Rapidly Rotating Rayleigh-Bénard Convection

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27 February 2008

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Outline

- Rotationally constrained flow
  - Rayleigh-Bénard convection (Bénard 1900; Rayleigh 1916)
    - Challenges for experiments and direct numerical simulation (DNS) of full Navier-Stokes equations in rapidly rotating limit
  - Geophysical fluid dynamics

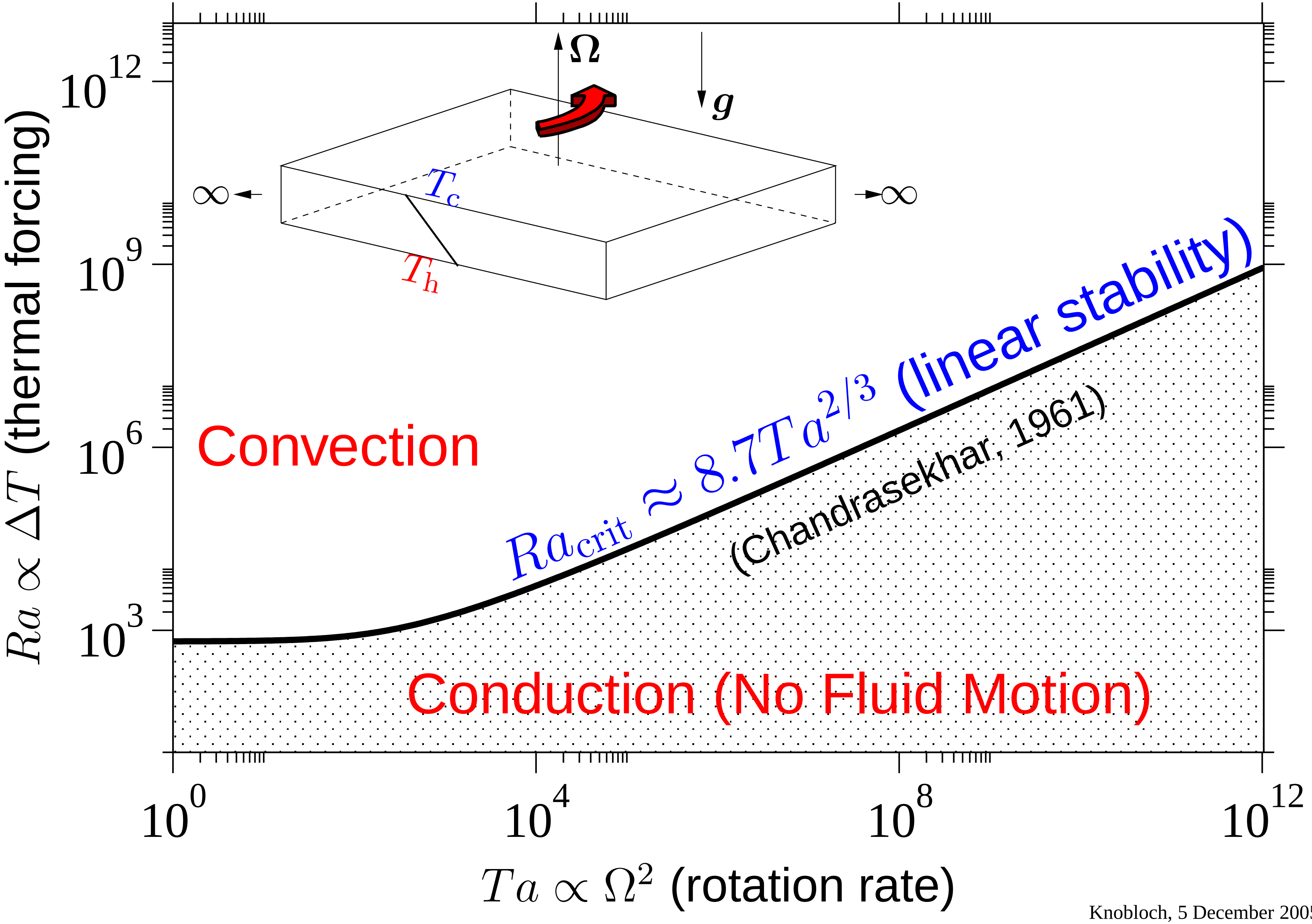
- Derivation of a reduced model for convection in rapidly rotating limit

- DNS of reduced system: method & results
  Julien *et al*. JFM vol 555 (2006); Sprague *et al*. JFM vol 551 (2006)

- Future work

Rotationally Constrained Convection

Ra–Ta Parameter Space



Rotationally Constrained Convection

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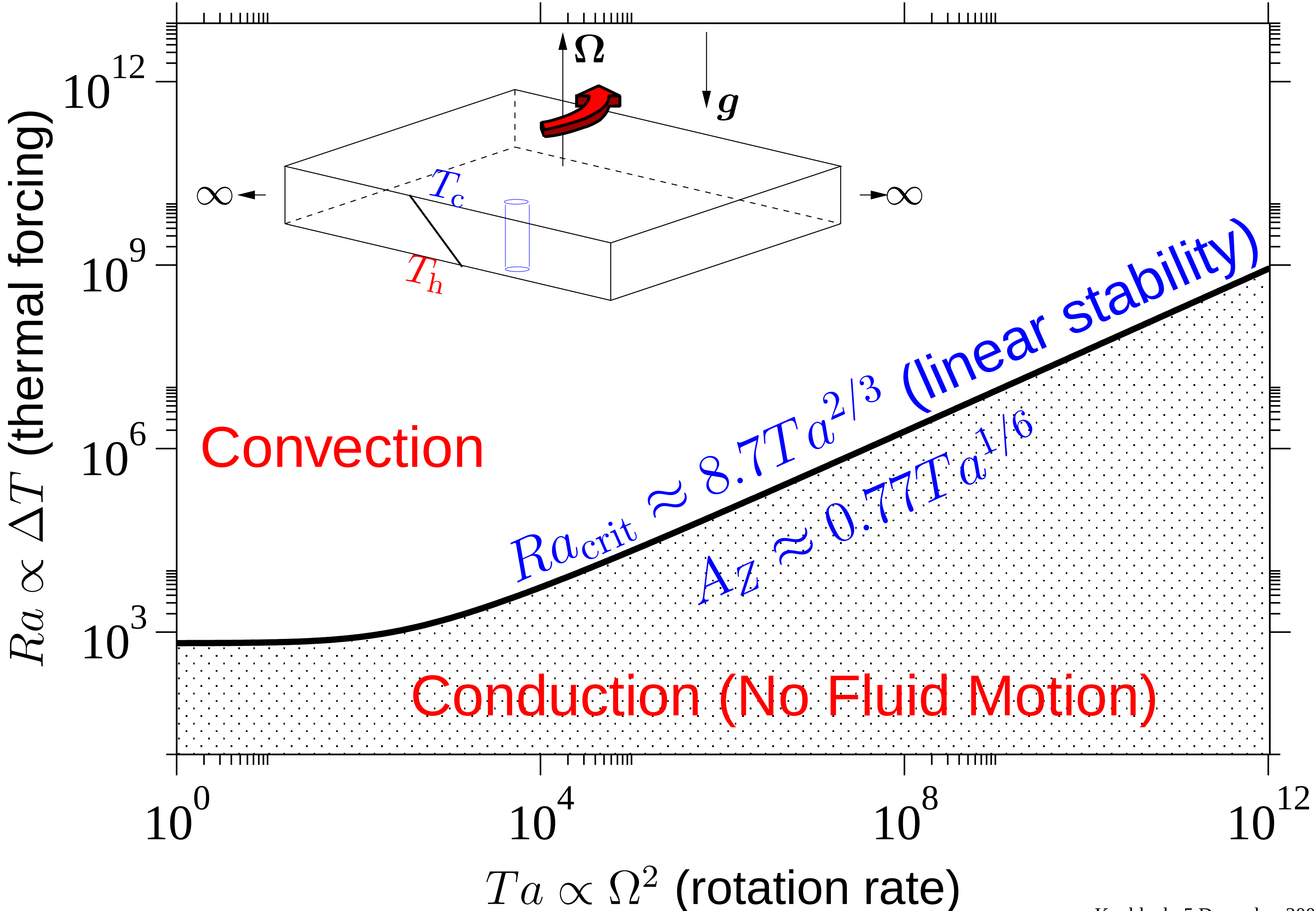

