Large Scale Coherent Structure under Random Small Scale Bombardment

Emergence of Large Structure

Xiaoming Wang

wxm@math.fsu.edu

joint work with Andrew J. Majda

paper in *Comm. Pure and Applied Mathematics* 2006

Overview

Overview

- Introduction

Overview

- Introduction
- Heuristics

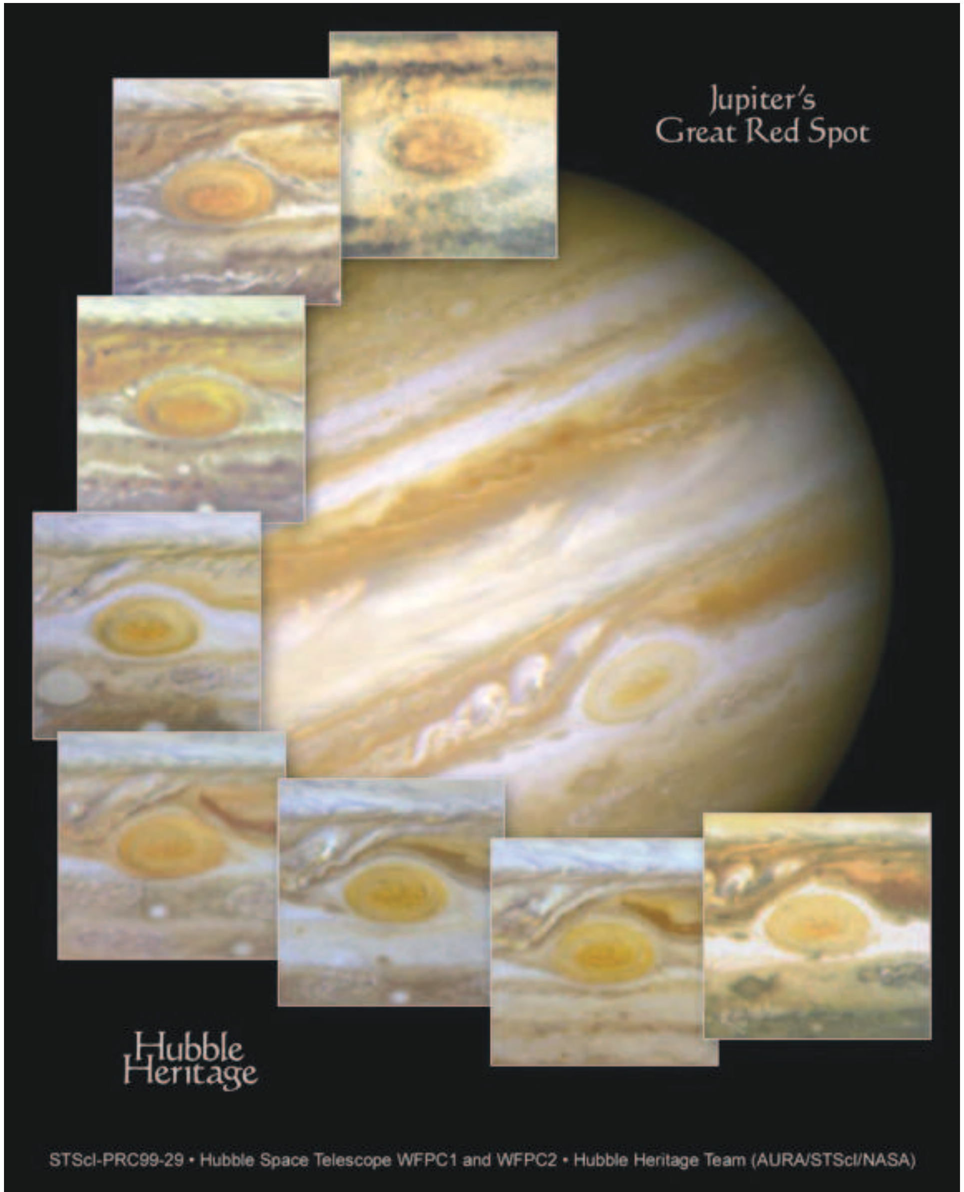
Overview

- Introduction
- Heuristics
- Rigorous result

Overview

- Introduction
- Heuristics
- Rigorous result
- Summary and comments

Great Red Spot



Goal

Goal

- Understand the emergence and persistence of such large scale coherent structure

Goal

- Understand the emergence and persistence of such large scale coherent structure
- Prediction

Mathematical model

Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

$$
\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \mathcal{D}(-\Delta)\psi + \mathcal{F}, \\
q &= \Delta\psi + \beta y - F\psi + h \\
\mathcal{D}(-\Delta)\psi &= \sum_{j\geq 1} d_j (-\Delta)^j \psi
\end{aligned}
$$

Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

$$\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \mathcal{D}(-\Delta)\psi + \mathcal{F}, \\
q &= \Delta\psi + \beta y - F\psi + h \\
\mathcal{D}(-\Delta)\psi &= \sum_{j\geq 1} d_j(-\Delta)^j\psi
\end{aligned}$$

d_1: Ekman damping, d_2:Newtonian viscosity, $d_j, j \geq 3$: hyper-viscosity

Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

$$\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \mathcal{D}(-\Delta)\psi + \mathcal{F}, \\
q &= \Delta\psi + \beta y - F\psi + h \\
\mathcal{D}(-\Delta)\psi &= \sum_{j\geq 1} d_j(-\Delta)^j \psi
\end{aligned}$$

d_1: Ekman damping, d_2:Newtonian viscosity, $d_j, j \geq 3$: hyper-viscosity
Rationale: 1 fast rotation, (Charney, Bourgeois-Beale, Embid-Majda, Lions-Temam-Wang ...), 2. relative thinness (Raugel-Sell, Temam-Ziane, ...)

