Anisotropic constraints on energy in rotating and stratified flows

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Question:

- How does potential enstrophy constrain the energy of rotating and stratified flow in regimes *other* than quasi-geostrophy?
  - strong rotation, strong stratification (equal)
  - strong rotation, moderate stratification
  - moderate rotation, strong stratification

Rotating and stratified turbulence

- Boussinesq equations for rotating, stably stratified flow (periodic or infinite boundary conditions)

$$\begin{aligned}\frac{D}{Dt}\boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla p + N\theta\hat{\boldsymbol{z}} &= \nu\nabla^2\boldsymbol{u} + \mathcal{F} \\ \frac{D}{Dt}\theta - Nw &= \kappa\nabla^2\theta \\ \nabla \cdot \boldsymbol{u} &= 0,\end{aligned}$$

- conserved quantities (inviscid)

$$\text{total energy } E_T = E + P, \quad \frac{D}{Dt}\int E_T \, d\boldsymbol{x} = 0,$$

$$E = ½ \, u^2$$
$$P = ½ \, \theta^2$$

$$\text{potential vorticity } q = \left(\boldsymbol{\omega}_a \cdot \nabla\rho_T\right), \quad \frac{Dq}{Dt} = 0,$$

$$\text{potential enstrophy } Q = \frac{1}{2}q^2, \quad \frac{DQ}{Dt} = \frac{D}{Dt}\int Q \, d\boldsymbol{x} = 0.$$

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Potential enstrophy in *quasi-geostrophic* turbulence

- inviscid quasi-geostrophic dynamics is described by evolution of potential vorticity:

$$\frac{\partial q_{qg}}{\partial t} + \boldsymbol{u}_{0h} \cdot \nabla q_{qg} = 0,$$

$$\text{where} \quad q_{qg} = f\frac{\partial \theta}{\partial z} - N\omega_3,$$

$$Q = \frac{1}{2}|q|^2$$

- Charney 1971: Forward (downscale) transfer of total energy is suppressed in favor of (isotropic) forward transfer of potential enstrophy, with resultant scaling of energy spectrum:

$$E_T(k) \propto \varepsilon_Q^{2/3} k^{-3}$$

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Potential enstrophy in Boussinesq rotating and stratified flows

- non-dimensional potential vorticity: *Ro = U/Lf, Fr = U/LN*

$$q = \boldsymbol{\omega} \cdot \nabla\theta + Ro^{-1}\frac{\partial\theta}{\partial z} - Fr^{-1}\omega_3$$

- Linear limit of potential vorticity is approached when either rotation or stratification are strong

$$\tilde{q}(\boldsymbol{k}) \simeq f k_z \tilde{\theta} + i N k_h \tilde{u}_h$$

Limiting cases in *N* and *f* (*Fr* and *Ro*)

$$\tilde{q}(\boldsymbol{k}) \simeq f k_z \tilde{\theta} + i N k_h \tilde{u}_h$$

- strongly rotating and strongly stratified

  $N = f$; N and f large (Fr and Ro small)

- strongly rotating and moderately stratified

  $N >> f$; N large, $f \sim O(1)$ (Fr small, $Ro \sim O(1)$)

- moderately rotating, strongly stratified

  $N << f$; $N \sim O(1)$, f large ($Fr \sim O(1)$, Ro small)

Numerical simulations

- 3d Boussinesq, unit aspect ratio, periodic, stochastic forcing of momentum at *k=3*,4,5, variable rotation and stratification, hyperviscous dissipation (diffusion) (∇^{16}) of momentum (density)

# gridpoints	case	Ro	Fr
512^3	$N = f$	0.004	0.004
640^3	$N << f$	0.004	1
640^3	$N >> f$	1	0.004

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N=f, potential enstrophy suppresses potential energy in the large aspect-ratio modes

- as $k_z/k_h \gg 1$ (large aspect-ratio modes) $\tilde{q} \simeq f k_z \tilde{\theta}$

$$Q(k_h, k_z) = \frac{1}{2}|\tilde{q}|^2 = f^2 k_z^2 P(k_h, k_z) \text{ where } P(k_h, k_z) = \frac{1}{2}|\tilde{\theta}|^2$$

