Spatially localized analysis of dynamically adaptive spectral-element simulations

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Turbulent theory and modeling, IMAGe Theme for 2008,

Atmospheric vortices contain significant multiscale nonlinear interactions

Figure 1. Left: space-shuttle STS-66 photograph of cloud formations due to von Karmen vortices near Heard Island, 1994/11 (NASA Science Photo Library E120/457). Right: same, chromatically adjusted to suppress some non-vortical features, and annotated to show a scale $\backslash \approx 20$km, (anti-)cyclonic vortex centers $+$ ($-$) and approximate hyperbolic points (**x**).

Spectral method: some cons

(thanks to M. Taylor)

Spectral methods are excellent, except:

- Nonlinear terms must be computed in physical space.
- An fFt costs $\mathcal{O}[M \log M]$, but other geometries require transforms costing $\mathcal{O}[M^2]$ for each coordinate.
- Transforms require all-to-all communication that reduces parallel-computation scaling.
- Global Fourier analysis obscures physical-location information.

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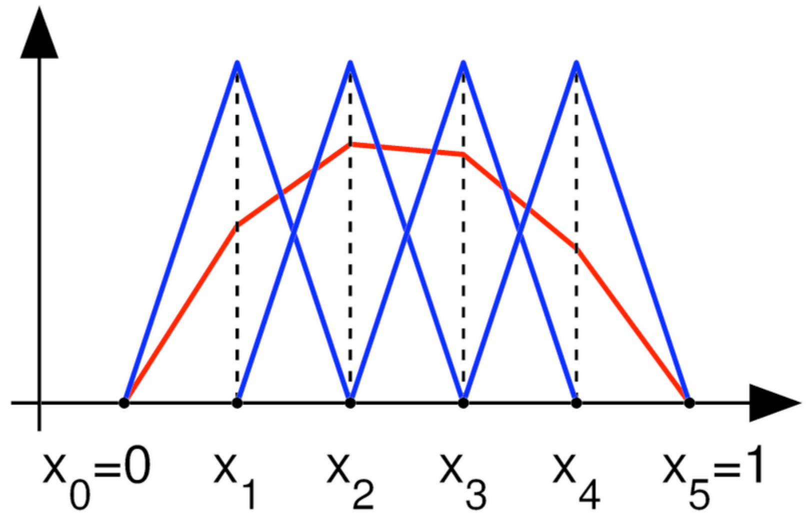
Finite-element method, roughly (wikipedia image)

The red curve approximates a smooth function $\psi[x]$ as a weighted sum of 4 blue "tent" functions $\phi_k[x]$. One can state *exactly* :



$$\psi[x] = \sum_{k=1}^{4} \Big(\psi[x_k]\phi_k[x] + \tfrac{1}{2}(x-x_k)(x-x_{k\pm1})\psi''[\xi_k^{\pm}[x]]\Big).$$

FEM pros & cons

- Nonlinear terms are straightforward.
- Complicated geometries and bcs can be treated.
- Efficient parallelization.
- Generally, error goes like h^1 or h^2, where h is the size of the largest element.
- Recovered spectral information tends to be poor.

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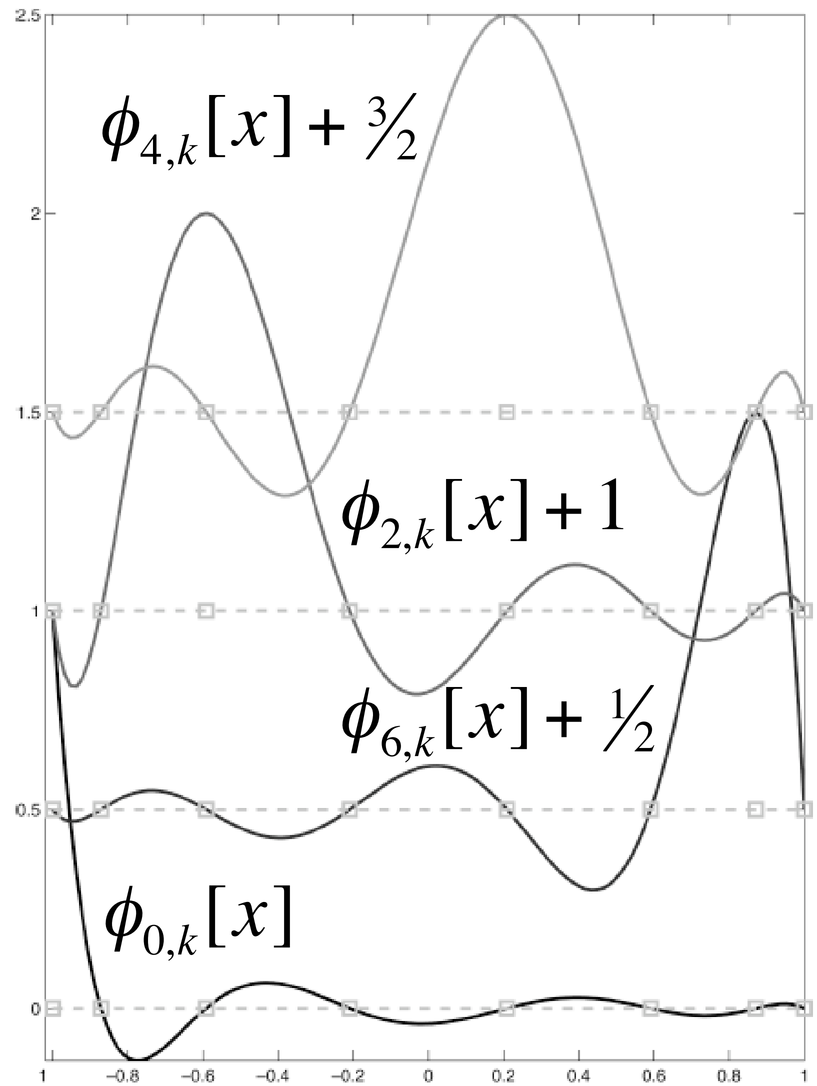
SEM, roughly

(e.g., Fournier et al. *MWR* 2004)

Where FEM uses a basis $\phi_k[\vec{x}]$ that is piecewise linear and interpolating, SEM uses $\phi_{\vec{\jmath},k}[\vec{x}]$ that is piecewise degree-p polynomial and interpolates p-1 additional interior points per direction: $\phi_{\vec{\imath},k}[\vec{x}_{\vec{\jmath},l}] = \delta_{\vec{\imath},\vec{\jmath}}\,\delta_{k,l}$.



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SEM, roughly … (e.g., Fournier et al. *MWR* 2004)

The Gauss-node distribution enables the $\phi_{j,k}[x]$ representation to be as accurate as a Fourier-Legendre expansion in each direction.