Equilibrium/empirical statistical mechanics

Equilibrium/empirical statistical mechanic

- undamped/unforced setting customary

Equilibrium/empirical statistical mechanic

- undamped/unforced setting customary
- Information theoretical approach: Maximize Shannon entropy with given information

Equilibrium/empirical statistical mechanics

- undamped/unforced setting customary
- Information theoretical approach: Maximize Shannon entropy with given information
- conserved quantity becomes constraints on ρ

Equilibrium/empirical statistical mechanics

- undamped/unforced setting customary
- Information theoretical approach: Maximize Shannon entropy with given information
- conserved quantity becomes constraints on ρ
- **Mean field equation**

$$\bar{q} = \mathcal{G}(\bar{\psi})$$

Equilibrium/empirical statistical mechanic

- undamped/unforced setting customary
- Information theoretical approach: Maximize Shannon entropy with given information
- conserved quantity becomes constraints on ρ
- **Mean field equation**

$$\bar{q} = \mathcal{G}(\bar{\psi})$$

- Most of them are stable under appropriate assumptions

Equilibrium/empirical statistical mechanic

- undamped/unforced setting customary
- Information theoretical approach: Maximize Shannon entropy with given information
- conserved quantity becomes constraints on ρ
- **Mean field equation**

$$\bar{q} = \mathcal{G}(\bar{\psi})$$

- Most of them are stable under appropriate assumptions
- Majda and W., **Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows**, CUP, 2006

Dynamical approach

Dynamical approach

- Forcing on the largest scale (Yudovitch, Marchioro, Constantin-Foias-Temam)

Dynamical approach

- Forcing on the largest scale (Yudovitch, Marchioro, Constantin-Foias-Temam)
- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)

Dynamical approach

- Forcing on the largest scale (Yudovitch, Marchioro, Constantin-Foias-Temam)
- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)
- Large scale structure: ground energy shell

Damped driven environment

Damped driven environment

- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans' mixing layer)

Damped driven environment

- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans' mixing layer)
- random small scale forcing (in Jupiter's case: predominantly positive)

Damped driven environment

- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans' mixing layer)
- random small scale forcing (in Jupiter's case: predominantly positive)
- Newtonian viscosity needed

Simple model

Two dimensional Navier-Stokes equation (vorticity-stream function)

$$
\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \nu \Delta q + \mathcal{F}, \\
\Delta \psi &= q, \\
q|_{t=0} &= q_0 (\geq 0) \\
\psi = q &= 0, \text{on } \partial Q (Q = [0,\pi] \times [0,\pi])
\end{aligned}
$$

Impulse(kick) random small scale forcing

Impulse(kick) random small scale forcing

$$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t)\, A\omega_r(\vec{x} - \vec{x}_j)$$

Impulse(kick) random small scale forcing

- $$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t)\, A\omega_r(\vec{x} - \vec{x}_j)$$

- $$\omega_r(\vec{x}) = \begin{cases} \left(1 - |\vec{x} - \vec{x}_j|^2/r^2\right)^2, & |\vec{x} - \vec{x}_j|^2 \le r^2 \quad , \\ 0, & |\vec{x} - \vec{x}_j|^2 > r^2 \quad . \end{cases}$$

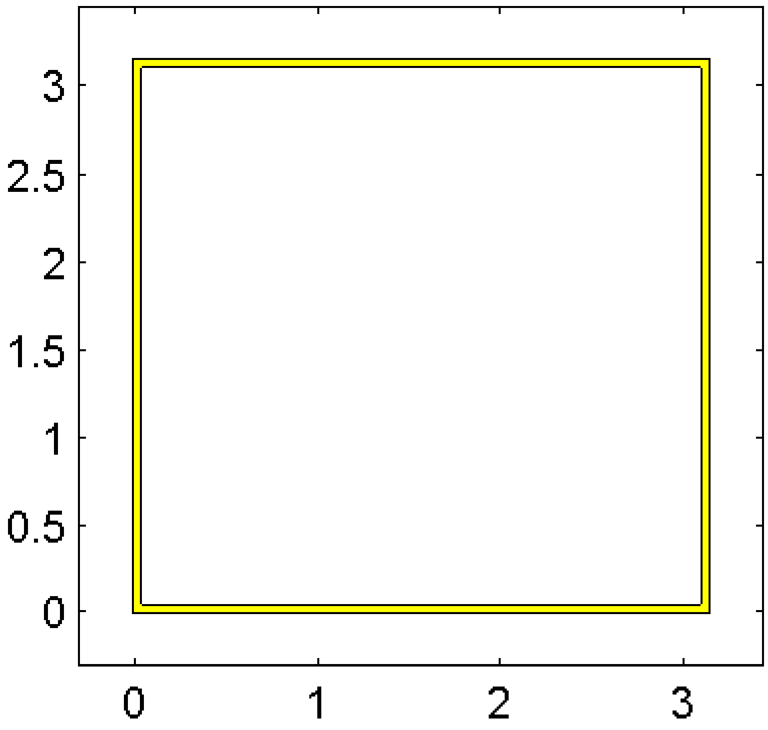
Impulse(kick) random small scale forcing