- potential enstrophy must dominate potential energy as $\kappa_z \to \infty$

$$\int_{\kappa_z}^{\infty} Q(k_h, k_z) dk_z \gg f^2 \kappa_z^2 \int_{\kappa_z}^{\infty} P(k_h, k_z)\, dk_z$$

- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis (Kraichnan, Charney)

$$P(k_h, k_z) \sim \varepsilon_Q^{2/5} k_z^{-3}$$

<u>*N=f*</u>, potential enstrophy suppresses horizontal kinetic energy in the small aspect-ratio modes

- in the limit as $k_z/k_h \ll 1$ (small aspect-ratio modes)

$$\tilde{q} = iNk_h\tilde{u}_h$$

- potential enstrophy must dominate over horizontal kinetic energy $E_h(k_h, k_z) = \frac{1}{2}|\tilde{u}_h|^2$ as $k_h \rightarrow \infty$
- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:

$$E_h(k_h, k_z) \sim \varepsilon_Q^{2/5} k_h^{-3}$$

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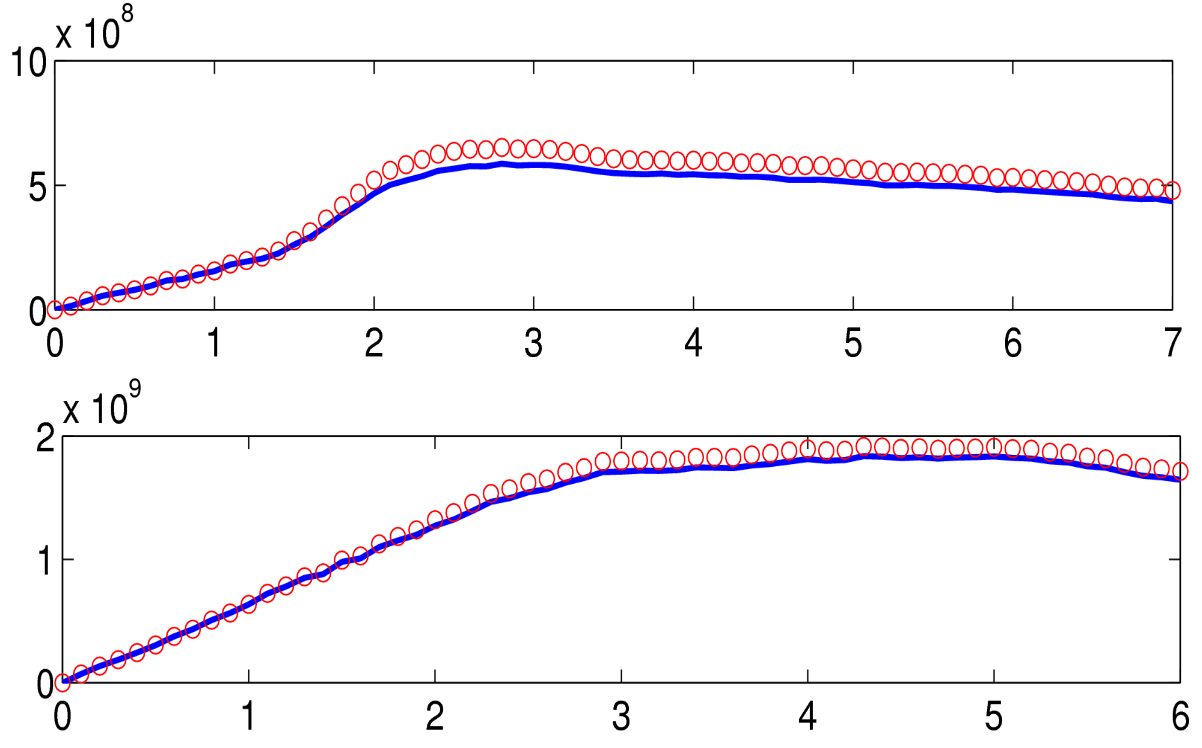
$\underline{N = f}$, total potential enstrophy (lines) and quadratic part (circles) as a function of time.