For example, the error $\tilde{\psi}[x]$ in solving $\psi'' + \omega^2\psi = f$ is bounded as $\left\|(\mathrm{d}/\mathrm{d}x + \mathrm{i}\omega)\tilde{\psi}\right\| \leq C_s\, h^{\min[p,s]} p^{-s} \left\|\psi_{\text{true}}^{(s+1)}\right\|$ assuming $\left\|f^{(s\text{-}1)}\right\| < \infty$, <u>similar to spectral method!</u>

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Fourier analysis on spectral elements

(Fournier *J. Comp. Sci.* 2006)

- $\mathbb{C}^0$-continuous and $\mathbb{C}^1$-discontinuous implies that standard N^d-point uniform cubature for the Fourier coefficient $\boldsymbol{u_q}$ potentially commits an $\mathcal{O}(N^{-d-1})$ error.

- *This* error can be *completely* eliminated starting from known (Legendre polynomial)$_{\boldsymbol{q}}$.

Analysis of sinqx using 1D spectral elements

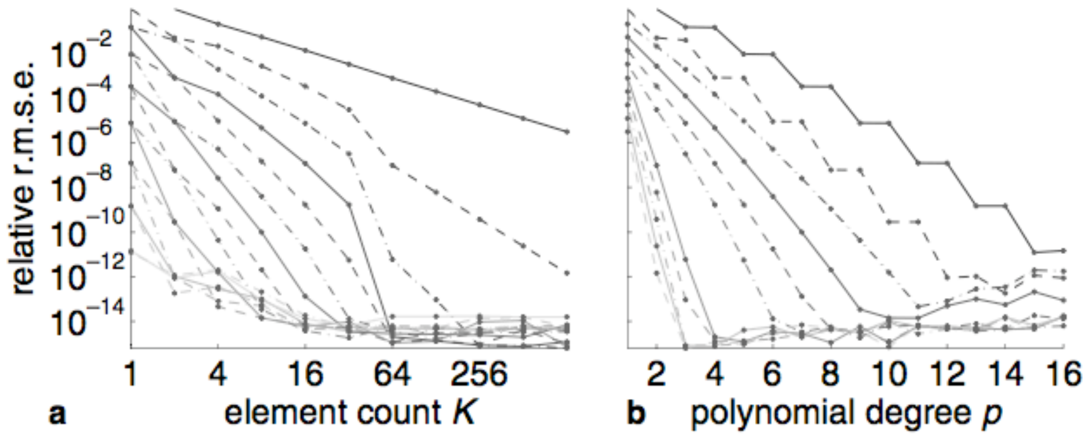
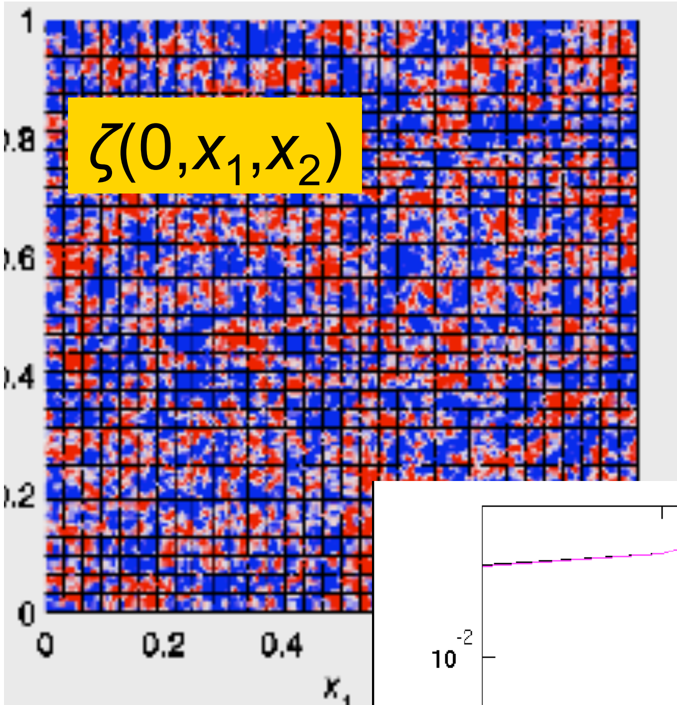


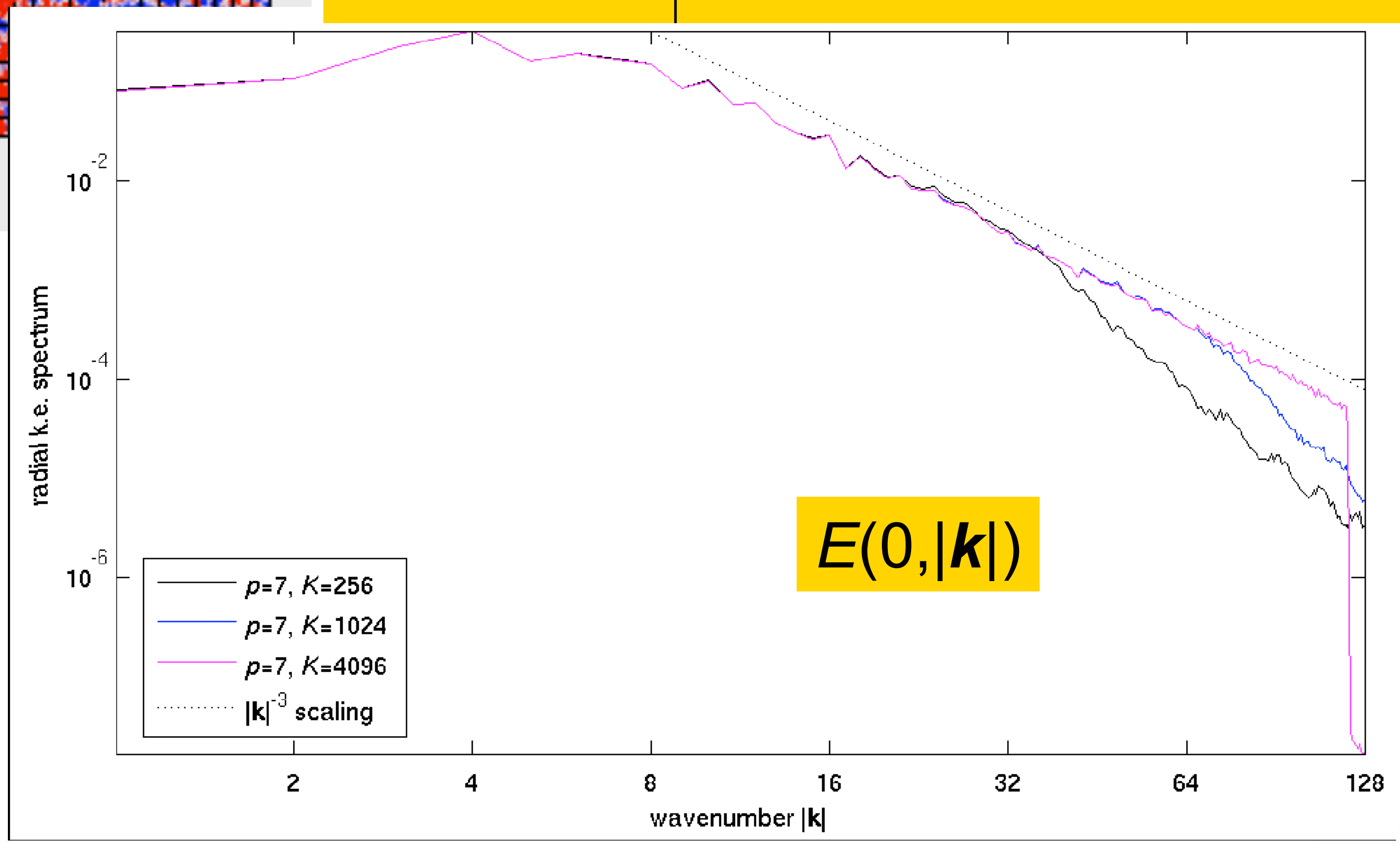
Fig. 1. Relative r.m.s. error in (3) for $u(x) = \sin x$, (a) vs. $K = 2\pi/h$ for $p \in \{1, \ldots, 16\}$ (dark to light), and (b) vs. p for $\log_2 K \in \{0, \ldots, 10\}$ (dark to light).

