Reduced Equations for Langmuir Turbulence

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Langmuir Circulation (LC) Windrows

Related Work – Theory and Simulation

Quasi-Laminar Simulations of 2D Craik–Leibovich (CL) Equations
- Gnanadesikan & Weller (*JPO* 1995)

Weakly Nonlinear 2D and 3D Investigations
- 2D: Leibovich, Lele & Moroz (*JFM* 1989)

Small Wavenumber Finite-Amplitude 3D Investigations

Simulations of Full 3D Craik–Leibovich (CL) Equations
- DNS: Tandon & Leibovich (*JGR*, 1995)
Goals and Motivation

**Objective** Obtain reduced PDE model capable of describing coarse-grained, strongly anisotropic but otherwise turbulent LC dynamics.

**Motivation**

- Secondary stability analysis by Tandon & Leibovich (*JPO*, 1995)

**Purpose**

- Reveal dominant 3D physics.
- Enable simpler (e.g. upper-bound) analysis.
- Less expensive numerical simulations for multi-scale process studies.
- Incorporation into formal multiscale numerical scheme.
Isotropically Scaled CL Equations

- Consider full 3D, isotropically-scaled CL equations, where two parameters \( R_* \equiv u_* H / \nu_e \), \( La_t = \sqrt{u_*/u_{s0}} \) replace single parameter \( La \equiv La_t R_*/3/2 \):

\[
\frac{Du}{Dt} = -\nabla p + \frac{1}{La_t^2} (U_s \times \omega) + \frac{1}{R_*} \nabla^2 u
\]

- Two turbulence regimes:

  **Shear flow turbulence regime:** \( La_t \gg 1 \) with \( R_* \gg 1 \).
  **Langmuir turbulence regime:** \( La_t = O(0.1) \) with \( La \ll 1 \).

- Motivates consideration of formal limit \( La_t \to 0 \) with \( R_* \) fixed or \( R_* \to \infty \):

  **2D dynamics:** \( \Omega \neq 0 \), \( \partial_x \Omega = 0 \), \( u \)-fluctuations \( \ll (v, w) \)-fluctuations.
Anisotropic Velocity Scalings

- Employ anisotropic velocity scales to capture nonlinear, spatially anisotropic reduced dynamics:

\[
L_x = H, \quad (L_y, L_z) = H, \quad T = H/V \\
\mathcal{U} = u^* R^*, \quad (V, \mathcal{W}) = \sqrt{\mathcal{U} u_{s0}}, \quad \mathcal{P} = \rho \mathcal{V}^2
\]

- In essence, perturbing off of strictly 2D \([\partial(\cdot)/\partial x = 0]\) problem.

- Identify \(\varepsilon \equiv \mathcal{U}/\mathcal{W} = R^{1/2}_{*} L a_t\) (cf. Tejada-Martinez & Grosch 2007).
Rescaled CL Equations in Strong Wave–Forcing Limit

\[
\begin{align*}
\partial_t u + \varepsilon u \partial_x u + (\mathbf{v}_\perp \cdot \nabla \perp) u &= -\varepsilon^{-1} \partial_x P + \varepsilon R_*^{-2} \left[ \partial_x^2 + \nabla_\perp^2 \right] u \\
\partial_t \mathbf{v}_\perp + \varepsilon u \partial_x \mathbf{v}_\perp + (\mathbf{v}_\perp \cdot \nabla \perp) \mathbf{v}_\perp &= -\nabla \perp P + U_s \left( \nabla \perp u - \varepsilon^{-1} \partial_x \mathbf{v}_\perp \right) \\
&\quad + \varepsilon R_*^{-2} \left[ \partial_x^2 + \nabla_\perp^2 \right] \mathbf{v}_\perp \\
\varepsilon \partial_x u + \nabla \perp \cdot \mathbf{v}_\perp &= 0
\end{align*}
\]

- Wind stress BC: \( \partial_z u = 1 \) along \( z = 0, -1 \).
- \( x \)-invariance at leading-order: \( \partial_x P = \partial_x v = \partial_x w = 0 \) and \( \nabla_\perp \cdot \mathbf{v}_\perp = 0 \).
**Multiple Scale Expansion**

