Inviscid limit for damped and driven Navier-Stokes equations in $\mathbb{R}^2$

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Incompressible Navier-Stokes Equations

We consider an incompressible Newtonian flow in $\mathbb{R}^N$, $N = 2, 3$, and driven by a force $\mathbf{f}$. More precisely, the velocity vector field $\mathbf{u} = (u_1, u_2, u_3)$ of the fluid satisfies the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0,$$

The scalar p is the kinematic pressure, and the parameter $\nu > 0$ is the kinematic viscosity.

Law of finite energy dissipation rate

If in an experiment on turbulent 3D flow, all the control parameters are kept the same, except for the viscosity, which is lowered as much as possible, the average energy dissipation per unit mass, $\epsilon = \nu\langle|\nabla \mathbf{u}^{\nu}|^2\rangle$, behaves in a way consistent with a finite positive limit.

Fixed a characteristic velocity, and a characteristic length, what is the behavior of the inviscid limit, $\lim_{\nu\to 0} \mathbf{u}^{\nu}(t) = \mathbf{u}^{(E)}$? Is $\mathbf{u}^{(E)}$ a solution of the incompressible Euler equation? Does $\mathbf{u}^{(E)}$ develop singularities? This law is consistent with large gradients of velocity.

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Finite time zero viscosity limit

- For smooth solutions in $\mathbb{R}^2$ and $\mathbb{R}^3$, the zero viscosity limit is given by solutions of the Euler equations, for short time, in classical (Swann), and Sobolev (Kato) spaces
- The limit holds for as long as the Euler solution is smooth (Constantin). The convergence occurs in the Sobolev space H^s as long as the solution remains in the same space (Masmoudi). The rates of convergence are optimal in the smooth regime, $O(\nu)$.
- In some nonsmooth regimes (smooth vortex patches), the finite time inviscid limit exists and optimal rates of convergence can be obtained ([Abidi, Danchin], [Masmoudi]) but the rates deteriorate when the smoothness of the initial data deteriorates ([Constantin, Wu])

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Averages

Most traditional and modern descriptions of turbulence are not deterministic. This is because the many degrees of freedom that arise from hydrodynamic instability displays a complex behavior.

Averages

Time average: $\langle \mathbf{u}^{\nu} \rangle(x) \approx \frac{1}{T} \int_0^T u^{(\nu)}(t,x)dt$

Ensemble average: $\langle \mathbf{u}^{\nu} \rangle(x) \approx \frac{1}{N} \Sigma_{n=1}^N u^{(\nu,n)}(t,x)$

Spatial average: $\langle \mathbf{u}^{\nu} \rangle(x) \approx \frac{1}{N} \Sigma_{n=1}^N u^{(\nu)}(t, x + r^{(n)})$

Ergodic Hypothesis (not used here)

The averaged quantities are independent of the kind of average considered

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One possible meaning of $\langle\Phi(\mathbf{u})\rangle$ is the mathematical expectation of the functional Φ with respect to a measure in function space. The measure is supported on Navier-Stokes solution paths, and should be stable with respect to small random perturbations.

In practice, in order to be relevant to an experiment, the meaning of $\langle\Phi(u)\rangle$ has to be a specific empirical average (long time average, or long time and space average).

"Don't read the times, read the eternities" (Thoreau)

- In 1941 Kolmogorov introduced the idea of universality of the statistical properties of small scales. Infinite time behavior at finite but larger and larger Reynolds numbers.
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Invariant measures

In the case of finite dimensional dynamical systems

$$\frac{du^{(\nu)}}{dt} = N(u^{(\nu)}), \quad u^{(\nu)}(t) \in \mathbb{R}^N,$$

invariant measures μ obey

$$\int \nabla_u \Psi(u) N(u) d\mu^{(\nu)}(u) = 0.$$

for any test function Ψ. In infinite dimensions we need to restrict the test functions to a limited class of admissible functions. Among them are generalizations of the characters $\exp i\langle \omega, \mathbf{w} \rangle$.

A stationary statistical solution is a Borel probability measure μ^ν on L^2 such that:

(1) $\displaystyle\int_{L^2} \|u\|^2_{H^1}\, d\mu^\nu(u) < \infty;$

(2) $\displaystyle\int_{L^2} \langle u\cdot\nabla u - \mathbf{f}, \Psi'(u)\rangle + \nu\langle\nabla_x u, \nabla_x \Psi'(u)\rangle d\mu^\nu(u) = 0$

for any test functional $\Psi \in \mathcal{T}$, and

(3) $\displaystyle\int_{E_1 \le \|u\|_{L^2} \le E_2} \left\{\nu\,\|u\|^2_{H^1} - (\mathbf{f}, u)\right\} d\mu^\nu(u) \le 0., E_1, E_2 > 0.$

Stationary Statistical Solutions of the Navier-Stokes equations and long time averages.