E.g., at degree p=2 or 8, need K=1024q or 8q elements (Kp points) to compute Fourier coefficient to 12 digits.
Fournier 2006.



Decaying incompressible Navier-Stokes: scaling $E(0,|\boldsymbol{k}|)$, random phase i.c.

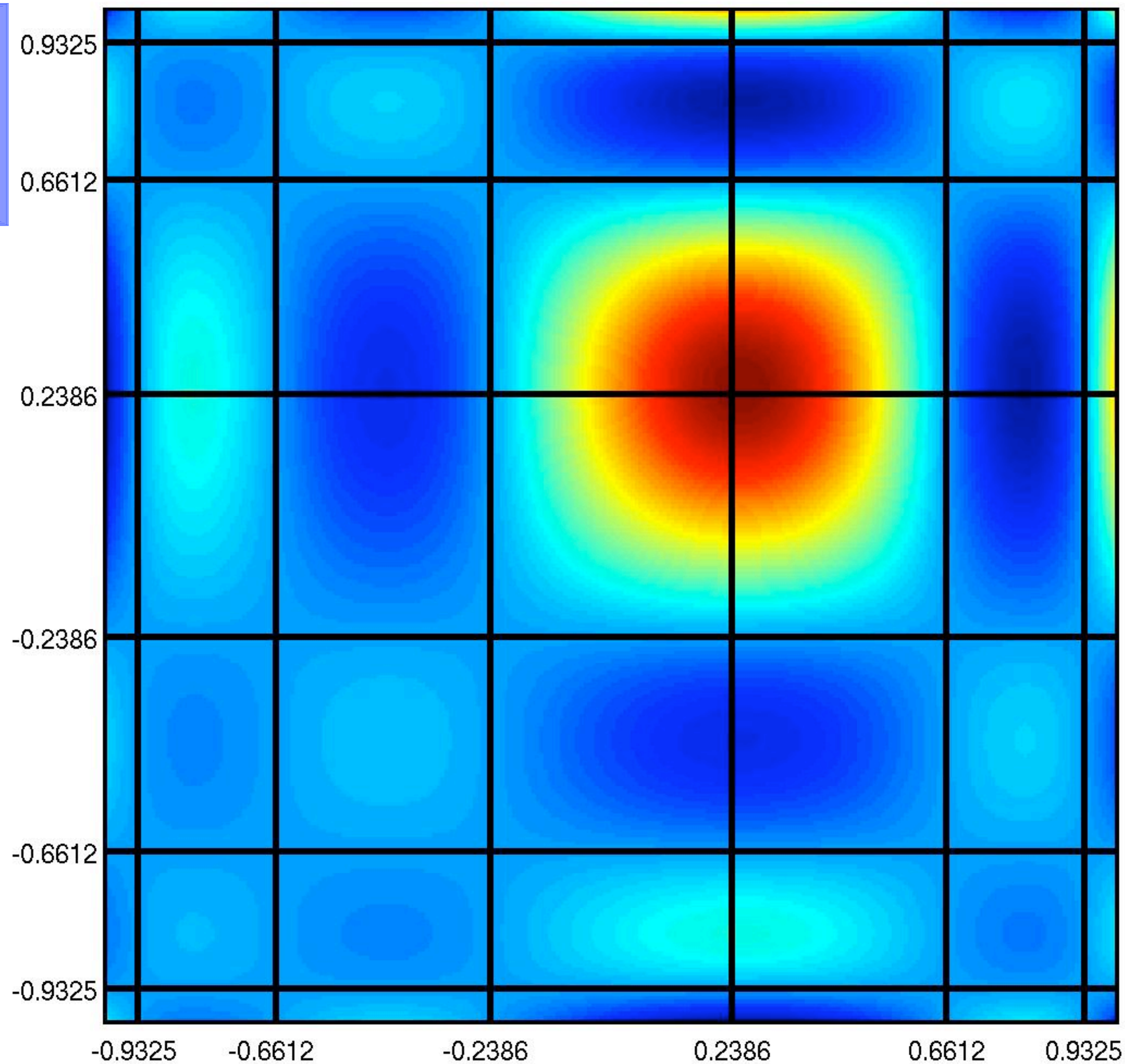
$p = 7$	Initial condition: Matthaeus, Stribling, Martinez, Oughton & Montgomery 1991 (dealiased pseudospectral, 512^2 d.o.f.).
$K = 16^2, 32^2, 64^2$	GASpAR simulation code: Rosenberg, Fournier, Fischer & Pouquet 2006.