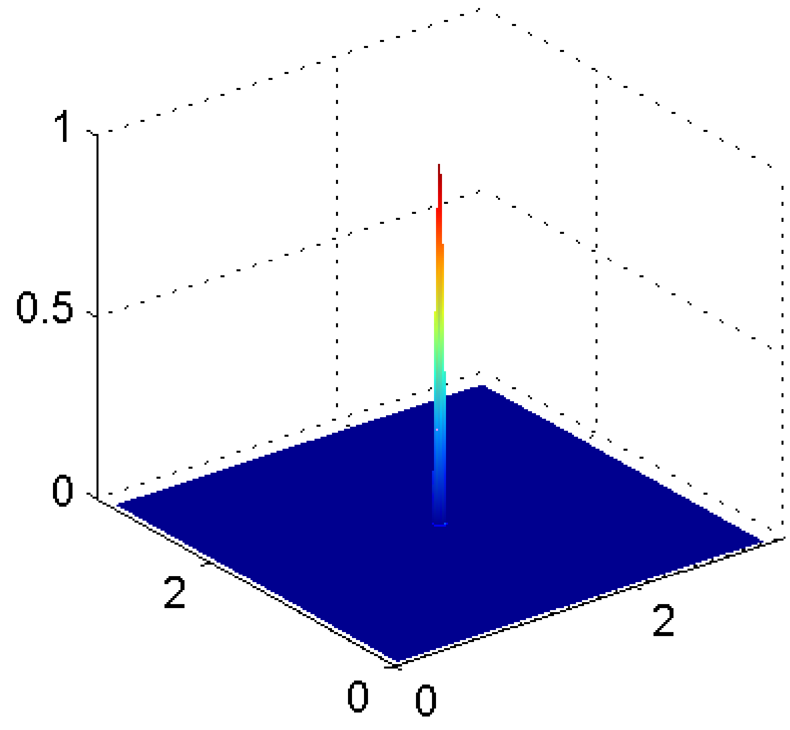
- $$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t)\, A\omega_r(\vec{x} - \vec{x}_j)$$

- $$\omega_r(\vec{x}) = \begin{cases} \left(1 - |\vec{x} - \vec{x}_j|^2/r^2\right)^2, & |\vec{x} - \vec{x}_j|^2 \leq r^2 \quad , \\ 0, & |\vec{x} - \vec{x}_j|^2 > r^2 \quad . \end{cases}$$

- x_j: uniform distribution on $Q_{r_0} = [r_0, \pi - r_0] \times [r_0, \pi - r_0]$

Forcing figure





ediction via statistical theory (Grote-Maj

ediction via statistical theory (Grote-Maj

- EEST leads to the ground state $\sin x \sin y$

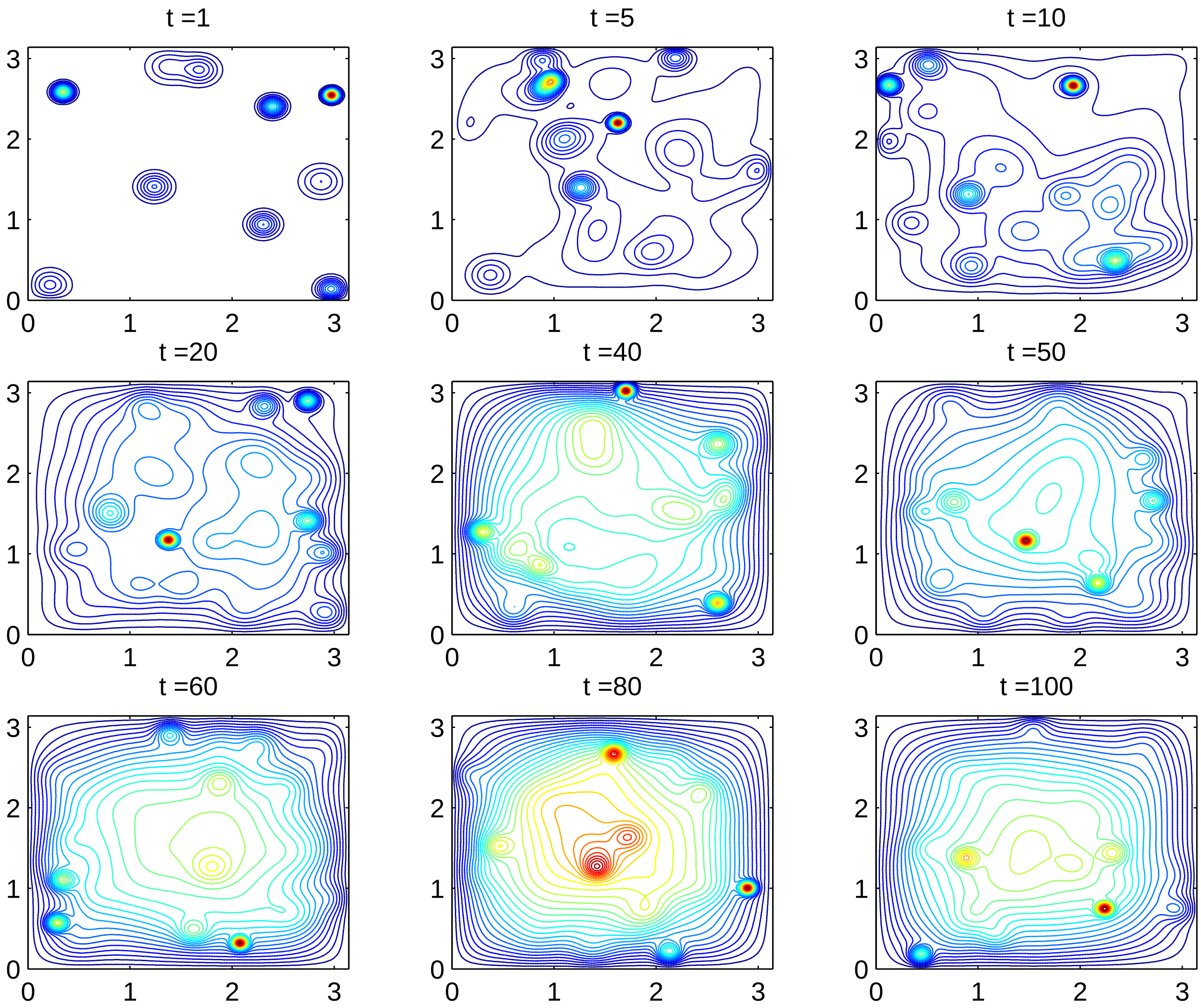
ediction via statistical theory (Grote-Maj