Rotationally Constrained Convection

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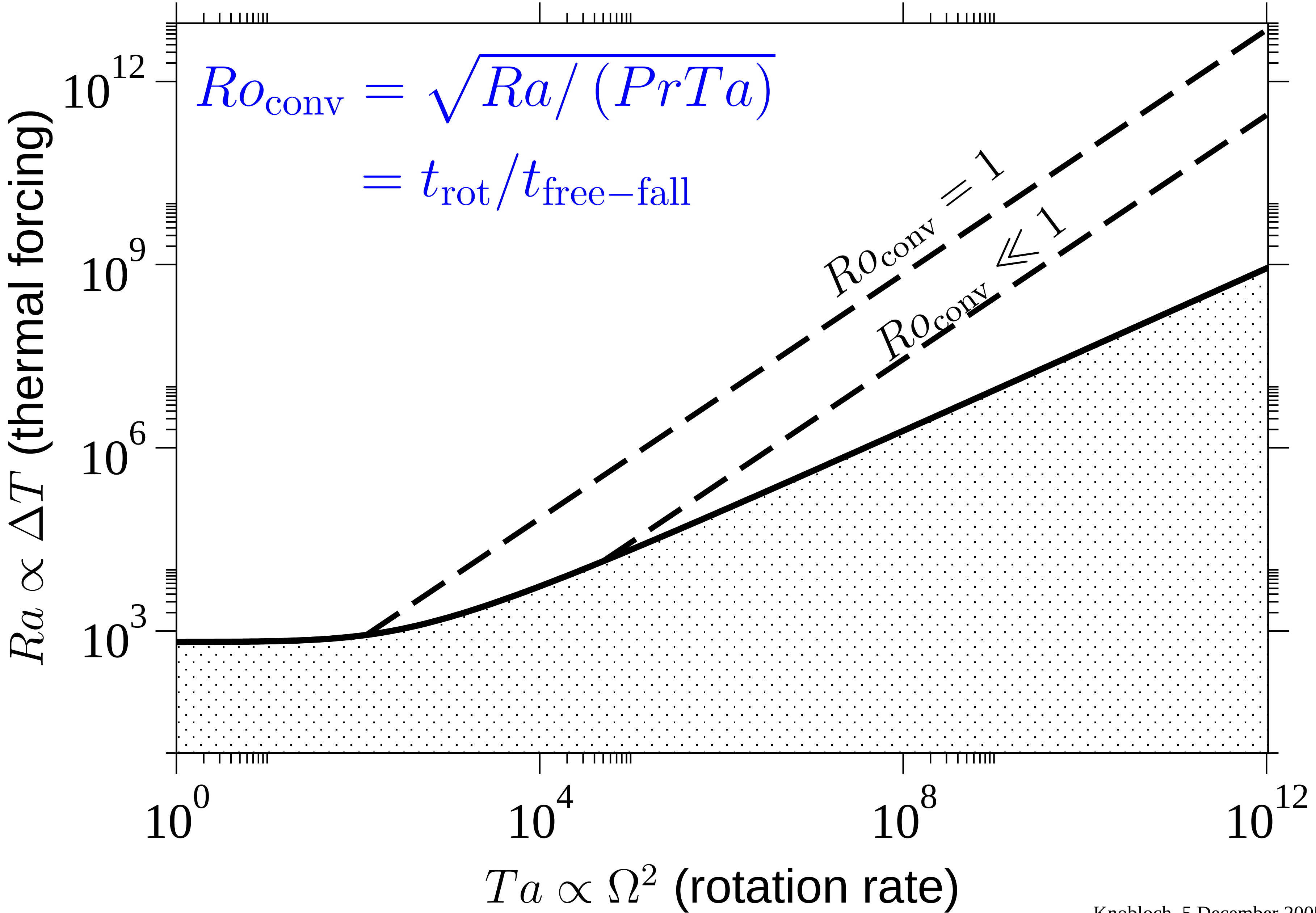
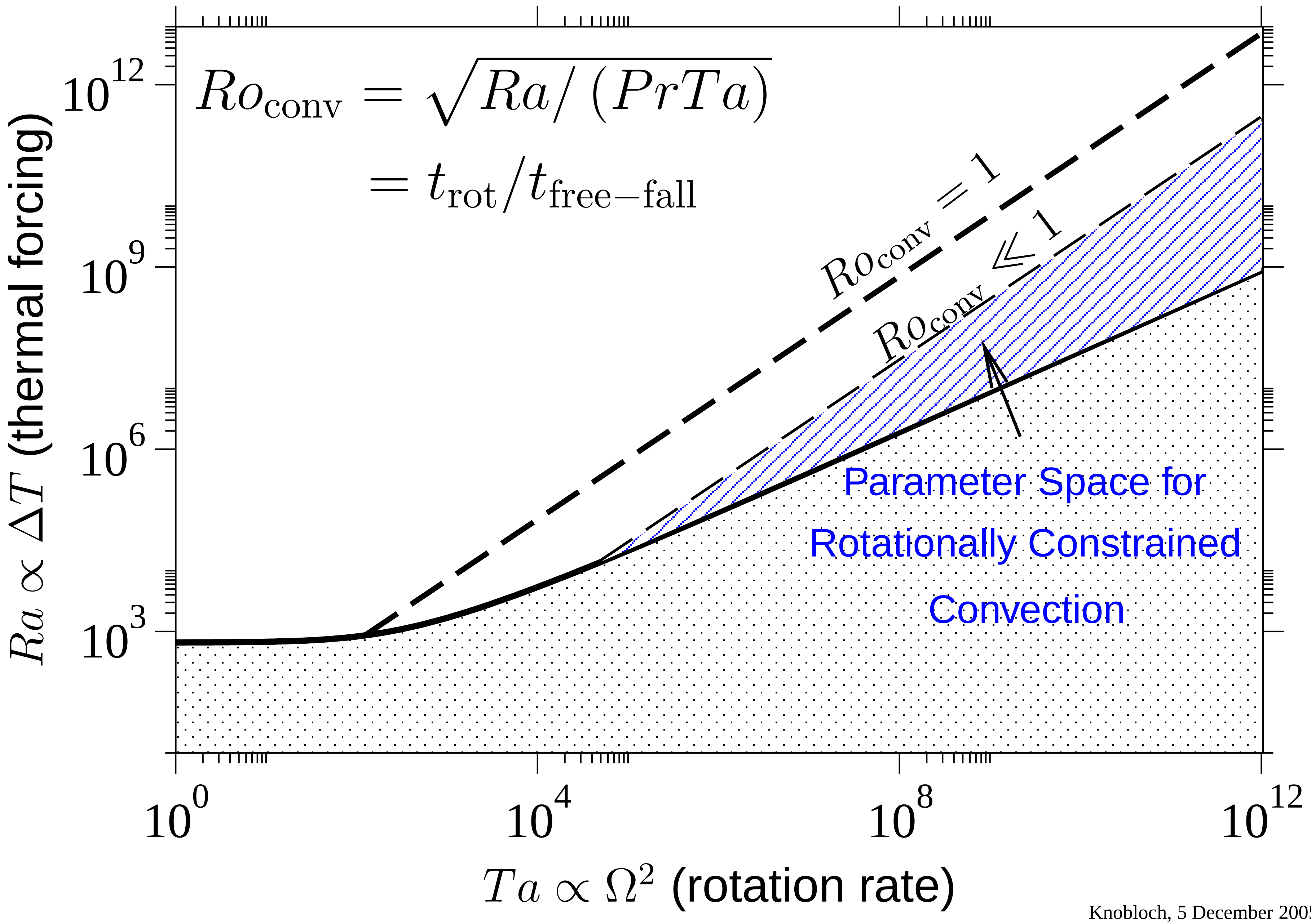