2D Spectral elements

- Patera 1984,
- Karniadakis & Sherwin 1999,
- Deville et al. 2002

Here's 1 of 36 basis functions ϕ_j:



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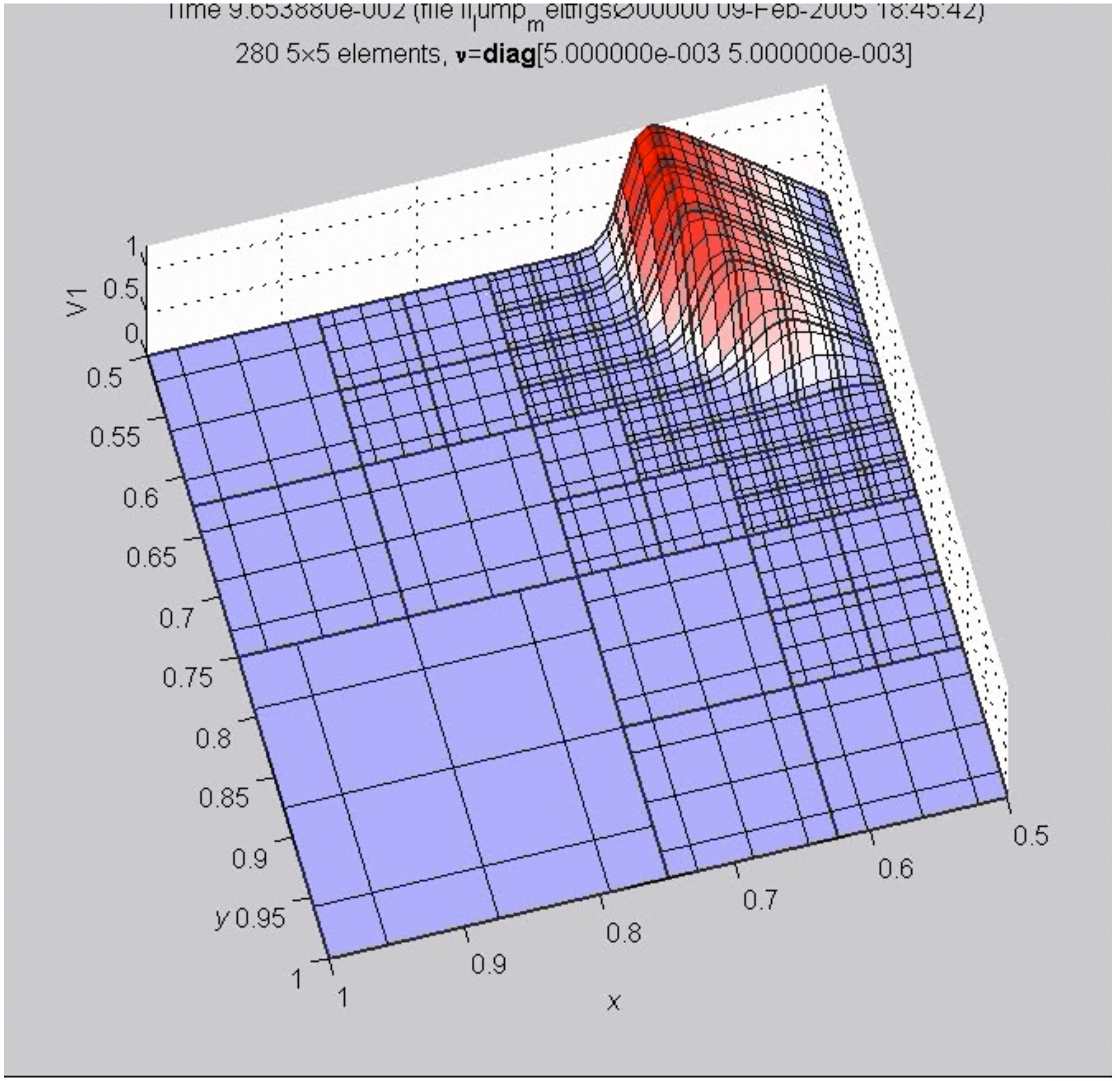
2D Burgers eq., *Re*=200

Adaptive *nonconforming* refinement for a nonlinear radial "N-wave". Each element has degree p=4.

Rosenberg, Fournier, Fischer & Pouquet 2006.



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Decaying incompressible Navier-Stokes: 3 vortices

Degree $p = 7$, element count K varies.

Viscosity ν , 1-vortex circulation Γ

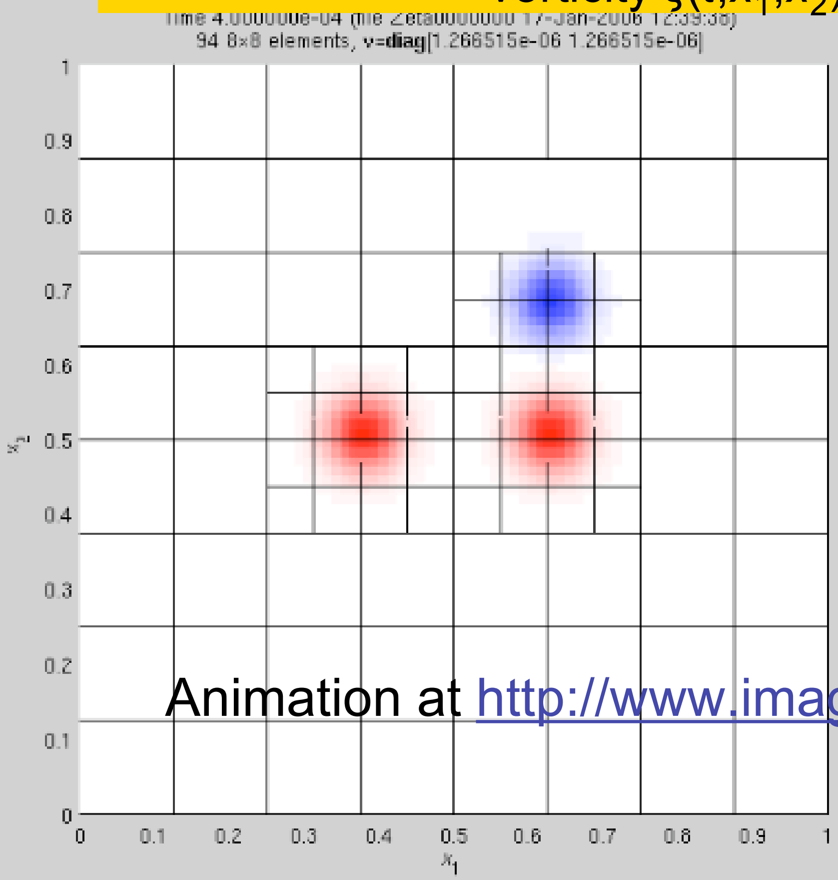
Reynolds nu. Re = $\Gamma / \nu = 2\times10^4$

Initial condition: Schneider, Kevlahan & Farge 1997.