- EEST leads to the ground state $\sin x \sin y$
- PVST or ESTP leads to sinh-Poisson

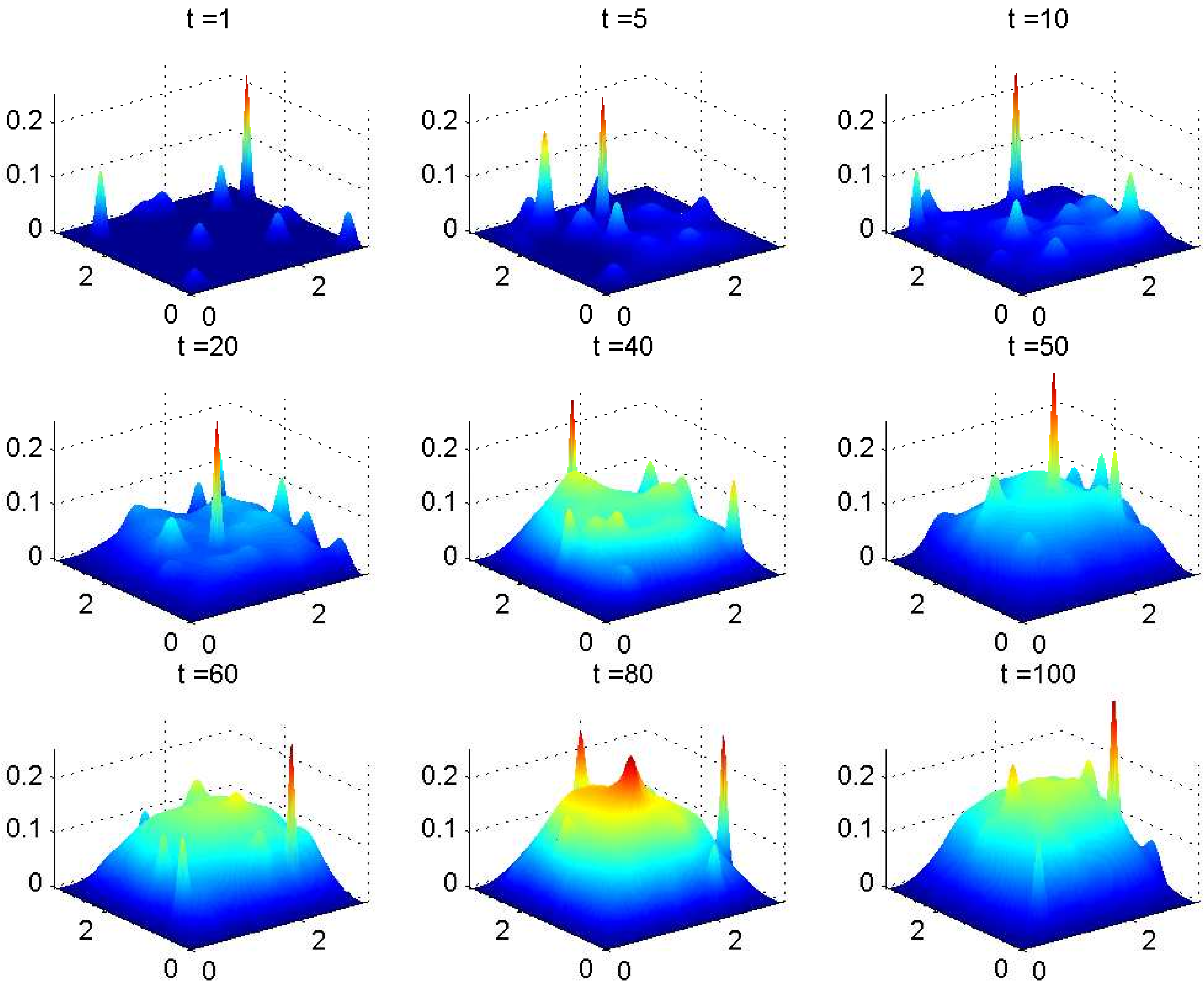
ediction via statistical theory (Grote-Maj

- EEST leads to the ground state $\sin x \sin y$
- PVST or ESTP leads to sinh-Poisson
- crude closure (tracking energy and circulation only) works very well

