From Quasigeostrophic Turbulence to Stratified Turbulence

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Mesoscale atmospheric spectrum

- Gage (1979) then Lilly (1983) argued that there was an inverse cascade from convective scales. Rotation unimportant.

- Van Zandt (1982) said that it was a “gravity-wave spectrum”.

An early test of the Gage-Lilly idea

No inverse cascade with strong stratification and no rotation.

Flow reached statistical stationarity (except for shear modes). Therefore energy went downscale.

Later, Métais et al. added strong rotation and obtained an inverse cascade.
Boussinesq Normal Mode Decomposition

Let $G_k = B_k^{(0)}$ and $A_k^{(\pm)}(\epsilon t)e^{\pm i\omega_k t} = B_k^{(\pm)}$, where $\epsilon = \min(Ro, Fr)$.

\[
\frac{\partial}{\partial(\epsilon t)} \langle |G_k|^2 \rangle = \sum_{\Delta} N_1 \langle G_k G_p G_q \rangle + \langle G_k G_p A_q \rangle e^{i\omega_p t} + \langle G_k A_p A_q \rangle e^{i(\omega_q + \omega_p) t}
\]

\[
\frac{\partial}{\partial(\epsilon t)} \langle |A_k|^2 \rangle = \sum_{\Delta} M_1 \langle A_k G_p G_q \rangle e^{i\omega_k t} + M_2 \langle A_k A_p G_q \rangle e^{i(\omega_k + \omega_p) t} + M_3 \langle A_k A_p A_q \rangle e^{i(\omega_k + \omega_p + \omega_q) t}
\]

Energy, $E = \sum_k |G_k|^2 + |A_k|^2 = E_G + E_A$.

Potential Enstrophy, $V = \sum_k \sigma_k^2 |G_k|^2$ as.

Gravity-wave dispersion relation, $\sigma_k = \left( f^2 c \right)$. 
Strong rotation and strong stratification

Geostrophic, $G$ and ageostrophic, $A$ spectra

- $G$ modes cascade upscale, $A$ modes downscale → balance
- Balance requires dissipation of balanced modes to be less than that of unbalanced ones (e.g. inverse cascades).
The atmosphere?

- Boussinesq with $N/f = 10$ + Rayleigh damping
- Grid: $500 \times 500 \times 50$ with $\Delta x = \Delta y = \Delta z$
- A set of baroclinic QG modes with $L/H = N/f$ are forced.
Stratification only: Dimensionless equations

- Nondimensionalise using \( u \sim U, \, x, \, y \sim L, \, z \sim H \), etc.

- Wave: \( T \sim L/NH \) \quad Vortical: \( T \sim L/U \).

- Two Froude numbers:
  \[
  Fr_h = \frac{U}{NL}, \quad Fr_z = \frac{U}{NH}.
  \]

Two scaling ideas:

- Integral scale vertical Froude number \( \rightarrow O(1) \), i.e. \( H_i \sim \sqrt{E}/N \)

- Vertical Froude is unity at all vertical scales, i.e.
  \[
  H(k_z) \sim [k_z E(k_z)]^{1/2}/N \quad \text{or equivalently} \quad E(k_z) \sim N^2 k_z^{-3}
  \]
Limit of Strong Stratification ($\Omega = 0$)

- The vertical scale collapses until dissipation scale reached.
- Two limits:
  1. $Fr_h \to 0$, $Fr_h/Fr_z$ fixed (RMW 1981)
     - Appropriate if flow is initially isotropic or when vertical scale determined by dissipation
     - Potential enstrophy approximately quadratic
  2. $Fr_h \to 0$, $Fr_z$ fixed
     - Implies $H \sim U/N$ set by stratification. Due to
       i) Thorpe (1977) (introduced characteristic scale for overturning. $U/N$ is an upper bound.)
       ii) Munk (1981) (identified it as small-scale end of Garret-Munk spectrum)
       iii) Hines (1996) (transition scale between unsaturated and saturated waves)
       iv) Billant & Chomaz (2001)
Stratification only

- Vectors and $\nabla$, etc. are horizontal components
- Use vortical time scale $T \sim L/U$:

$$
\frac{\partial u}{\partial t} + u \cdot \nabla u + Fr_z^2 w \frac{\partial u}{\partial z} = -\nabla p,
$$

$$
Fr_h^2 \left( \frac{\partial w}{\partial t} + u \cdot \nabla w + Fr_z^2 w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + b,
$$

$$
\nabla \cdot u + Fr_z^2 \frac{\partial w}{\partial z} = 0,
$$

$$
\frac{\partial b}{\partial t} + u \cdot \nabla b + Fr_z^2 w \frac{\partial b}{\partial z} = -w.
$$