1. Limit process: $\varepsilon \to 0$, i.e. $La_t \to 0$, $R_* = La_t^{-2\alpha/(1-\alpha)}$, $0 \leq \alpha < 1/2$.

2. Introduce slow $x$ scale: $X \equiv \varepsilon x$ so that $\partial_x \to \partial_x + \varepsilon \partial_X$.

3. Expand fields:

   \[
   u(x, y, z, t) = u_0(x, X, y, z, t) + \varepsilon u_1(x, X, y, z, t) + \ldots
   \]
   \[
   v_\perp(x, y, z, t) = v_\perp(X, y, z, t) + \varepsilon v_1(\perp x, X, y, z, t) + \ldots
   \]
   \[
   P(x, y, z, t) = P_0(X, y, z, t) + \varepsilon P_1(x, X, y, z, t) + \ldots
   \]

4. Substitute into PDEs, collect terms of like order and **average** over fast $x$.

5. Obtain closed set of equations for $\bar{u}_0 \equiv U(X, y, z, t)$, $v_\perp \equiv V_\perp(X, y, z, t)$ and $P_0 \equiv \Pi(X, y, z, t)$.
Reduced PDEs

- Define:
  \[ D_t^\perp(\cdot) \equiv \partial_t(\cdot) + (\mathbf{V}_\perp \cdot \nabla_\perp)(\cdot) \equiv \partial_t(\cdot) + J[(\cdot), \psi], \]
  where \( J[(\cdot), \psi] = \partial_z \psi \partial_y(\cdot) - \partial_y \psi \partial_z(\cdot). \)

- Reduced dynamics governed by:

\[
\begin{align*}
D_t^\perp U &= -\partial_X \Pi + La \nabla_\perp^2 U \\
D_t^\perp \Omega + U_s(z) \partial_X \Omega &= U_s'(z)(\partial_X V - \partial_Y U) + La \nabla_\perp^2 \Omega \\
\nabla_\perp^2 \Pi &= 2J[\partial_y \psi, \partial_z \psi] + \nabla_\perp \cdot (U_s(z) \nabla_\perp U) + U_s'(z) \partial_X (\partial_y \psi) \\
\nabla_\perp^2 \psi &= -\Omega, \quad \mathbf{V}_\perp \equiv \nabla_\perp \times \psi \hat{\imath} 
\end{align*}
\]

- Fast \( x \) averaged BCs along \( z = 0, -1: \) \( \partial_z U = 1, \quad \Omega = 0, \quad \psi = 0. \)

- Advection by \( U \) and stretching of \( \Omega \) are subdominant processes.
Strongly Nonlinear, Strictly 2D Convective States

$\psi(y, z)$  

$U(y, z)$  

$\Omega(y, z)$

- Steady-state $U(y, z)$ profiles show excellent qualitative agreement with $x-t$ averaged LES profiles of Tejada–Martinez & Grosch (2007).
Asymptotic analysis predicts core vorticity $|\tilde{\Omega}| \sim 1 \ \forall \ k$ as $La \to 0$.

With $\psi(y, z)$ known, boundary/interior layer problems linearize and entire solution can be approximated asymptotically.

Reduced PDEs — Linear and Secondary Stability Results

Conclusions

- Derived reduced PDEs for anisotropic Langmuir turbulence in strong wave-forcing limit that capture dominant linear and secondary instabilities.

- Ongoing investigations:
  - time-dependent simulations of reduced system.
  - incorporation of stratification (interactions b/w LC and internal waves) and rotation.
  - exploration of scalings yielding complementary reduced PDE models of Langmuir turbulence.
  - application of methodology to other shear-flow instability phenomena.

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Future Directions — Submesoscale–Mesoscale Interactions

- Investigate multiscale interactions b/w mixed-layer LC and submeso- and mesoscale eddies.

- Develop modulation (or homogenization) theory for asymptotic strongly-nonlinear 2D LC solutions.

- Construction of a hierarchy (algebraic, ODE, reduced PDE) of LC vertical-flux parameterizations for use in OGCMs.