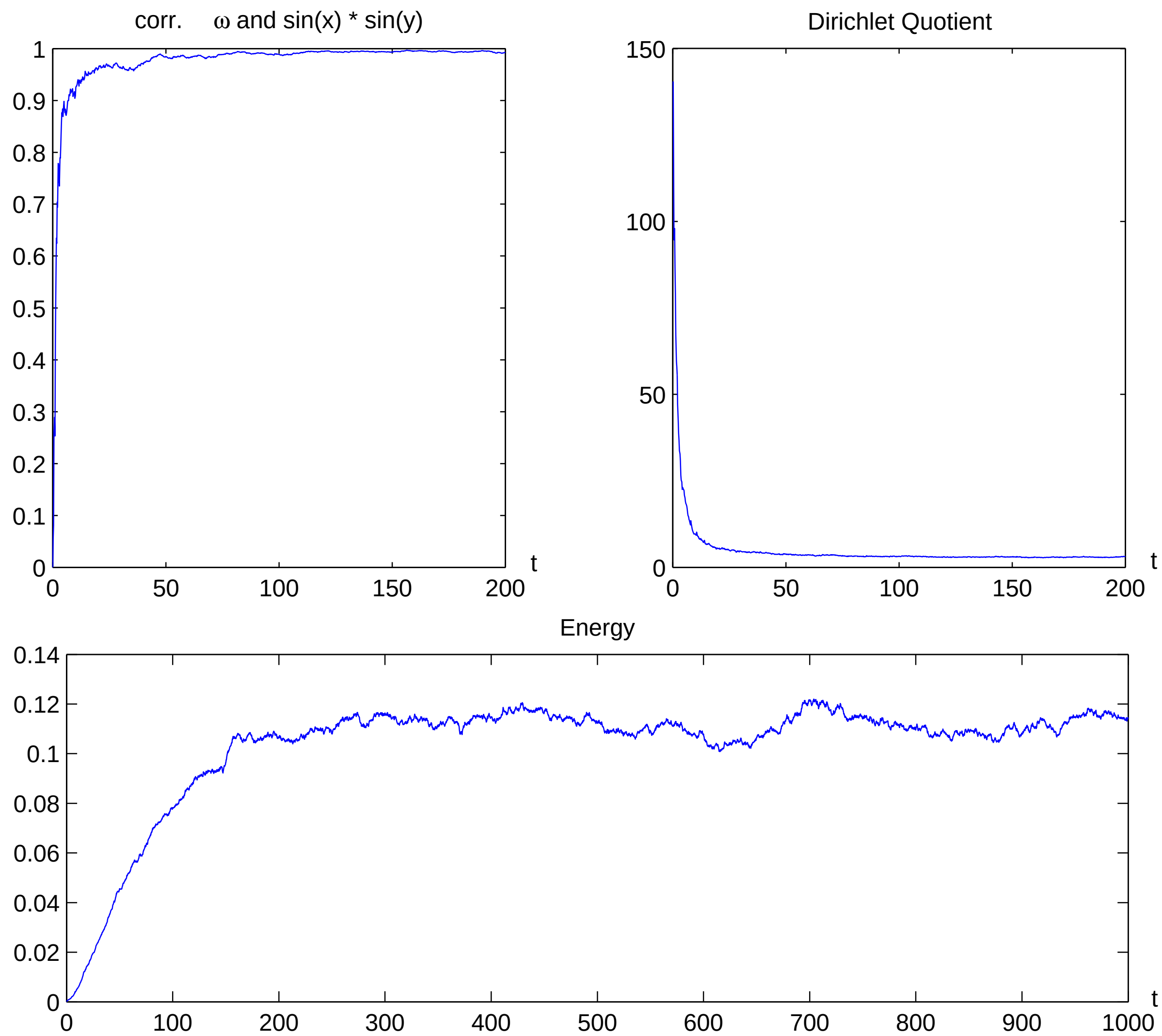
Numerical results (contours)



Numerical results (vorticity)



nerical results (correlation, D quotient, en



Stochastic approach

Stochastic approach

- Decomposition of the kick as mean plus fluctuation

$$\omega_r = \bar{\omega}_r + \omega_r', \quad \bar{\omega}_r = \mathbb{E}\omega_r$$

Stochastic approach

- Decomposition of the kick as mean plus fluctuation

$$\omega_r = \bar{\omega}_r + \omega'_r, \quad \bar{\omega}_r = \mathbb{E}\omega_r$$

- cumulative forcing effect (deterministic part)

$$\lfloor \frac{t}{\Delta t} \rfloor A\bar{\omega}_r$$

Stochastic approach

- Decomposition of the kick as mean plus fluctuation

$$\omega_r = \bar{\omega}_r + \omega_r', \quad \bar{\omega}_r = \mathbb{E}\omega_r$$

- cumulative forcing effect (deterministic part)

$$\lfloor \frac{t}{\Delta t} \rfloor A \bar{\omega}_r$$

- deterministic part remain order one requires

$$A \approx \Delta t, \text{or } A = c_r \Delta t$$

stochastic forcing (fluctuation part)

stochastic forcing (fluctuation part)

- cumulative forcing effect (fluctuation part)

$$\int_0^t \mathcal{F}' = A \frac{\omega_r'(1) + \cdots + \omega_r'(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \sqrt{\frac{1}{\Delta t}}$$

stochastic forcing (fluctuation part)

- cumulative forcing effect (fluctuation part)

$$\int_0^t \mathcal{F}' = A \frac{\omega_r'(1) + \cdots + \omega_r'(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \sqrt{\frac{1}{\Delta t}}$$

- Donsker's invariance principle

$$\int_0^t \mathcal{F}' \approx \frac{A}{\sqrt{\Delta t}} G(t) = c_r \epsilon G(t)$$

$$\epsilon = \sqrt{\Delta t}$$

Stochastic continuous version

Stochastic continuous version

- The continuous equation

$$\begin{aligned}\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}, \\ q &= \Delta \psi\end{aligned}$$

Stochastic continuous version

- The continuous equation

$$\begin{aligned} \frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}, \\ q &= \Delta \psi \end{aligned}$$

- existence and uniqueness of solutions well known, existence of invariant measure, random dynamical system, existence of random attractor well-known (Benssouson-Temam, Vishik-Fursikov, Schmalfuss, Crauel-Debussche-Flandoli...)

Heuristic limit

Heuristic limit

- heuristic limit as $\epsilon \to 0$

$$\begin{aligned} \frac{\partial q^0}{\partial t} + \nabla^\perp \psi^0 \cdot \nabla q^0 &= \nu \Delta q^0 + c_r \bar{\omega}_r, \\ q^0 &= \Delta \psi^0 \end{aligned}$$

Heuristic limit

- heuristic limit as $\epsilon \to 0$

$$\begin{aligned} \frac{\partial q^0}{\partial t} + \nabla^\perp \psi^0 \cdot \nabla q^0 &= \nu \Delta q^0 + c_r \bar{\omega}_r, \\ q^0 &= \Delta \psi^0 \end{aligned}$$

- limiting behavior in time for relatively small $c_r \bar{\omega}_r$

$$\nabla^\perp \psi^0 \cdot \nabla q^0 = \nu \Delta^2 \psi^0 + c_r \bar{\omega}_r$$