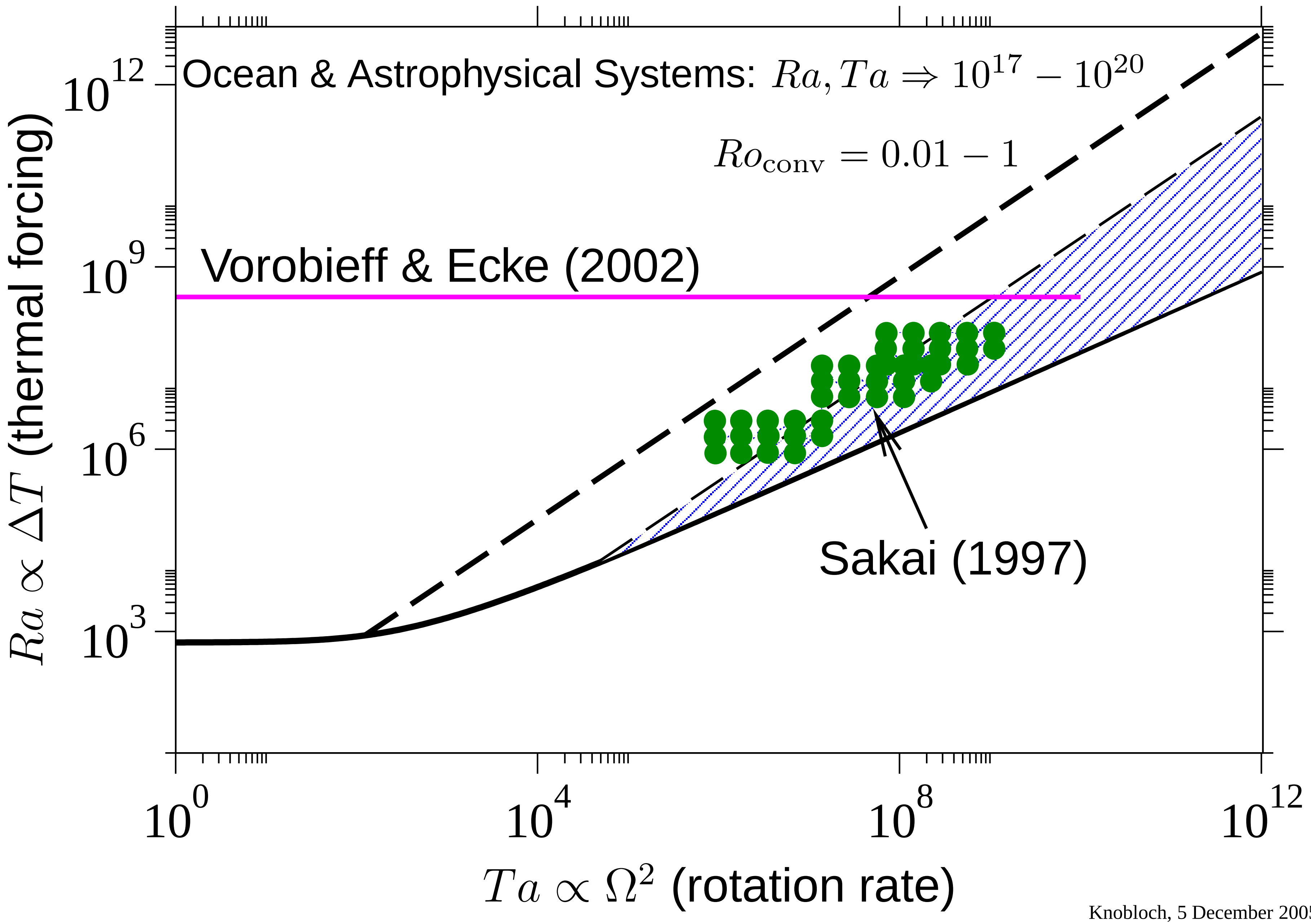
Rotationally Constrained Convection

Ra–Ta Parameter Space



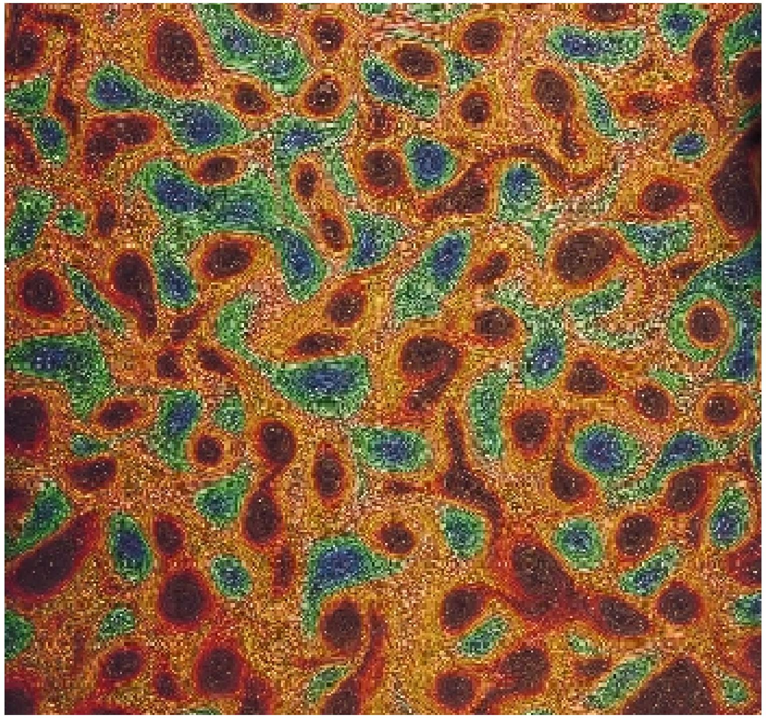
Rotationally Constrained Convection

Ra–Ta Parameter Space: Experiments

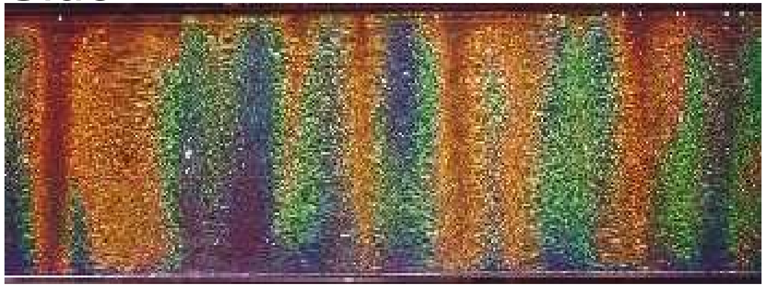


Rotationally Constrained Convection ($Ro_{\mathrm{conv}} \ll 1$)

Top



Side

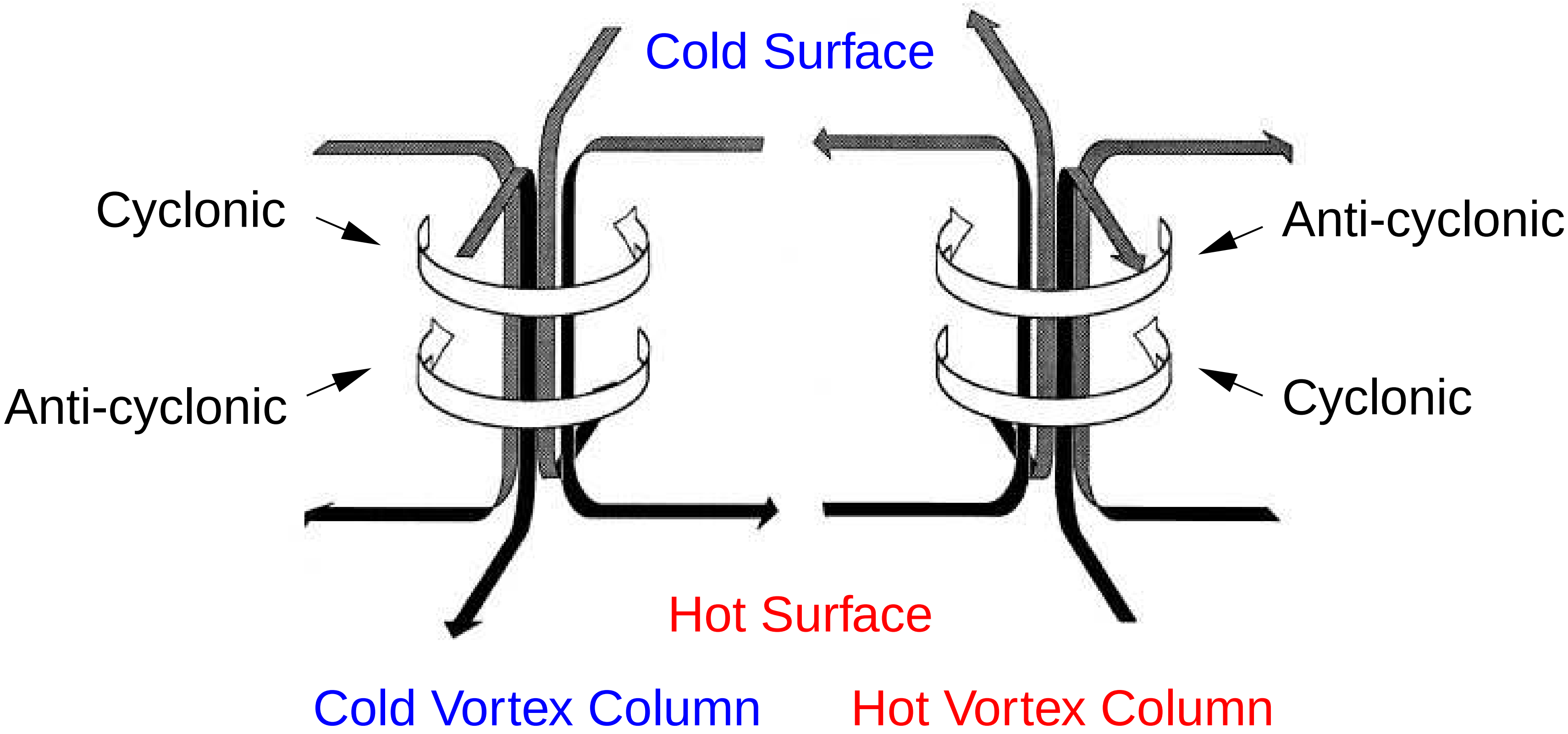


Temperature (Sakai, 1997)

($Ra \approx 10^7$, $Ro_{\mathrm{conv}} \approx 0.1$, $Pr \approx 7$)

- Experiments by Sakai (1997); Vorobieff & Ecke (2002) show features of rotationally constrained convection:
  - intense vortical structures spanning layer of fluid
  - cyclonic and anticyclonic vortical structures
  - vortex-vortex interaction
- Experimental Challenge: Visualization/measurement of 3-D data

Rapidly Rotating Convection: Vortex Structure



(Sakai, 1997)

- Hot vortices have cyclonic vorticity below mid-plane; anti-cyclonic above mid-plane (opposite for cold vortices)
- Sakai alludes to geostrophic balance in interior: pressure forces balance Coriolis forces (our results support this!)

