, measurable})\)

\[
\varphi(t, \omega)A(\omega) = A(\theta_t \omega)
\]

\[
\lim_{t \to \infty} \text{dist}(\varphi(t, \theta_{-t} \omega)B, A(\omega)) = 0
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Convergence of attractors (Majda-W.)

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genarlization of Caraballo-Langa-Robinson
Commutative diagram (Majda-W.)

Theorem

\[
q(t, \omega) \rightarrow q_\infty(\omega)
\]

\[
\downarrow \quad \downarrow
\]

\[
q^0(t) \rightarrow q^0_\infty
\]
Uniqueness of invariant measure
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Invariant measure $\mu_0(du)$

$$\int_H F(u) \mu_0(du) = \int_H \mathbb{E} F(\varphi(t, \omega, u)) \mu_0(du)$$
Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$
  \[ \int_H F(u) \mu_0(du) = \int_H \mathbb{E} F(\varphi(t, \omega, u)) \mu_0(du) \]
- Invariant measure is unique for small data
Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

\[\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t, \omega, u))\mu_0(du)\]

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- $q' = q^2 - q^1$

\[\frac{d}{dt} \|q'\|^2 \leq (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu})\|q'\|^2\]
Uniqueness of invariant measure

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$$\frac{d}{dt}||q'||^2 \leq (-2\nu - c\frac{||\nabla q^1||^2}{\nu})||q'||^2$$

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)
Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$
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- Invariant measure is unique for small data

- $q' = q^2 - q^1$
  \[ \frac{d}{dt} ||q'||^2 \leq (-2\nu - c \frac{||\nabla q^1||^2}{\nu}) ||q'||^2 \]

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)

- E, Flandoli, Kuksin, Mattingly, Maslowski, Schmalfuss, Sinai, Shirikyan, ...
Discrete case
Discrete case

Discrete time Markov processes

\[ \eta_{-}^{j+1} = S(\Delta t)(\eta_{-}^j + A\omega_r(j)), \]

\[ \eta_{+}^{j+1} = S(\Delta t)(\eta_{+}^j) + A\omega_r(j) \]
Discrete case

- Discrete time Markov processes

\[
\eta_{-}^{j+1} = S(\Delta t)(\eta_{-}^{j} + A\omega_r(j)), \\
\eta_{+}^{j+1} = S(\Delta t)(\eta_{+}^{j}) + A\omega_r(j)
\]

- existence and uniqueness of invariant measure, random attractor ...
Discrete case

- Discrete time Markov processes

\[ \eta_{\downarrow}^{j+1} = S(\Delta t)(\eta_{\downarrow}^{j} + A\omega_{r}(j)), \]
\[ \eta_{\uparrow}^{j+1} = S(\Delta t)(\eta_{\uparrow}^{j}) + A\omega_{r}(j) \]

- existence and uniqueness of invariant measure, random attractor ...

- Invariant measure concentrated around a large coherent structure (Majda-W.)
Discrete case

- Discrete time Markov processes
  \[
  \eta_{j+1}^- = S(\Delta t)(\eta_j^- + A\omega_r(j)), \\
  \eta_{j+1}^+ = S(\Delta t)(\eta_j^+) + A\omega_r(j)
  \]

- Existence and uniqueness of invariant measure, random attractor ...

- Invariant measure concentrated around a large coherent structure (Majda-W.)

- Kuksin-Shirikyan, Masmoudi-Young, etc
Summary
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- Random small scale bombardments could induce large coherent structure
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- Large structures well predicted by equilibrium statistical theory
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- Generalizes to other geometry and more general one layer system, or multi-layer system
Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
- Different large coherent structure could emerge depending on different distribution of small scale forcing
- Generalizes to other geometry and more general one layer system, or multi-layer system
- Long way to go to reach our goal
Questions
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- Geophysical effects ($\beta$, $F$, topography, Ekman, ...)?
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- What if smallness assumption is violated?
- What if the mean of the forcing is zero?
- Vanishing viscosity and noise?
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- Geophysical effects ($\beta, F$, topography, Ekman, …)?
- What if smallness assumption is violated?
- What if the mean of the forcing is zero?
- Vanishing viscosity and noise?
- Convergence from discrete to the continuous case?
Vanishing noise (SRB)

\[ \mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt} \]
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- non-trivial global attractor for deterministic NSE
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- unique invariant measure with appropriate noise
Vanishing noise (SRB)

\[ \mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt} \]

- non-trivial global attractor for deterministic NSE
- unique invariant measure with appropriate noise
- Does the invariant measures converge?
Vanishing noise (SRB)

\[ \mathcal{F} = \mathcal{F} + \epsilon \frac{dG}{dt} \]

- non-trivial global attractor for deterministic NSE
- unique invariant measure with appropriate noise
- Does the invariant measures converge?
- Converge to what?
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The End