Heuristic limit

- heuristic limit as $\epsilon \to 0$

$$\begin{aligned} \frac{\partial q^0}{\partial t} + \nabla^\perp \psi^0 \cdot \nabla q^0 &= \nu \Delta q^0 + c_r \bar{\omega}_r, \\ q^0 &= \Delta \psi^0 \end{aligned}$$

- limiting behavior in time for relatively small $c_r \bar{\omega}_r$

$$\nabla^\perp \psi^0 \cdot \nabla q^0 = \nu \Delta^2 \psi^0 + c_r \bar{\omega}_r$$

limiting behavior as $c_r \to 0$

$$q^0 \approx \frac{c_r}{\nu} (-\Delta)^{-1} (\bar{\omega}_r)$$

Heuristic limit (approximation)

Heuristic limit (approximation)

- $$\bar{\omega}_r \approx r^2$$

Heuristic limit (approximation)

- $$\bar{\omega}_r \approx r^2$$

- $$q^0 \approx \frac{r^2 c_r}{\nu}(-\Delta)^{-1}(1)$$

Heuristic limit (approximation)

- $$\bar{\omega}_r \approx r^2$$

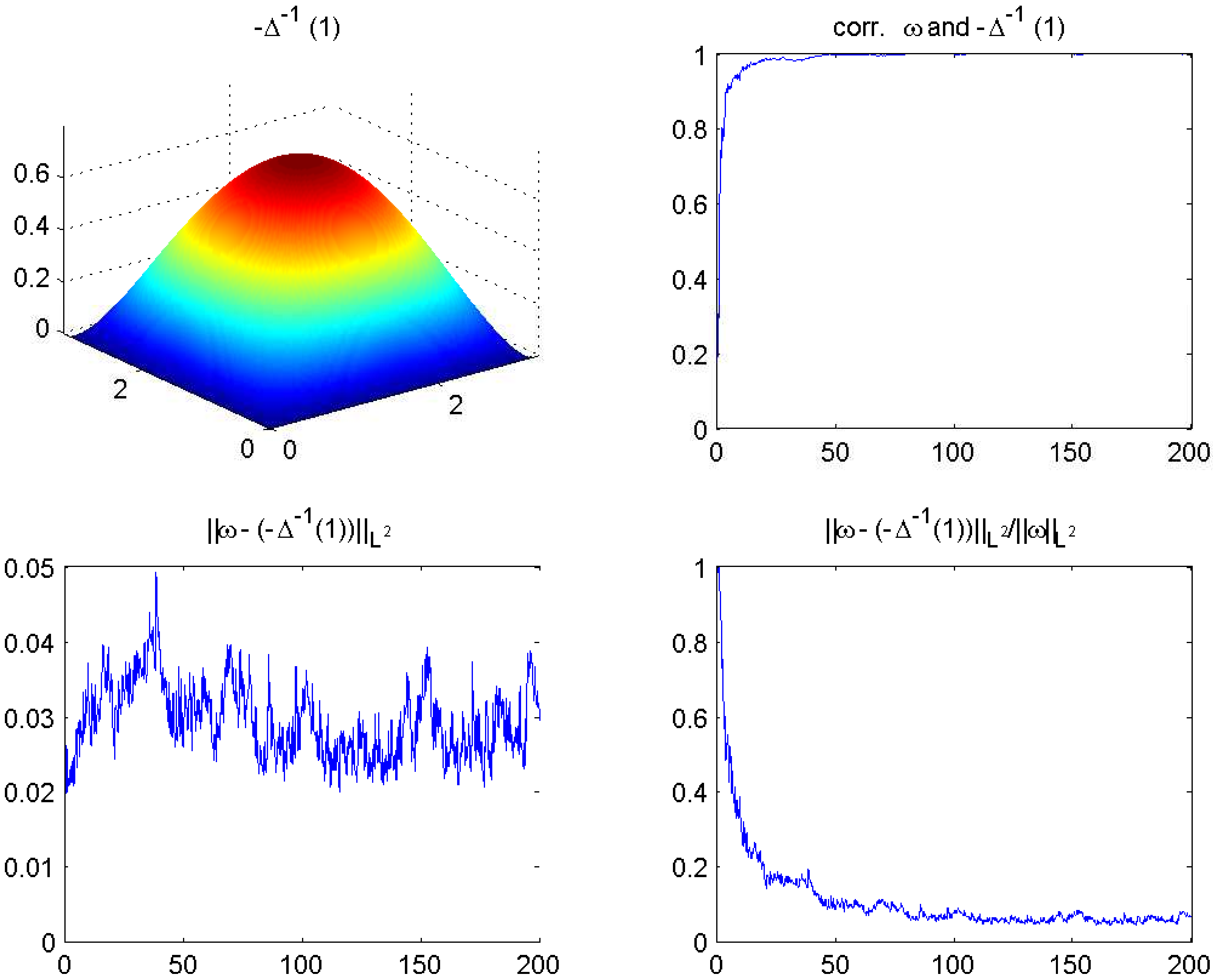
- $$q^0 \approx \frac{r^2 c_r}{\nu}(-\Delta)^{-1}(1)$$

- $$(-\Delta)^{-1}(1) = \sum_{k_j pos.odd, j=1,2} \frac{16}{\pi^2 k_1 k_2 |\vec{k}|^2} \sin(k_1 x) \sin(k_2 y)$$

Heuristic limit (approximation)

- $$\bar{\omega}_r \approx r^2$$
- $$q^0 \approx \frac{r^2 c_r}{\nu}(-\Delta)^{-1}(1)$$
- $$(-\Delta)^{-1}(1) = \sum_{k_j pos.odd, j=1,2} \frac{16}{\pi^2 k_1 k_2 |\vec{k}|^2} \sin(k_1 x) \sin(k_2 y)$$
- $$corr(\sin x \sin y, (-\Delta)^{-1}(1)) \approx 0.99$$

Numerical results and prediction



Pathwise convergence (Majda-W.)