Rapidly Rotating Convection: Challenges for Direct Numerical Simulation (DNS)

- Ekman boundary layers become increasingly thin as the rotation rate is increased ($\delta_E \sim E^{1/2} \ll 1$): must resolve in DNS
- Fast inertial waves exist ($\omega \sim E^{-1}$), which hinder explicit time integration

Rapidly Rotating Convection: Challenges for Direct Numerical Simulation (DNS)

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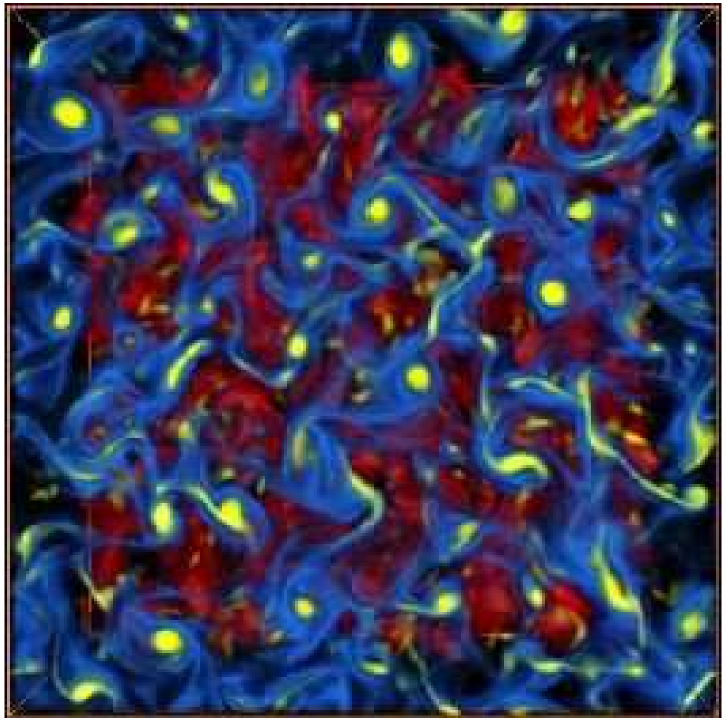

DNS Simulation (Julien *et al.* 1996)

- $Ro_{\rm conv} = 0.75$, $Ra \approx 10^7$, $Pr \approx 1$
- Temperature (red/blue) and vertical cyclonic vorticity (yellow)
- In physical experiments, anti-cyclonic vortices emerge as $Ro_{\rm conv} \lesssim 0.2$ (Vorobieff & Ecke 2002; Sakai 1997)

So, how do we numerically investigate convection in the regime $Ro_{\rm conv} \ll 1$?

Governing Equations

- Scales used for nondimensionalization: L, U, $\widetilde{T}$, P
- Boussinesq approximation in a rotating coordinate frame $\widehat{\mathbf{z}}$:

$$D_t\mathbf{u} + Ro^{-1}\widehat{\mathbf{z}} \times \mathbf{u} = -\overline{P}\nabla p + \Gamma\theta\widehat{\mathbf{z}} + Re^{-1}\nabla^2\mathbf{u}$$

$$D_t\left(\theta - \frac{1}{\Gamma Fr^2}\overline{\rho}(z)\right) = Pe^{-1}\nabla^2\theta$$

$$\nabla\cdot\mathbf{u} = 0$$

where $\mathbf{u} = (u, v, w)$ is the velocity, $D_t = \partial_t + \mathbf{u}\cdot\nabla$, p is pressure, and θ is the buoyancy anomaly (temperature)

- Important Nondimensional Parameters:

$Ro = U/2\Omega L$	Rossby Number	$Fr = \frac{U}{N_0 L}$	Froude Number
$Re = \frac{UL}{\nu}$	Reynolds Number	$Pe = \frac{UL}{\kappa}$	Péclet Number
$\Gamma = \frac{BL}{U^2}$	Buoyancy Number	$\overline{P} = \frac{P}{\rho_0 U^2}$	Euler Number

Asymptotic Theory: NH-QGE

- Multiple scales expansion in the vertical direction and in time:

$$\partial_z \to \frac{1}{A_Z}\partial_Z, \quad \partial_t \to \partial_t + \frac{1}{A_\tau}\partial_\tau$$

Large Scale: $Z = A_Z^{-1} z$

Slow Time: $\tau = A_\tau^{-1} t$

- Field variables are separated into average (over fast/short scales) and fluctuating components:

$$\mathbf{v}(\mathbf{x}, Z, t, \tau) = (\mathbf{u}, p, \theta)^T = \overline{\mathbf{v}}(Z, \tau) + \mathbf{v}'(\mathbf{x}, Z, t, \tau),$$

where

$$\overline{\mathbf{v}} := \lim_{\tilde{t}, V \to \infty} \frac{1}{\tau V} \int_{\tilde{t}, V} \mathbf{v} \mathrm{d}\mathbf{x} \mathrm{d}t, \qquad \overline{\mathbf{v}'} = 0.$$

Asymptotic Theory: NH-QGE

- Relate aspect ratio to $Ro \equiv \epsilon$: $A_Z = \epsilon^{-1}$
- Find:

$$A_\tau = \epsilon^{-2}, \qquad \overline{P} = O(\epsilon^{-2}), \qquad \Gamma = O(\epsilon^{-1})$$

- Scaling chosen
  - For isotropic velocity field: $u_0 \sim v_0 \sim w_0$
  - For fluid motions to feed back and adjust mean stratification: $Fr = \epsilon^{\frac{1}{2}}$
- Remark I: If $A_Z < \mathcal{O}(\epsilon^{-1})$ vertical motions are weak. Hydrostatic-QGE recovered for columnar regime.
- Remark II: If $\overline{P} \sim \epsilon^{-1}, \Gamma = 1$ no feedback occurs. Dynamics consists of nonlinear propagating inertial-gravity waves (Smith & Waleffe JFM 2002).
- Expand all fields in powers of ϵ:

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + \ldots$$

Asymptotic Theory: NH-QGE

Leading-Order Results:

- Hydrostatic balance: $\partial_Z \overline{p}_0 = \widetilde{\Gamma}\overline{\theta}_0, \qquad \overline{\mathbf{u}}_0 = 0$
- Temp. & press. fluctuations occur at first order ($\theta_0' = 0, p_0' = 0$)
- Momentum (geostrophic balance)

$$\widehat{\mathbf{z}} \times \mathbf{u}_0' = -\nabla p_1' \quad \Rightarrow \quad \begin{cases} (\widehat{\mathbf{z}} \cdot \nabla)\, p_1' = 0 \\ (\widehat{\mathbf{z}} \cdot \nabla)\, \mathbf{u}_0' = 0 \\ \nabla_\perp \cdot \mathbf{u}_{0\perp}' = 0 \end{cases}$$

All dependent variables are governed by Taylor-Proudman constraint on small scales (invariance along axis of rotation):

$$\text{Solution}: \quad \mathbf{u}_0' = \widehat{\mathbf{z}} \times \nabla\psi(x,y,Z,t) + W(x,y,Z,t)\widehat{\mathbf{z}}, \quad p_1' = \psi(x,y,Z,t)$$

Asymptotic Theory: NH-QGE

- Geostrophy:

$$\widehat{\mathbf{z}} \times \mathbf{u}_0' + \nabla p_1' = \mathbf{0},$$

- Nonhydrostatic Quasigeostrophic equations obtained from solvability conditions applied to:

$$\widehat{\mathbf{z}} \times \mathbf{u}_1' + \nabla p_2' = \mathbf{F}(\mathbf{u}_0', \theta_1'), \qquad \nabla \cdot \mathbf{u}_1' + \partial_Z w_0' = 0.$$