512^3
Ro = Fr = 0.014

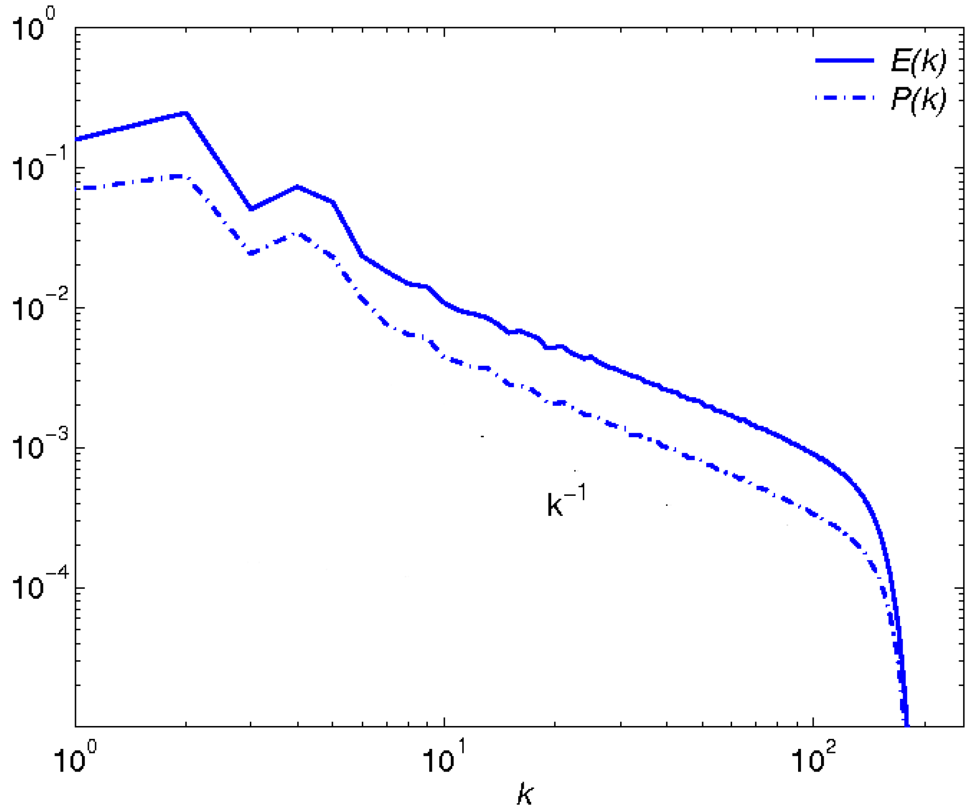
512^3
Ro = Fr = 0.007



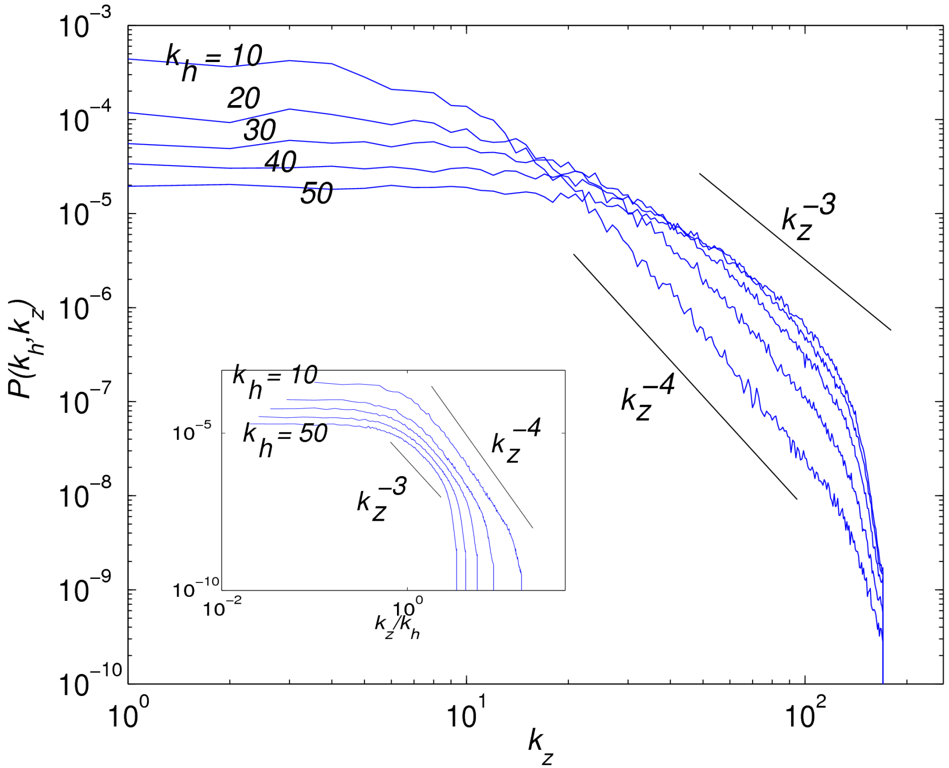
time

Shell-averaged kinetic and potential energy spectra

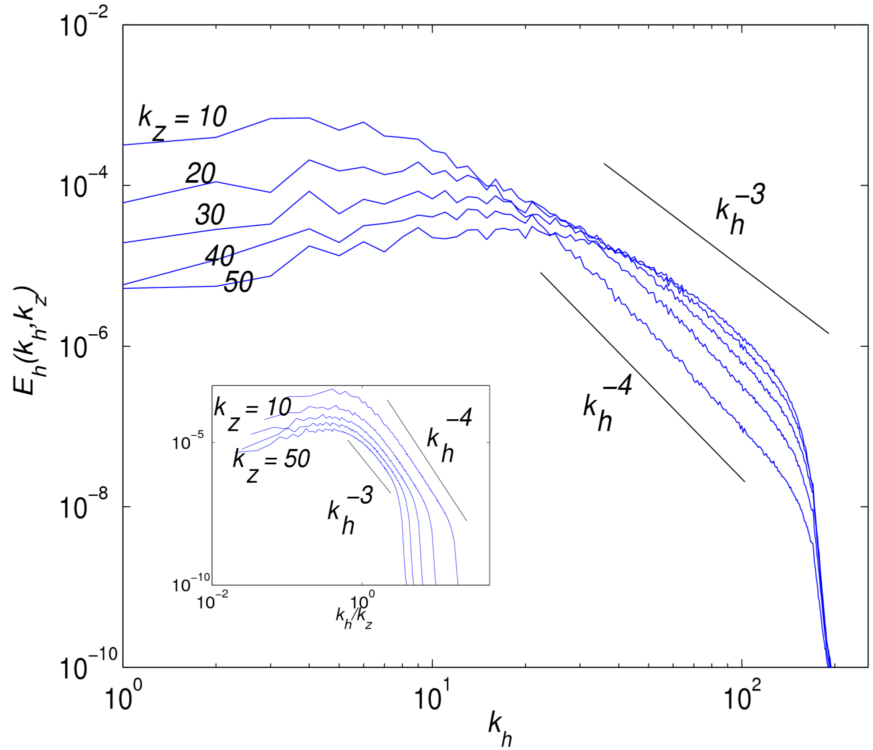
- 512^3 $Ro = Fr = 0.007$



Scaling of potential energy and horizontal kinetic energy spectra for N=f, aspect-ratio dependence



$k_z/k_h \gg 1$



$k_h/k_z \gg 1$

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$\underline{N \ll f}$: potential enstrophy suppresses potential energy dependence on k_h