Pathwise convergence (Majda-W.)

- **Theorem**

$$\|q - q^0\|_{L^\infty(0,T;L^2(\Omega))} \to 0, a.s.$$

Pathwise convergence (Majda-W.)

- **Theorem**

$$\|q - q^0\|_{L^\infty(0,T;L^2(\Omega))} \to 0, a.s.$$

-

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}$$

Pathwise convergence (Majda-W.)

- **Theorem**

$$\|q - q^0\|_{L^\infty(0,T;L^2(\Omega))} \to 0, a.s.$$

-

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}$$

- **For** $\tilde{q} = q - c_r \epsilon G$

$$\begin{aligned} & \frac{\partial \tilde{q}}{\partial t} + \nabla^\perp(\tilde{\psi} + c_r \epsilon \Delta^{-1} G) \cdot \nabla(\tilde{q} + c_r \epsilon G) \\ = & \quad \nu \Delta \tilde{q} + c_r \bar{\omega}_r + \nu c_r \epsilon \Delta G \end{aligned}$$

Pathwise convergence (Majda-W.)

- **Theorem**

$$\|q - q^0\|_{L^\infty(0,T;L^2(\Omega))} \to 0, a.s.$$

-

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}$$

- **For** $\tilde{q} = q - c_r \epsilon G$

$$\begin{aligned} & \frac{\partial \tilde{q}}{\partial t} + \nabla^\perp(\tilde{\psi} + c_r \epsilon \Delta^{-1} G) \cdot \nabla(\tilde{q} + c_r \epsilon G) \\ = & \quad \nu \Delta \tilde{q} + c_r \bar{\omega}_r + \nu c_r \epsilon \Delta G \end{aligned}$$

- **For** $q' = \tilde{q} - q^0$

$$\begin{aligned} & \frac{\partial q'}{\partial t} + \nabla^\perp \psi \cdot \nabla q' + \nabla^\perp(\psi' + c_r \epsilon \Delta^{-1} G) \cdot \nabla q^0 \\ = & \quad \nu \Delta q' + \nu c_r \epsilon \Delta G \end{aligned}$$

Rate of convergence (Majda-W.)

Rate of convergence (Majda-W.)

- **Theorem**

$$\mathbb{E}(\|q - q^0\|_{L^2}^2) \le \kappa \varepsilon^2$$

Rate of convergence (Majda-W.)

- **Theorem**

$$\mathbb{E}(\|q - q^0\|_{L^2}^2) \leq \kappa\varepsilon^2$$

- $q' = q - q^0$

$$dq' + (-\nu\Delta q' + \nabla^{\perp}\psi \cdot \nabla q' + \nabla\psi' \cdot \nabla q^0)dt = c_r\epsilon dG$$

Rate of convergence (Majda-W.)

- **Theorem**

$$\mathbb{E}(\|q - q^0\|_{L^2}^2) \leq \kappa\varepsilon^2$$

- $q' = q - q^0$

$$dq' + (-\nu\Delta q' + \nabla^\perp\psi \cdot \nabla q' + \nabla\psi' \cdot \nabla q^0)dt = c_r \epsilon dG$$

- **Ito's formula** $\Rightarrow$

$$\frac{d}{dt}\mathbb{E}(\|q'\|_{L^2}^2) \leq -(2\nu - \frac{c}{\nu^3}\|q^0\|_{L^2}^8)\mathbb{E}(\|q'\|_{L^2}^2) + c_r^2\epsilon^2 \sum b_{\vec{k}}^2$$

where

$$G(\vec{x}, t) = \sum b_{\vec{k}} e_{\vec{k}}(\vec{x})\beta_{\vec{k}}(t)$$

$\{e_{\vec{k}}(\vec{x})\}$ **o.n.b.,** $\{\beta_{\vec{k}}(t)\}$ **Brownians**

Convergence of attractors (Majda-W.)

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon\rightarrow 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon \to 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

- **random dynamical system**

$$\varphi : R^+ \times \Omega \times H \to H, (t, \omega, u) \rightarrowtail \varphi(t, \omega)u$$

$$\varphi(0, \omega) = id, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$$

$$(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R}),$$

θ_t **measure preserving,** $\theta_0 = id$**,** $\theta_{t+s} = \theta_t \theta_s$

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon\longrightarrow 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0,\ a.s.$$

- **random dynamical system**

$$\varphi : R^+ \times \Omega \times H \longrightarrow H, (t, \omega, u) \rightarrowtail \varphi(t, \omega)u$$

$$\varphi(0, \omega) = id, \quad \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$$

$$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R}),$$

θ_t **measure preserving,** $\theta_0 = id$**,** $\theta_{t+s} = \theta_t \theta_s$

- **random attractor** $\mathcal{A}(\omega)$**(compact, measurable)**

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon \to 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

- **random dynamical system**

$$\varphi : R^+ \times \Omega \times H \to H, (t, \omega, u) \rightarrowtail \varphi(t, \omega)u$$

$$\varphi(0, \omega) = id, \quad \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$$

$$(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R}),$$

θ_t **measure preserving,** $\theta_0 = id$**,** $\theta_{t+s} = \theta_t \theta_s$

- **random attractor** $\mathcal{A}(\omega)$**(compact, measurable)**

$$\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$$

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon \longrightarrow 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

- **random dynamical system**

$$\varphi : R^+ \times \Omega \times H \rightarrow H, (t, \omega, u) \rightarrowtail \varphi(t, \omega)u$$

$$\varphi(0, \omega) = id, \quad \varphi(t+s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$$

$$(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R}),$$

θ_t **measure preserving,** $\theta_0 = id$**,** $\theta_{t+s} = \theta_t \theta_s$

- **random attractor** $\mathcal{A}(\omega)$**(compact, measurable)**

$$\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$$

$$lim_{t \rightarrow \infty} dist(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0$$

Convergence of attractors (Majda-W.)