- Vertical Velocity (W) & Vertical Vorticity ($\omega = \nabla_\perp^2 \psi$):

$$\begin{aligned} \partial_t W + J(\psi, W) + \partial_Z \psi &= \widetilde{\Gamma}\theta_1' + Re^{-1}\nabla_\perp^2 W \\ \partial_t \omega + J(\psi, \omega) - \partial_Z W &= Re^{-1}\nabla_\perp^2 \omega \end{aligned}$$

Stream-Function Formulation: Closed System

- Vertical Velocity (W) & Vertical Vorticity ($\omega = \nabla_\perp^2 \psi$):

$$\begin{aligned} \partial_t W + J(\psi, W) + \partial_Z \psi &= \widetilde{\Gamma}\theta_1' + Re^{-1}\nabla_\perp^2 W \\ \partial_t \omega + J(\psi, \omega) - \partial_Z W &= Re^{-1}\nabla_\perp^2 \omega \end{aligned}$$

- Fluctuating and mean temperature equations:

$$\begin{aligned} \partial_t \theta_1' + J(\psi, \theta_1') + W\partial_Z\left(\overline{\theta}_0 - \widetilde{\Gamma}^{-1}\overline{\rho}\right) &= Pe^{-1}\nabla_\perp^2 \theta_1' \\ \partial_\tau \overline{\theta}_0 + \partial_Z\left(\overline{\theta_1' W}\right) &= Pe^{-1}\partial_{ZZ}\overline{\theta}_0 \end{aligned}$$

where $J(\psi, f) \equiv \partial_x \psi \partial_y f - \partial_x f \partial_y \psi = \mathbf{u}_{0\perp} \cdot \nabla_\perp f$

Stream-Function Formulation: Closed System

- Vertical Velocity ($W = \nabla_\perp^2 \phi$) & Vertical Vorticity ($\omega = \nabla_\perp^2 \psi$):

$$\begin{aligned} \partial_t W + J(\psi, W) + \partial_Z \psi &= \widetilde{\Gamma}\theta_1' + Re^{-1}\nabla_\perp^2 W \\ \partial_t \omega + J(\psi, \omega) - \partial_Z W &= Re^{-1}\nabla_\perp^2 \omega \end{aligned}$$

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- Conserves energy: $E = \dfrac{1}{2}\displaystyle\int_D |\nabla_\perp \psi|^2 + \widetilde{\Gamma}\frac{\theta_1'^2}{\partial_Z\left(\overline{\theta}_0 - \widetilde{\Gamma}^{-1}\overline{\rho}\right)}\,dx\,dy\,dZ$

- Conserves PV:

$$\Pi \equiv \nabla_\perp^2 \psi + J\left(\phi, \frac{\theta_1'}{\partial_Z\left(\overline{\theta}_0 - \widetilde{\Gamma}^{-1}\overline{\rho}\right)}\right) + \partial_Z\left(\frac{\theta_1'}{\partial_Z\left(\overline{\theta}_0 - \widetilde{\Gamma}^{-1}\overline{\rho}\right)}\right)$$

Application to Rotating RBC

- Vertical Velocity (W) & Vertical Vorticity ($\omega = \nabla_\perp^2 \psi$):

$$\begin{aligned}\partial_t W + J(\psi, W) + \partial_Z \psi &= \frac{\widetilde{Ra}}{Pr}\theta_1' + \nabla_\perp^2 W \\ \partial_t \omega + J(\psi, \omega) - \partial_Z W &= \nabla_\perp^2 \omega\end{aligned}$$

- Fluctuating- and mean-Temperature equations:

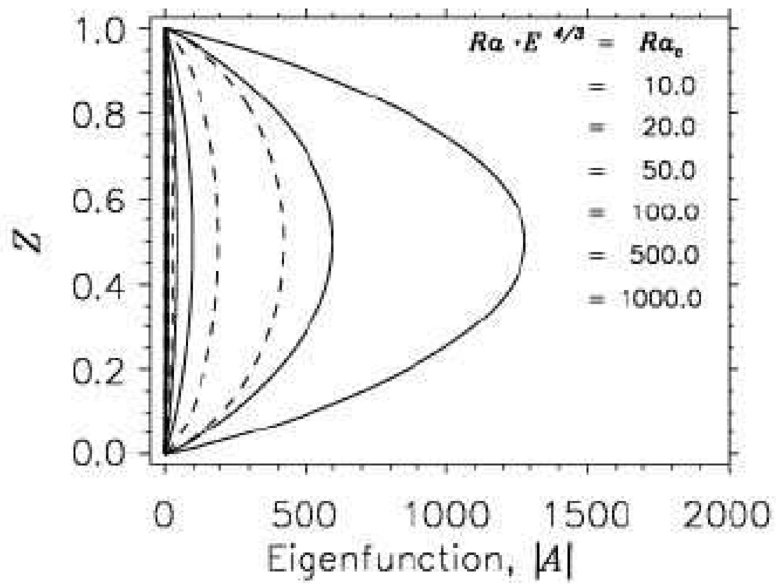
$$\begin{aligned}\partial_t \theta_1' + J(\psi, \theta_1') + W\partial_Z \overline{\theta}_0 &= Pr^{-1}\nabla_\perp^2 \theta_1' \\ \partial_\tau \overline{\theta}_0 + \partial_Z \left(\overline{\theta_1' W}\right) &= Pr^{-1}\partial_{ZZ}\overline{\theta}_0\end{aligned}$$

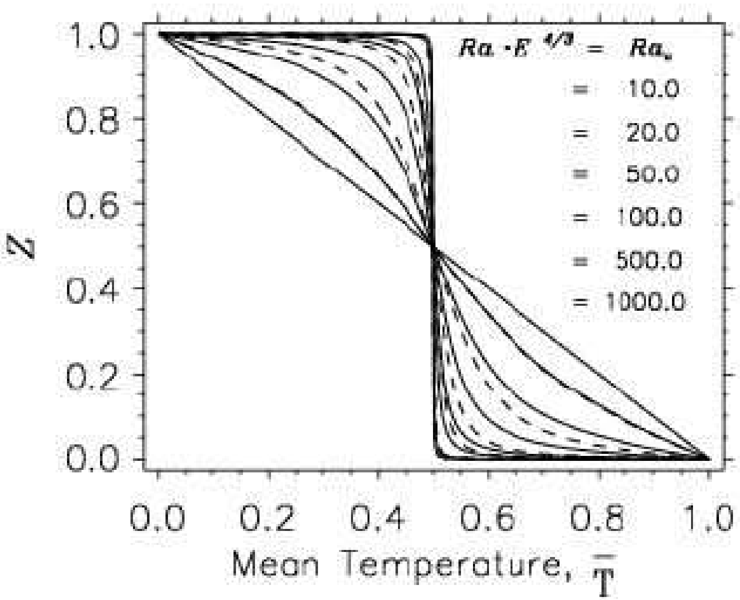
- RBC nondimensionalization
  - $Ta = 4\Omega^2 H^4/\nu^2, \qquad Ra = g\alpha\Delta T H^3/\nu\kappa, \qquad Pr = \nu/\kappa$
  - $Ro = Ta^{-1/6}, \quad Re = 1, \quad Pe = Pr, \quad \widetilde{\Gamma} = \epsilon^4 Ra/Pr \equiv \widetilde{Ra}/Pr$

Exact Single-Mode Solutions

- Bassom & Zhang GAFD '94; Julien & Knobloch PoF '96, '99; JFM '98
- Separable Solutions: $W = A(Z)h(x,y)$, with $\nabla_\perp^2 h + k_\perp^2 h = 0$ satisfy

$$\partial_{ZZ} A + \left(\frac{k_\perp^2 \widetilde{Ra} Nu}{1 + \frac{Pr^2}{k_\perp^2} A^2} - k_\perp^6 \right) A = 0, \qquad A(0) = A(1) = 0$$