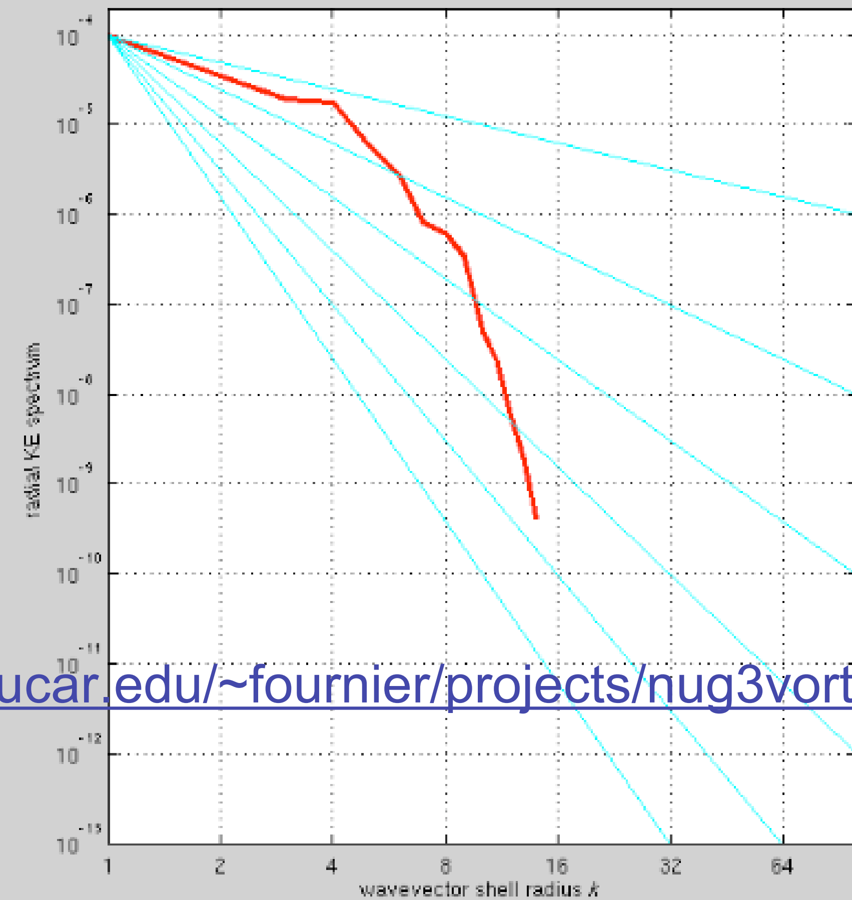
GASpAR simulation code: Rosenberg, Fournier, Fischer & Pouquet 2006; F, R & P *GAFD* **submitted** 2008.

Fourier analysis exact for SEM: Fournier 2006.

vorticity $\zeta(t,x_1,x_2)$



energy spectrum $E(t,|\boldsymbol{k}|)$



Animation at http://www.image.ucar.edu/~fournier/projects/nug3vort/

Note, $|((d/dt)_{O(\Delta t^2)}E)/2\nu Z+1| < 8\times10^{-3}\ \forall t$

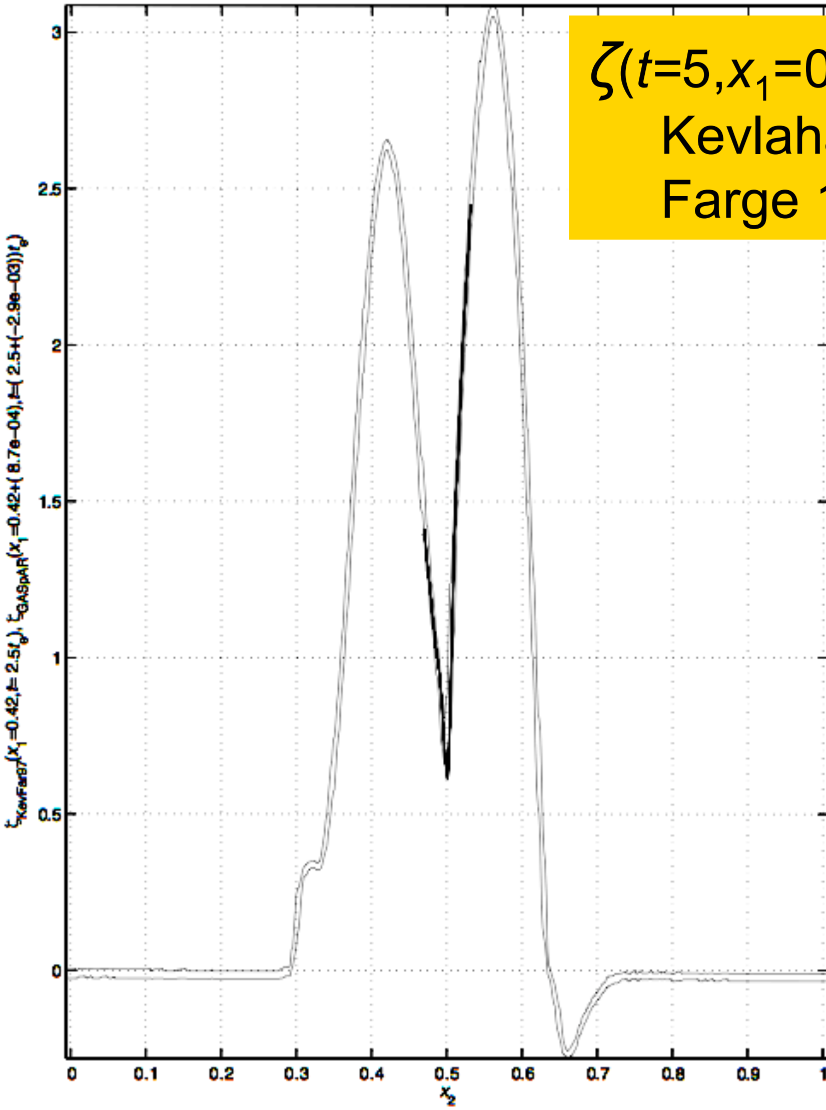
Decaying incompressible Navier-Stokes: 3-vortex slice comparison

Degree p = 7, element count K varies.