- in this limit, dependence of potential vorticity on k_h disappears

$$\tilde{q} \simeq f k_z \tilde{\theta} \qquad \text{for all } k_h$$

- potential enstrophy dominates over potential energy in large k_z, and assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:

$$P(k_z) \sim \epsilon_Q k_z^{-3} \qquad \text{for all } k_h$$

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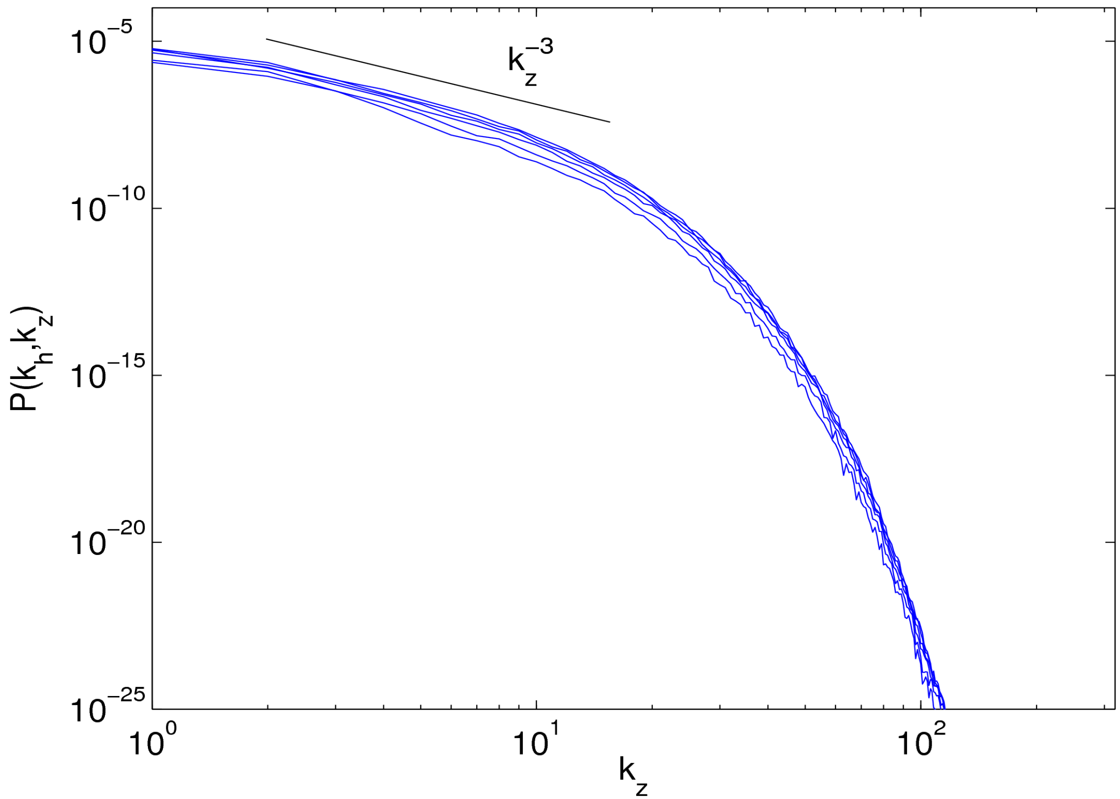
N << f: potential enstrophy suppresses potential energy dependence on k_h



$N \gg f$, potential enstrophy suppresses horizontal kinetic energy dependence on k_z

- in this limit, the dependence on k_z disappears since

$$\tilde{q} = iNk_h\tilde{u}_h \qquad \text{for all } k_z$$

- again potential enstrophy must dominate over horizontal kinetic energy E_h for large k_h
- assuming dependence on dissipation rate of potential enstrophy and dimensional analysis:

$$E_h(k_h) \sim \epsilon_Q k_h^{-3} \qquad \text{for all } k_z$$

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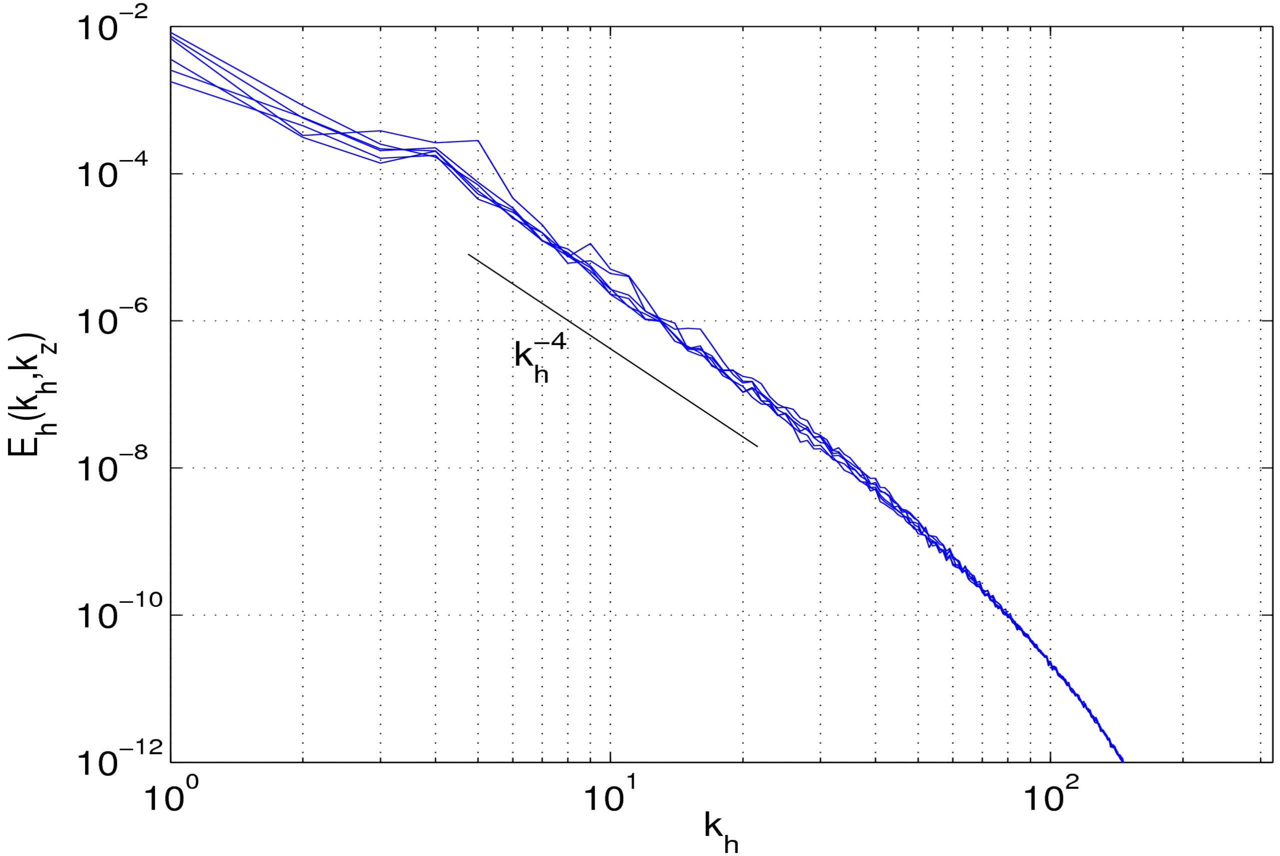
$\underline{N >> f}$, potential enstrophy suppresses horizontal kinetic energy dependence on k_z



Summary

- Potential enstrophy conservation for (equally) strong rotation and stratification: suppresses potential energy in the large aspect-ratio wavemodes and suppresses horizontal kinetic energy in the small aspect-ratio wavemodes (not QG, no projections used)

  (S. Kurien, M.Taylor and B. Wingate, submitted to PRL).
- Case of unequal rotation and stratification (SK, in preparation 2008)
  - potential enstrophy conservation in the limit of strong rotation and moderate stratification: suppresses potential energy in large k_z, eliminates dependence on k_h
  - potential enstrophy conservation in the limit of moderate rotation and strong stratification: suppresses horizontal kinetic energy in large k_h, eliminates dependence on k_z