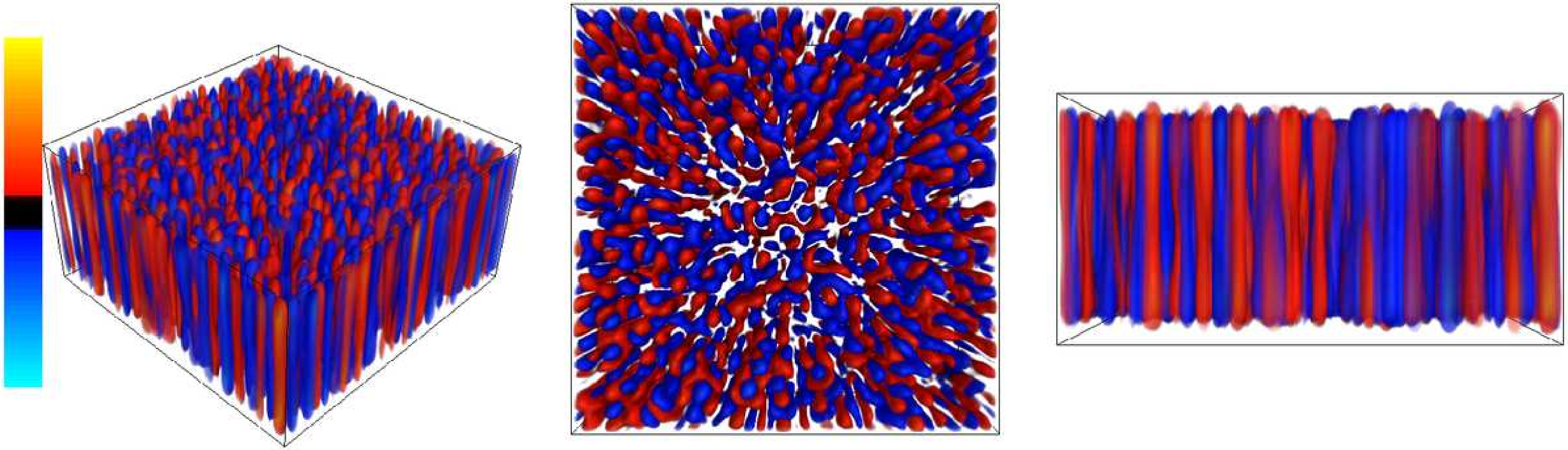


Numerical Method for DNS of Reduced Model

- Spectral spatial discretization: periodic Fourier modes in the horizontal; Chebyshev-Tau in the vertical
  - Nonuniform grid-point distribution in vertical is well suited to resolving the thin thermal boundary layers
- Impenetrable, stress-free boundary conditions
- Mixed implicit/explicit third-order Runge-Kutta time integration (Spalart *et al.*, JCP, 1991)
- Employs CRAY SHMEM libraries for parallelization; solved on CRAY and/or SGI supercomputers
- Typical models have 64^3 to $512^2 \times 256$ grid points
- Solutions evolve on fast (t) and slow (τ) time scales; we neglect variation on slow time scale $\Rightarrow$ numerical solutions only valid in statistically steady-state regime

Results: Topological Change of Flow: Columnar Regime

$\widetilde{Ra} = Ta^{-2/3} Ra = 20$, $Pr = 7$ (water), $\widetilde{Ra}_{\text{crit}} \approx 8.7$

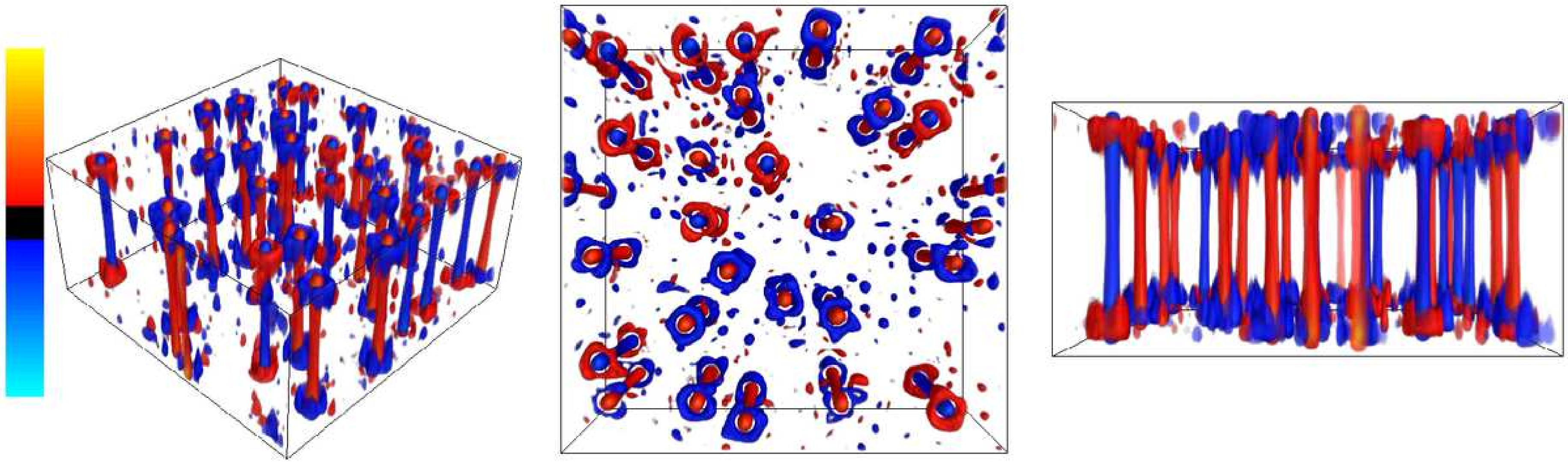


Iso. Top Side

- Above shows Temperature Anomaly θ_1'
- Columnar structure is clear
- Hot vortices have cyclonic vorticity below mid-plane; anti-cyclonic above mid-plane (opposite for cold vortices)
- Cyclonic and and anti-cyclonic vorticity balanced due to symmetry in governing equations; not present in full Boussinesq equations

Results: Topological Change of Flow: Shielded-Vortex Regime

$\widetilde{Ra} = Ta^{-2/3}Ra = 40$, $Pr = 7$ (Sakai '97, $\widetilde{Ra} \approx 35$)

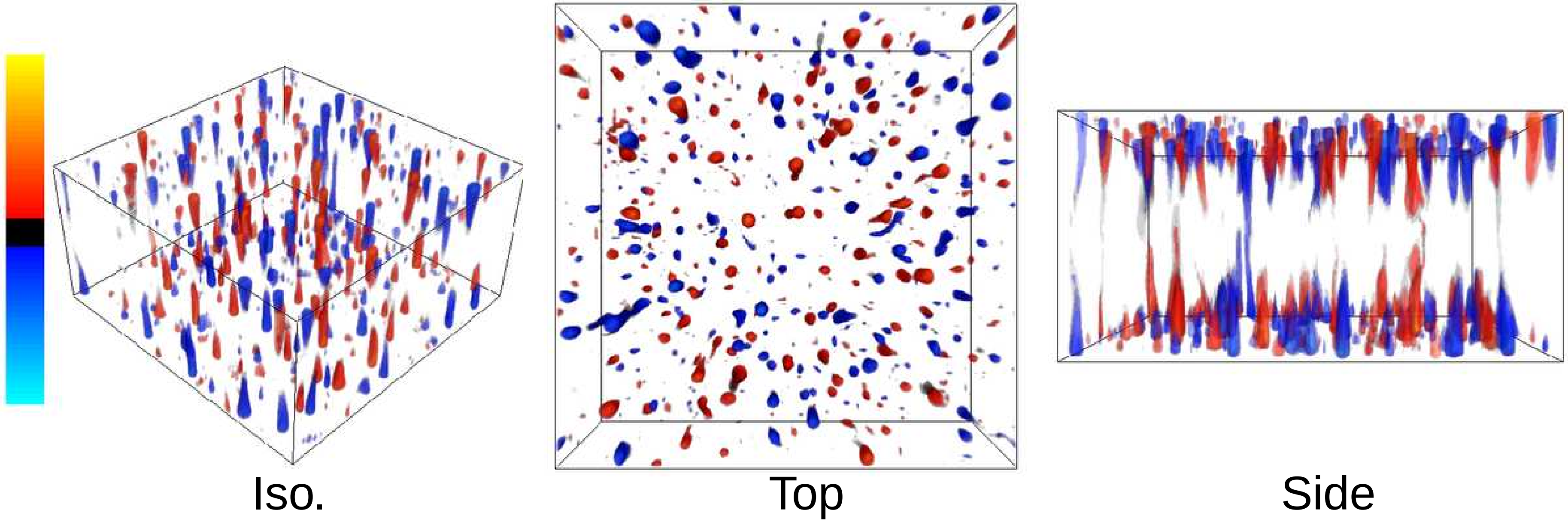


Iso. Top Side

- Columnar structure is clear; vortices are shielded by opposite-signed ‘sleeves’ extending across layer
- Vortices are in constant, but slow horizontal motion
- Columns highly efficient at heat transport; columns responsible for transporting 60% of heat flux