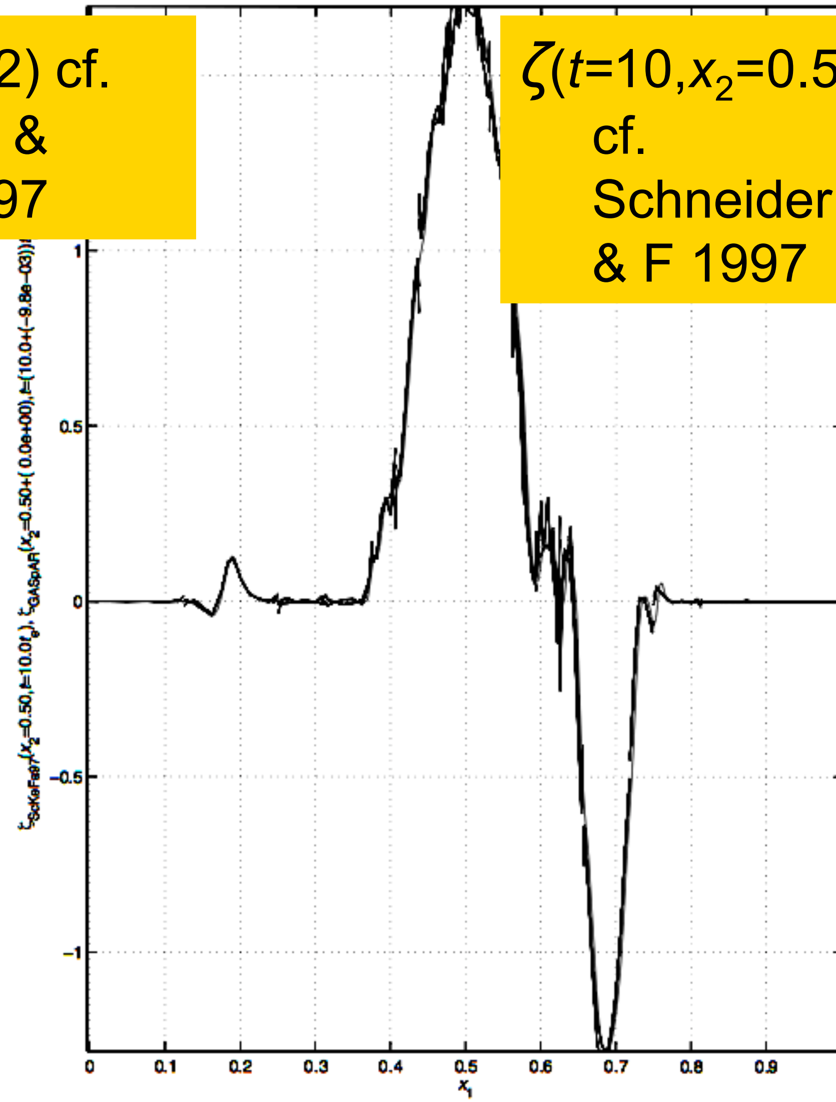
Reynolds nu. Re = $\Gamma / \nu = 2\times10^4$

Initial condition: Schneider, Kevlahan & Farge 1997.

GASpAR simulation code: Rosenberg, Fournier, Fischer & Pouquet 2006.



$\zeta(t=5, x_1=0.42)$ cf. Kevlahan & Farge 1997



$\zeta(t=10, x_2=0.5)$, cf. Schneider, K & F 1997

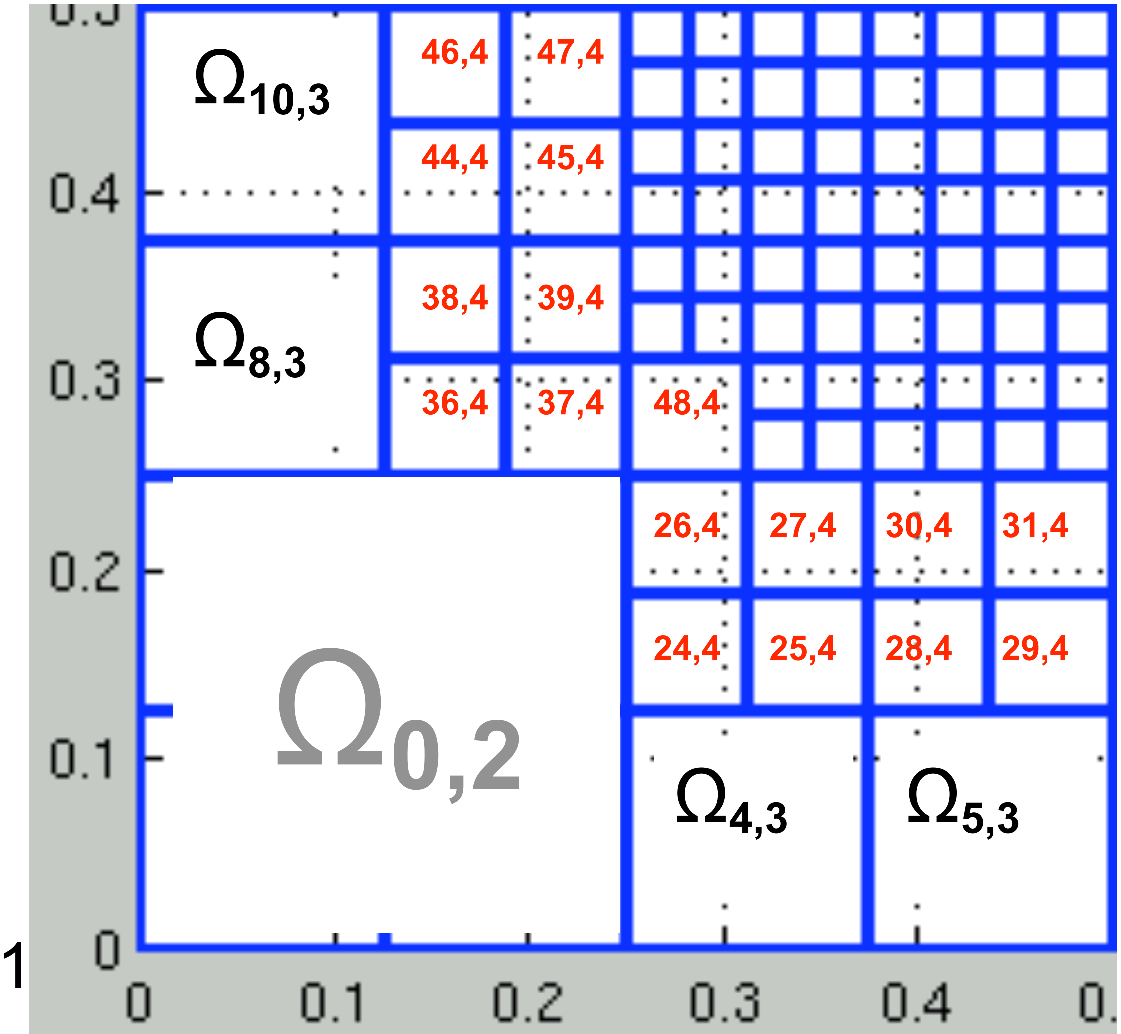
Multiresolution spectral elements

Fournier, Beylkin & Cheruvu 2005; Fournier 2008

New kind of MRA:

define $\tilde{\boldsymbol{u}}_{k,\ell}$ to contain the extra info. filtered out by merging element $\Omega_{k,\ell}$ with its 2^d—1 "sibling" elements.