- **Theorem**

$$\lim_{\epsilon \longrightarrow 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

- **random dynamical system**

$$\varphi : R^+ \times \Omega \times H \longrightarrow H, (t, \omega, u) \rightarrowtail \varphi(t, \omega)u$$

$$\varphi(0, \omega) = id, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$$

$$(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R}),$$

θ_t **measure preserving,** $\theta_0 = id$**,** $\theta_{t+s} = \theta_t \theta_s$

- **random attractor** $\mathcal{A}(\omega)$**(compact, measurable)**

$$\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$$

$$lim_{t \rightarrow \infty} dist(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0$$

generalization of Caraballo-Langa-Robinson

Commutative diagram (Majda-W.)

Theorem

$$\begin{array}{ccc} q(t,\omega) & \longrightarrow & q_\infty(\omega) \\ \Downarrow & & \Downarrow \\ q^0(t) & \longrightarrow & q^0_\infty \end{array}$$

Uniqueness of invariant measure

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

- Invariant measure is unique for small data

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

- Invariant measure is unique for small data
- $q' = q^2 - q^1$

$$\frac{d}{dt}\|q'\|^2 \le (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu})\|q'\|^2$$

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

- Invariant measure is unique for small data
- $q' = q^2 - q^1$

$$\frac{d}{dt}\|q'\|^2 \leq (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu})\|q'\|^2$$

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

- Invariant measure is unique for small data
- $q' = q^2 - q^1$

$$\frac{d}{dt}\|q'\|^2 \leq (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu})\|q'\|^2$$

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)
- E, Flandoli, Kuksin, Mattingly, Maslowski, Schmalfuss, Sinai, Shirikyan, ...

Discrete case

Discrete case

- Discrete time Markov processes

$$\eta_-^{j+1} = S(\Delta t)(\eta_-^j + A\omega_r(j)),$$
$$\eta_+^{j+1} = S(\Delta t)(\eta_+^j) + A\omega_r(j)$$

Discrete case

- Discrete time Markov processes

$$\eta_-^{j+1} = S(\Delta t)(\eta_-^j + A\omega_r(j)),$$
$$\eta_+^{j+1} = S(\Delta t)(\eta_+^j) + A\omega_r(j)$$

- existence and uniqueness of invariant measure, random attractor ...

Discrete case

- Discrete time Markov processes

$$\begin{aligned} \eta_-^{j+1} &= S(\Delta t)(\eta_-^j + A\omega_r(j)), \\ \eta_+^{j+1} &= S(\Delta t)(\eta_+^j) + A\omega_r(j) \end{aligned}$$

- existence and uniqueness of invariant measure, random attractor ...
- Invariant measure concentrated around a large coherent structure (Majda-W.)

Discrete case

- Discrete time Markov processes

$$
\begin{aligned}
\eta_-^{j+1} &= S(\Delta t)(\eta_-^j + A\omega_r(j)), \\
\eta_+^{j+1} &= S(\Delta t)(\eta_+^j) + A\omega_r(j)
\end{aligned}
$$

- existence and uniqueness of invariant measure, random attractor ...
- Invariant measure concentrated around a large coherent structure (Majda-W.)
- Kuksin-Shirikyan, Masmoudi-Young, etc

Summary

Summary

- Random small scale bombardments could induce large coherent structure

Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory

Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero

Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
- Different large coherent structure could emerge depending on different distribution of small scale forcing

Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
- Different large coherent structure could emerge depending on different distribution of small scale forcing
- Generalizes to other geometry and more general one layer system, or multi-layer system

Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
- Different large coherent structure could emerge depending on different distribution of small scale forcing
- Generalizes to other geometry and more general one layer system, or multi-layer system
- Long way to go to reach our goal

Questions

Questions

- Geophysical effects (β, F, topography, Ekman, $\cdots$)?

Questions

- Geophysical effects (β, F, topography, Ekman, $\cdots$)?
- What if smallness assumption is violated?

Questions

- Geophysical effects (β, F, topography, Ekman, $\cdots$)?
- What if smallness assumption is violated?
- What if the mean of the forcing is zero?

Questions

- Geophysical effects (β, F, topography, Ekman, $\cdots$)?
- What if smallness assumption is violated?
- What if the mean of the forcing is zero?
- Vanishing viscosity and noise?

Questions

- Geophysical effects (β, F, topography, Ekman, $\cdots$)?
- What if smallness assumption is violated?
- What if the mean of the forcing is zero?
- Vanishing viscosity and noise?
- Convergence from discrete to the continuous case?

Vanishing noise (SRB)

$$\mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt}$$

Vanishing noise (SRB)

$\mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt}$

- non-trivial global attractor for deterministic NSE

Vanishing noise (SRB)

$\mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt}$

- non-trivial global attractor for deterministic NSE
- unique invariant measure with appropriate noise

Vanishing noise (SRB)

$\mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt}$

- non-trivial global attractor for deterministic NSE
- unique invariant measure with appropriate noise
- Does the invariant measures converge?

Vanishing noise (SRB)

$\mathcal{F} = \bar{\mathcal{F}} + \epsilon \frac{dG}{dt}$

- non-trivial global attractor for deterministic NSE
- unique invariant measure with appropriate noise
- Does the invariant measures converge?
- Converge to what?

Acknowledgements

Acknowledgements

- Financial Support: NSF, NYU, FSU

Acknowledgements

- Financial Support: NSF, NYU, FSU
- People: Wendy Cheng, Weinan E, Marcus Grote, Nader Masmoudi, Charles Newman, Toufic Suidan

The End