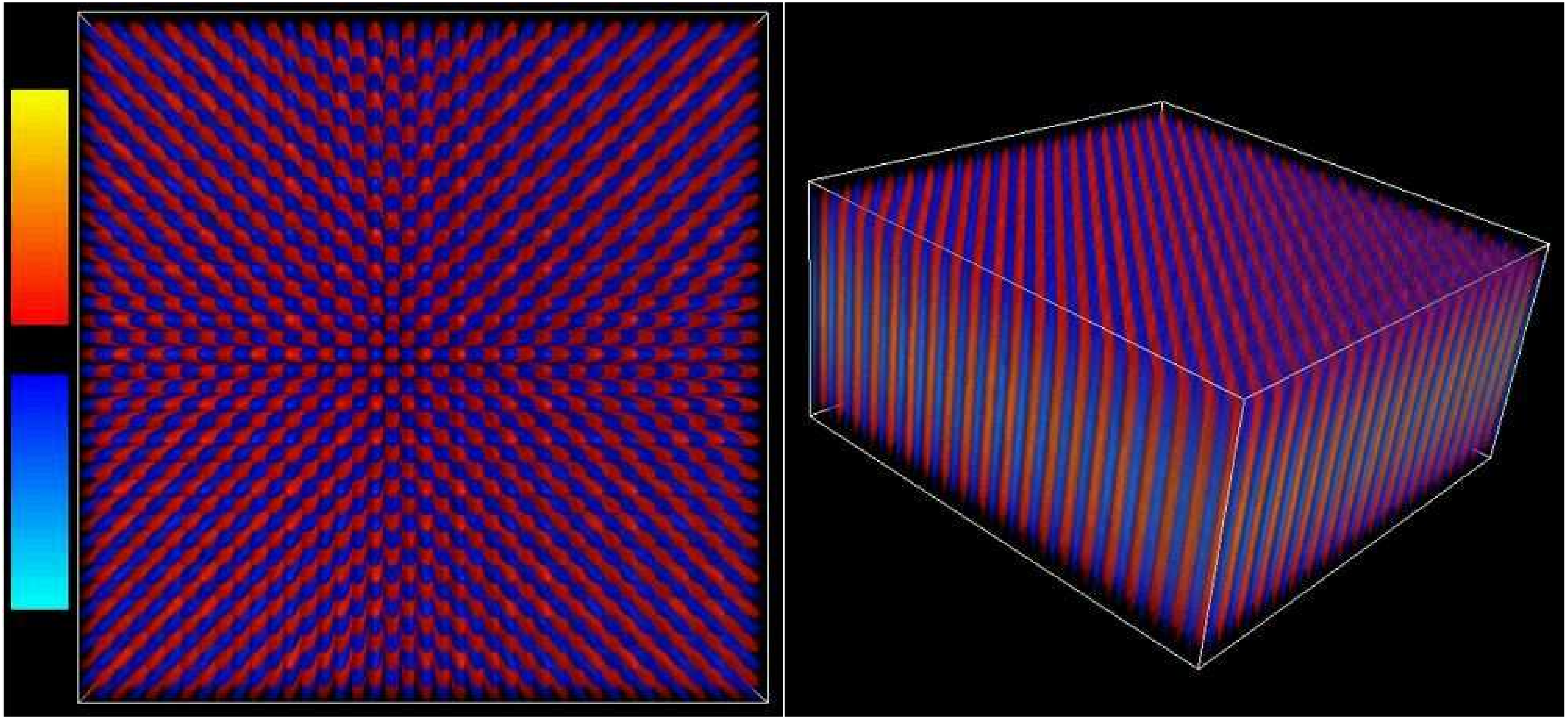
Results: Topological Change of Flow: Geostrophic Turbulence Regime

$\widetilde{Ra} = Ta^{-2/3}Ra = 80$, $Pr = 7$



- As thermal forcing is increased, lateral mixing plays significant role
- Columnar structure and shielding destroyed
- Geostrophic turbulence regime characterized by hot (cold) plumes emanating from the lower (upper) thermal boundary layers

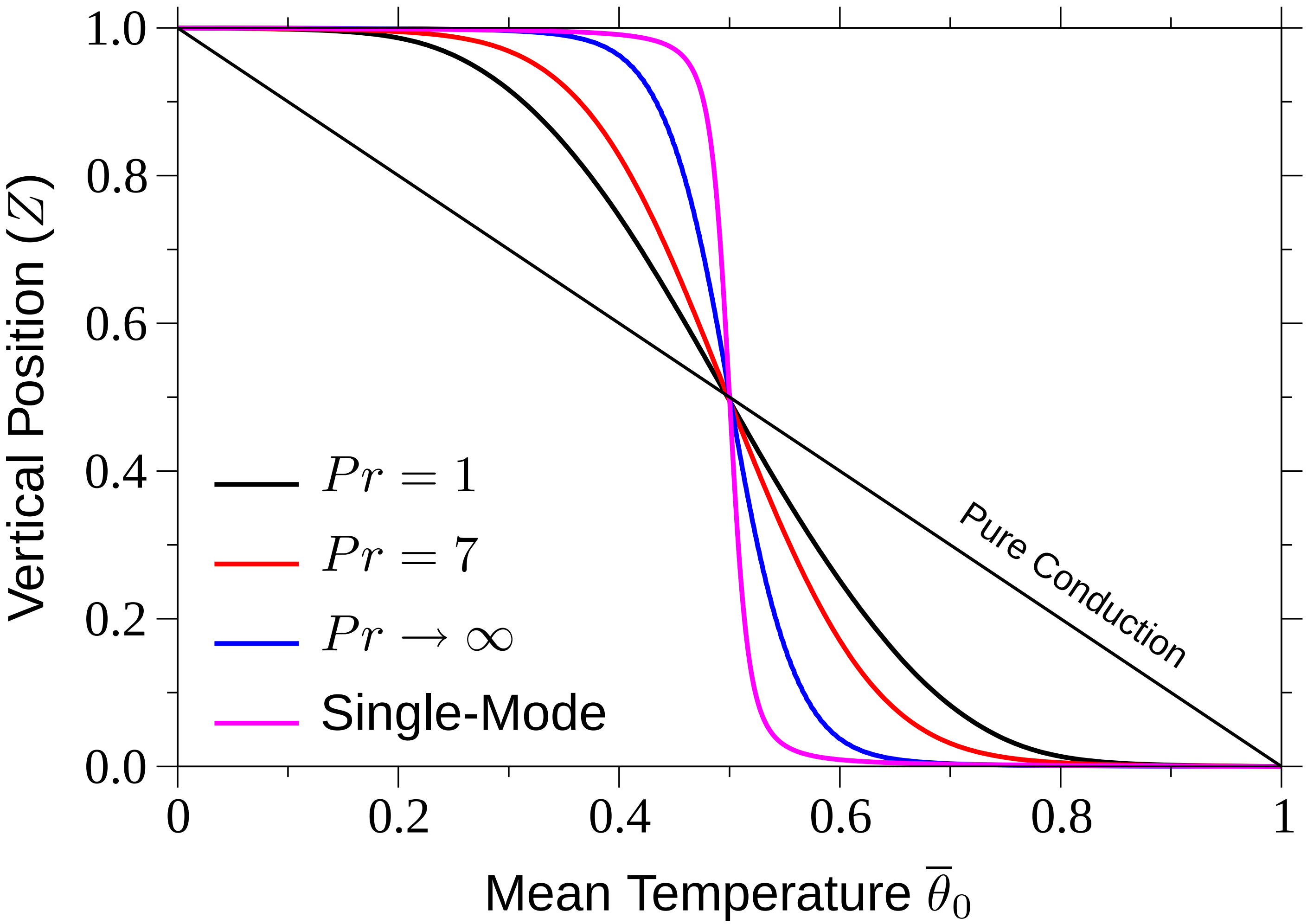
Results: $\widetilde{Ra} = Ro^4 Ra = 40$, $Pr = 7$ (water)