$$\phi_{k,\ell} = \sum_i \mathbf{H}_i \phi_{2^d k+i,\ell+1}$$



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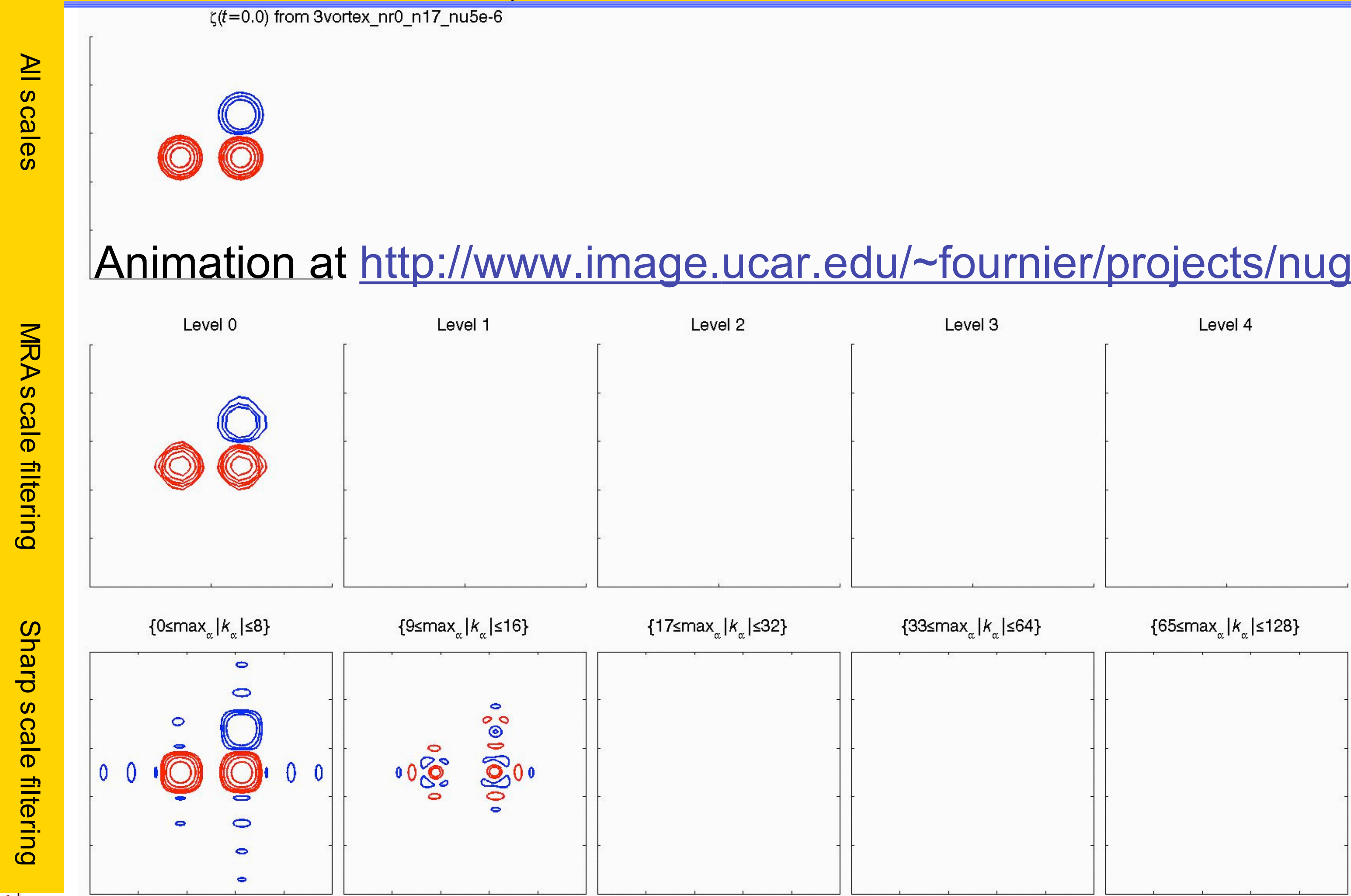
Decaying incompressible Navier-Stokes: 3 vortices (cont.)

$p = 16$, $K = 32^2$	Initial condition: Schneider, Kevlahan & Farge 1997.
$\mathrm{Re} = \Gamma / \nu = 5.07\times10^3$	GASpAR simulation code: Rosenberg, Fournier, Fischer & Pouquet 2006.
$\ell = 6$ multiresolution levels	Multiresolution analysis based on continuous SEM: Fournier 2006.



Animation at http://www.image.ucar.edu/~fournier/projects/nugmasse/

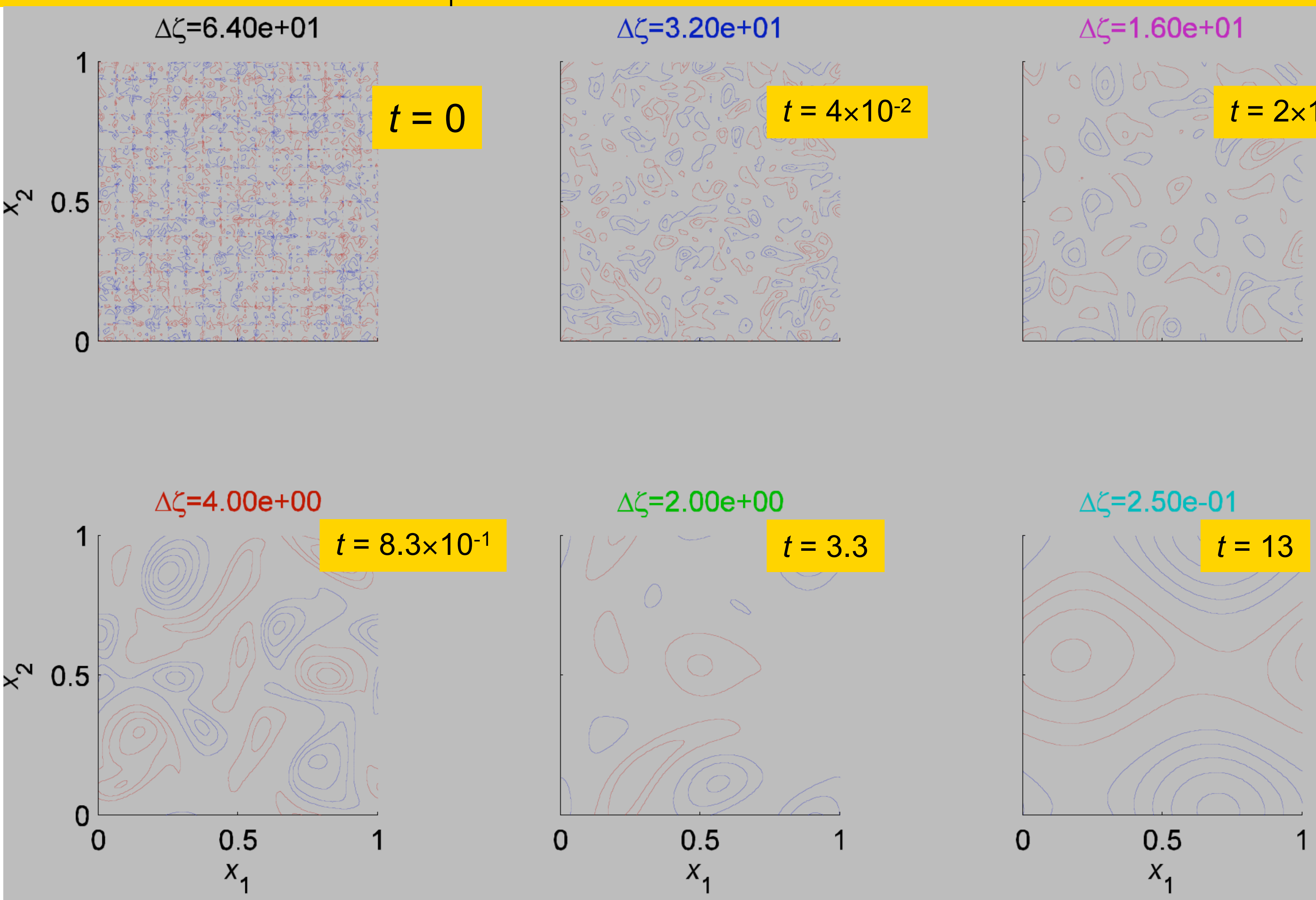
Decaying incompressible Navier-Stokes: scaling-*E*, random phase i.c.