- Decoupled layers when $Fr_h \to 0$, $Fr_z \to 0$.
- 3D hydrostatic turbulence if $H \sim U/N$
Wave/vortical decomposition without rotation

- Need to be careful: potential enstrophy only approximately quadratic when $Fr_z \ll 1$, but this may not be relevant.
- We need to restrict application to scales much larger than $H_i$ or to particular problems, e.g. early decay of isotropic i.c.’s.
- If $Fr_z \ll 1$ we can examine inviscid truncated statistical equilibrium

\[
\langle |G_k|^2 \rangle = \frac{1}{\lambda_1 + \lambda_2 k_z^2} \quad \langle |A_k|^2 \rangle = \frac{2}{\lambda_1}
\]

- Independent of $k_z$
- Can show that if $k_i \ll k_{\text{max}}$, where $k_i$ defined by the ratio of potential enstrophy to energy, then as $k_{\text{max}} \to \infty$, the spectrum is peaked at a wavenumber $\rightarrow \sqrt{2}k_i$.
- In 2D turbulence, this wavenumber goes to zero. This argument suggest that there is no inverse cascade.
Vortical forcing ($\Omega = 0$) : perpendicular vorticity

$x - z$ plane

$x - y$ plane
Vortical forcing ($\Omega = 0$): $k_z$ spectra

Vortical energy

Wave energy

offset by factors of 10
Vortical Forcing ($\Omega = 0$): Length scales

Vortical forcing:

- $k_b \sim \sqrt{N^3/\epsilon}$  Ozmidov
- $k_d$ is dissipation wavenumber
Wave forcing \((\Omega = 0)\): \(N^2 k_z^{-3}\) spectra?

Dashed line: \(0.1N^2 k_z^{-3}\) (Bouruet-Aubertot, Sommeria & Staquet 1996)

Figure 9. Vertical wavenumber spectra of wave energy, along with the hypothetical saturation spectrum \(0.1N^2 k_z^{-3}\), for (a) \(N = 4\), (b) \(N = 8\), (c) \(N = 16\) and (d) \(N = 32\) when \(M = 180\) and \(R = 0\).
Strong stratification at various rotations

No rotation $\rightarrow$ lots of rotation

This is the horizontal velocity component perpendicular to the screen.
Strong stratification at various rotations

Stratified turbulence. Vertical lengthscale $= U/N$

Quasigeostrophic turbulence. Vertical lengthscale $= fL/N$
(Charney 1971)

This is how atmosphere/ocean modellers will need to set the ratio $\Delta z/\Delta x$ as resolutions increase.
Back to strong stratification and rotation

Balance at Ro=0.09?

We have used the QG modes and the $\omega$ equation to diagnose vertical velocity (left).

The real vertical velocity is on the right.

If balance exists, it isn’t this simple.
Regarding recent dipoles... (see Snyder et al.)

- Boussinesq with $N/f = 10$, $Ro=0.7$. Grid: $256 \times 256 \times 16$ with $\Delta x = \Delta y = \Delta z$
- Initial flow is a 2D dipole + a little 3D noise.
Summary of the New Stuff

Stratified Turbulence without Rotation

- Vortical forcing:
  - Direct cascade.
  - Simulations confirm $H \sim U/N$.

- Wave forcing:
  - $N^2 k_z^{-3}$ not observed.

Both: When $H$ not resolved by $\Delta z$ weird things happen.

Stratified Turbulence with rotation

- Growth in the vertical integral scale as $\Omega$ increases
- Bands of ageostrophic energy remain localised near vortices.
- Steep vortical spectra, shallow ageo spectra very robust
Extra stuff
Observations: The Atmosphere

- GARP 1979
- Instruments on 7900 commercial flights
- Nastrom & Gage (1986)
Vortical forcing ($\Omega = 0$): $k_h$ spectra

Vortical energy

Wave energy
Wave forcing ($\Omega = 0$): dependence on $Re$

$Fr_h \sim 1$

$Fr_h \sim 0.1$

- Decreasing $Fr_h \rightarrow$ steeper spectra $\rightarrow$ nonlocal interactions.
Bottleneck at $\Delta^4$