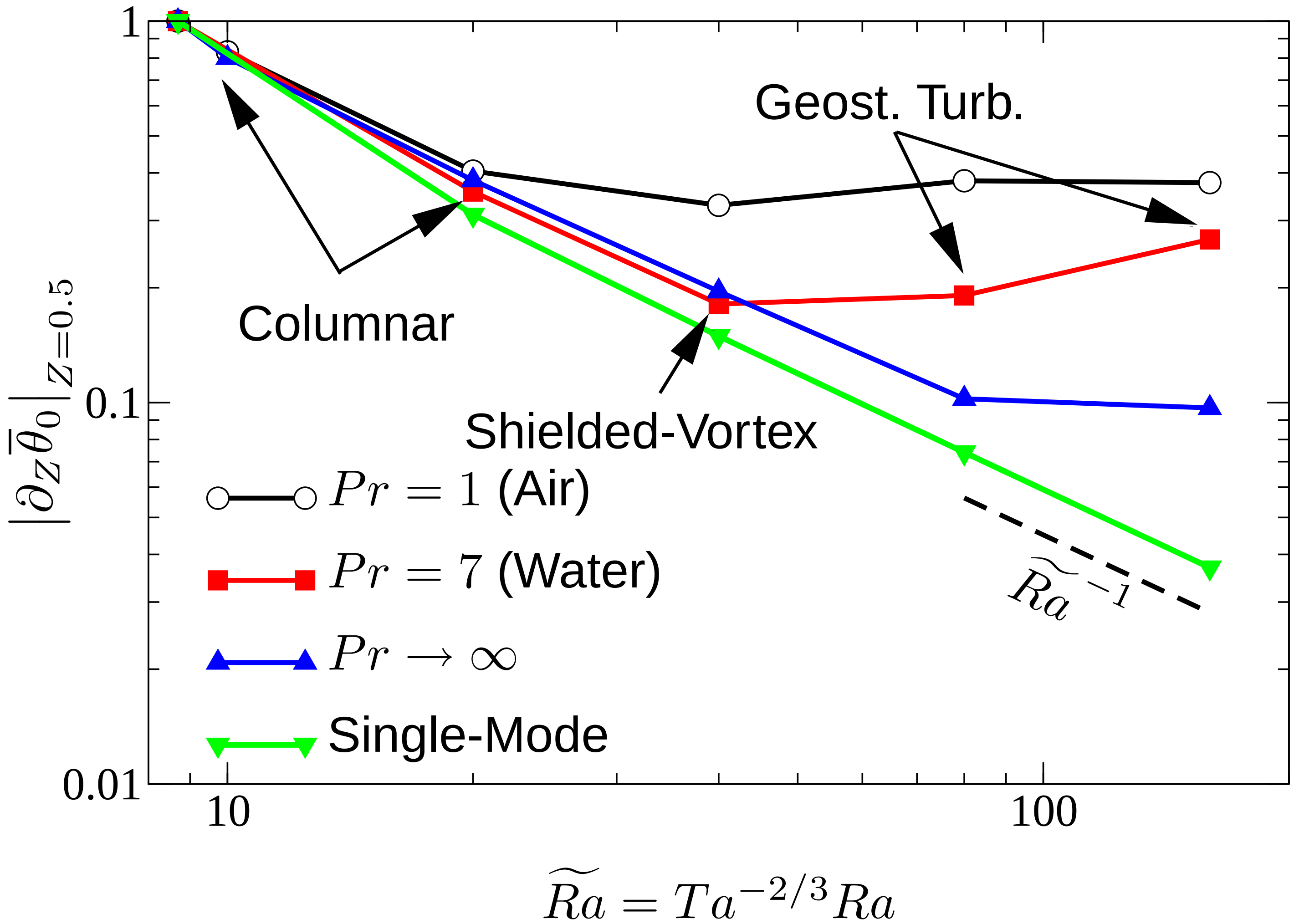
Top View Iso. View

- Vortical columns weakly interacting
- Zero vortical circulation $\int_0^{R_*} \omega r dr d\theta \approx 0$
- Particle model being pursued

Results: Mean Temperature ($\widetilde{Ra} = 160$)



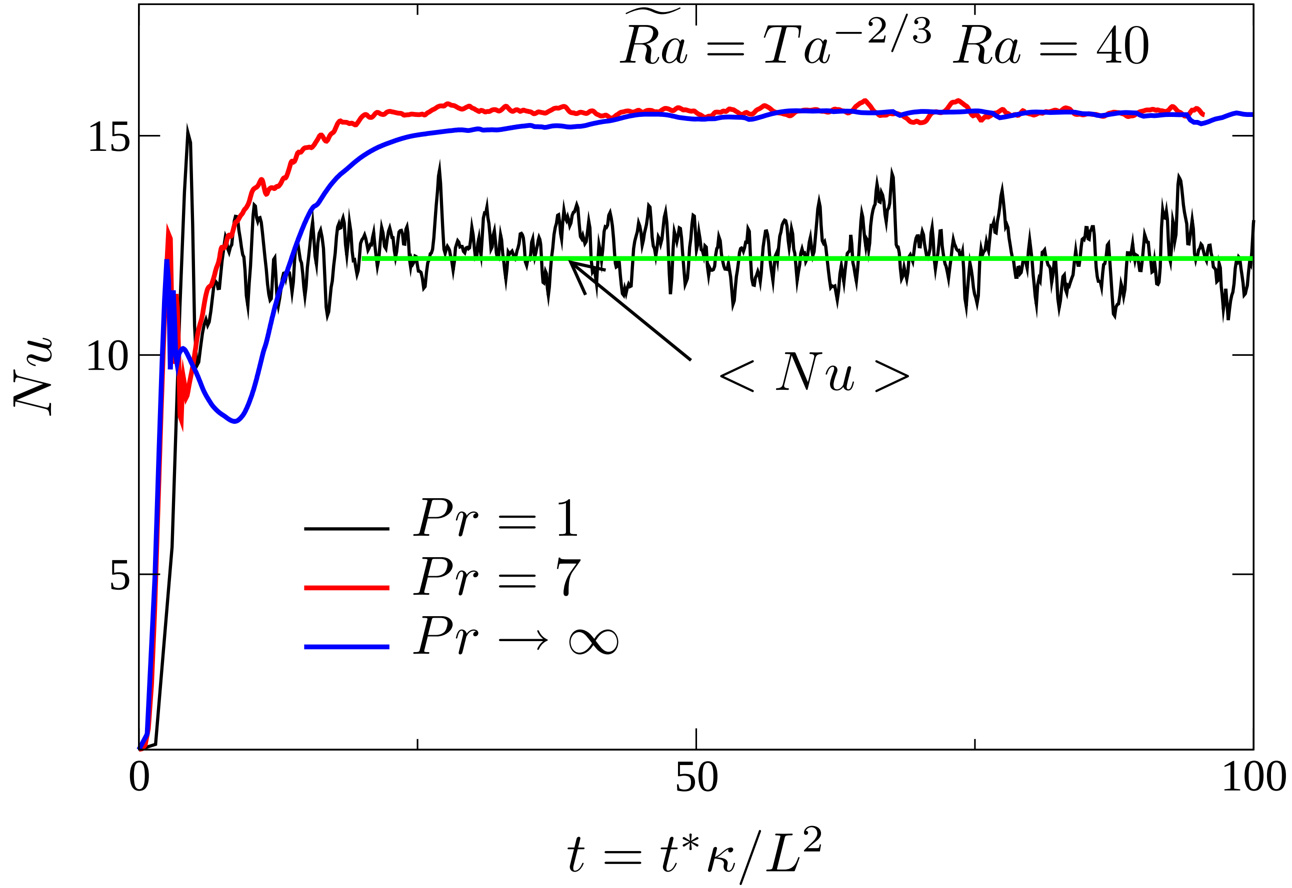
Results: Mean Temperature Gradient at Midplane



- Mean-temperature profile saturates at a non-isothermal interior for all finite Pr values studied

Results: Heat Transport

- Nusselt Number ($Nu = \partial_Z \overline{\theta}_0|_{Z=1}$): Measure of convective heat transport ($Nu = 1 \implies$ conductive heat transport)



- Results move quickly to statistically steady state
- $t_{rot} = 4\pi Pr^{-1} Ta^{-1/6}$; results shown for many rotation times

Summary

- Numerical simulation of reduced PDEs allows exploration of parameter range currently inaccessible with DNS of rotating Boussinesq equations
- Reduced PDEs capture dynamics seen in experiments
  - coherent structures spanning the layer of fluid
  - structures composed of cyclonic & anticyclonic vorticity
- Important findings:
  - Mean-temperature gradient saturates at nonzero value for **all** Prandtl numbers investigated due to increased lateral mixing
  - Found transition (with increasing $\widetilde{Ra}$) through three regimes:(*i*) columnar, (*ii*) shielded-vortex, and (*iii*) geostrophic turbulence
  - Illustrated importance of lateral mixing; should be incorporated in convection parametrizations for ocean circulation models

Future Work with Reduced PDEs

- Simulation of rapidly rotating convection on the tilted f-plane; more geophysically relevant
- Application of reduced PDEs to ocean deep convection (Legg, McWilliams & Gao, JPO 1998)
- Can we develop a particle model for the shielded-vortex regime of convection? (Legg & Marshall, JMR 1998),
- Simulation of recently developed equations for rapidly rotating convection in a cylinder (Sprague, *et al.*, TSFP4 2005)
- Application of Large-Eddy Simulation (LES) to reduced equations (Barbosa & Métais, JOT 2000)