$p = 7$, $K = 16^2$

$\mathrm{Re} = U_{rms}(0)L/\nu = 4000$

Initial condition: Matthaeus, Stribling, Martinez, Oughton & Montgomery 1991.

GASpAR simulation code: Rosenberg, Fournier, Fischer & Pouquet 2006.



Decaying incompressible Navier-Stokes: scaling-*E*, random phase i.c.

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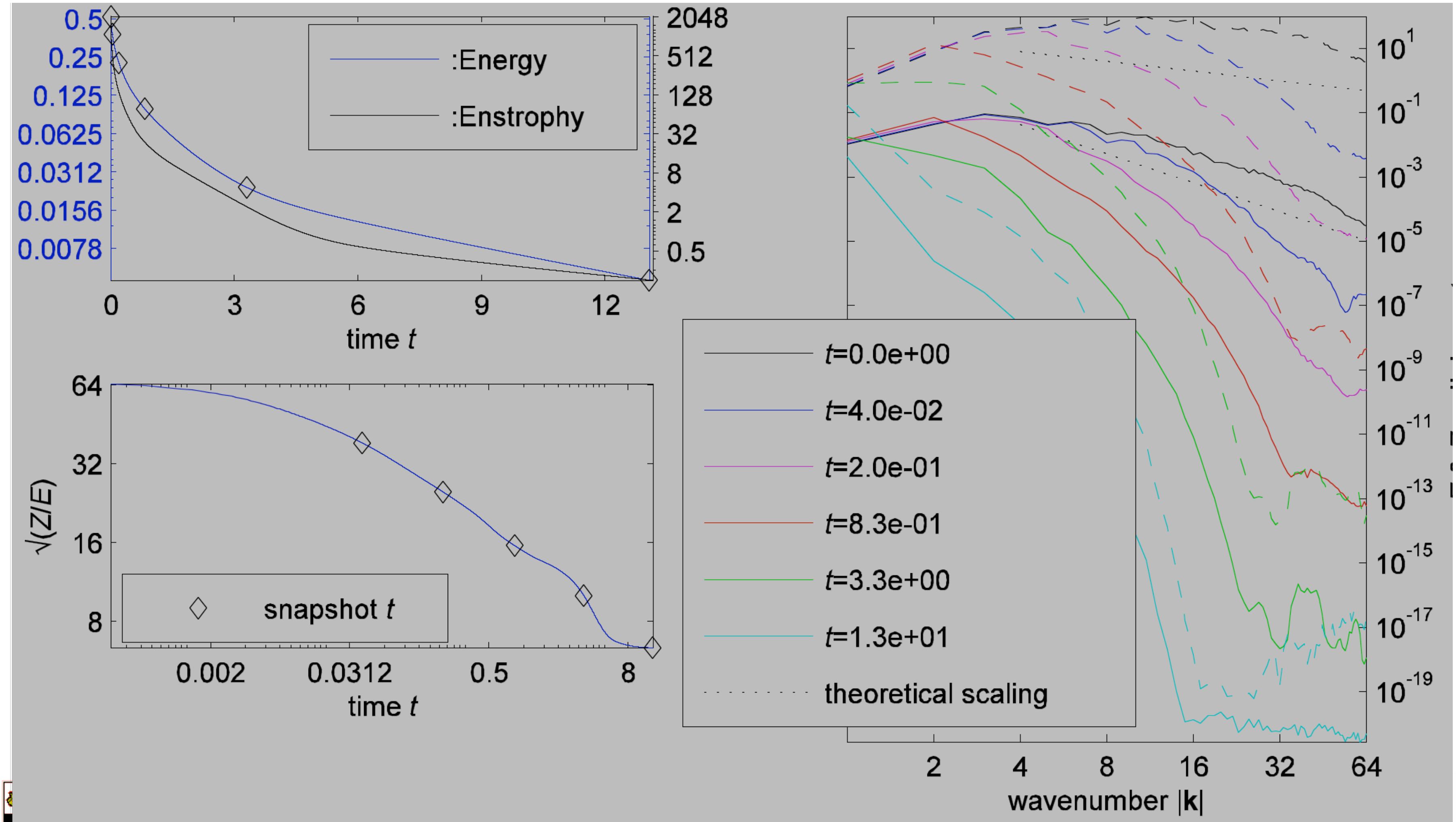

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Summary

1. Spectral-element method (SEM) reproduces some traditional pseudo-spectral-method (PSM) simulations, with same rate of accuracy increase with computational d.o.f. but adaptively, with greater geometric flexibility and better distributed c.p.u. efficiency.
2. Element-local polynomial spaces enable high-accuracy Fourier analysis.
3. Rigorous multiresolution analysis can be constructed w.r.t. element refinement (Fournier *in preparation* 2008).
4. (Not included) some local conservation laws can be enforced (Taylor & Fournier *in preparation* 2008).