We construct stationary statistical solutions Navier-Stokes equations by the Krylov-Bogoliubov procedure of taking long time averages.

$$\operatorname{LIM}_{T\to\infty} \frac{1}{T}\int_0^T \Phi(u^{(\nu)}(t))dt = \int_{L^2(\mathbb{R}^2)} \Phi(u)d\mu^{\nu}(u)$$

Inviscid limit: Time matters

- Finite time zero viscosity limit

$$\lim_{\nu \to 0} S_\nu^{NS}(t)u_0 = S^E(t)u_0, \quad \text{for } t \leq T \tag{1}$$

- Infinite time zero viscosity limit

$$\lim_{\nu \to 0} \operatorname{LIM}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(u^{(\nu)}(t))dt = \int \Phi(u) d\mu^E(u)$$

Time and zero viscosity limits do not commute.

2D NSE

In two dimensions there exist infinitely many integrals that are conserved by smooth Euler flows. One of them is the enstrophy, $\left|\omega^{(\nu)}(t)\right|_{L^2}^2$, where $\omega = \nabla \times \mathbf{u}$.

- The existence of anomalous dissipation of enstrophy is postulated in Kraichnan theory for two dimensional turbulence
$$\lim_{\nu \to 0} \nu \langle \left\| \nabla \omega^{(\nu)} \right\|_{L^2}^2 \rangle = \eta > 0,$$
- (**Finite Time**) This was studied in the framework of finite time inviscid limits with rough initial data ([Eyink], [M. Lopes Filho, A. Mazzucato, H. Nussenzveig Lopes]). It was established that, if the initial vorticity belongs to $L^2(\mathbb{R}^2)$ then rate of dissipation of enstrophy vanishes with viscosity, for finite time. The finite time inviscid limits are weak solutions of Euler equations.

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2D damped and driven Navier-Stokes

We consider damped, driven Navier-Stokes equations in the whole plane

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \gamma u + \nabla p = f, \\ \nabla \cdot u = 0 \end{cases} \tag{2}$$

with $\gamma > 0$ a fixed damping coefficient, $\nu > 0$, f time independent with zero mean and $f \in W^{1,\infty}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. The initial velocity is divergence-free and belongs to $(L^2(\mathbb{R}^2))^2$.

Some results concerning the damped case

- "It seems plausible that all the enstrophy could be removed by surface drag if the viscosity coefficient were zero or sufficiently small", Lilly in *(Geophys. and Astrophys. fluid Fluid Dynamics, 1972)*
- D. Bernard conducted a detailed analysis of the physics of damped equations, and also conjectured the absence of anomalous dissipation of enstrophy. *Europhys. Letters, 2000*
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Let $u_0 \in L^2(\mathbb{R}^2)$ and $\nabla^\perp u_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Let $f \in W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$, and let $\omega^\nu(t) = S^{NS,\gamma}(t)(\omega_0)$ be the vorticity of the solution of the damped and driven Navier-Stokes equation. Then,

$$\lim_{\nu \to 0} \nu \left(\limsup_{t \to \infty} \frac{1}{t} \int_0^t \|\nabla \omega^\nu(s + t_0)\| \, ds \right) = 0.$$

for any $t_0 > 0$.

Theorem

Let u_0 be smooth, divergence-free, $u_0 \in H^1(\mathbb{R}^2)^2$. Then the solution with initial datum u_0 exists for all time, is unique, smooth, and obeys the energy equality

$$\frac{d}{2dt}\int_{\mathbb{R}^2}|u|^2dx + \gamma\int_{\mathbb{R}^2}|u|^2dx + \nu\int_{\mathbb{R}^2}|\nabla u|^2dx = \int_{\mathbb{R}^2} f\cdot udx. \quad (3)$$

The kinetic energy is bounded uniformly in time, with bounds independent of viscosity:

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le e^{-\gamma t}\left\{\|u(\cdot,0)\|_{L^2(\mathbb{R}^2)} - \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R}^2)}\right\} + \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R}^2)}.$$

The vorticity ω (the curl of the incompressible two dimensional velocity)

$$\omega = \partial_1 u_2 - \partial_2 u_1 = \nabla^\perp \cdot u \quad (4)$$

obeys

$$\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega + \gamma \omega = g, \qquad (5)$$

with $g \in L^2(\mathbb{R}^2)$, the vorticity source, $g = \nabla^\perp \cdot f$. If the initial vorticity is in $L^p(\mathbb{R}^2)$, $p > 1$, and $g \in L^p(\mathbb{R}^2)$, then the p-enstrophy is bounded uniformly in time, with bounds independent of viscosity

$$\|\omega(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le e^{-\gamma t}\left\{\|\omega(\cdot,0)\|_{L^p(\mathbb{R}^2)} - \frac{1}{\gamma}\|g\|_{L^p(\mathbb{R}^2)}\right\} + \frac{1}{\gamma}\|g\|_{L^p(\mathbb{R}^2)}$$

Moreover, the solution does not travel: For every $\epsilon > 0$, there exists $R > 0$ such that,

$$\int_{|x| \ge R} |\omega(x,t)|^2 \, dx \le \epsilon$$

holds for all $t \ge 0$.

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$$\lim_{\nu \to 0} \nu \left(\limsup_{t \to \infty} \frac{1}{t} \int_0^t \|\nabla \omega^\nu(s + t_0)\| \, ds \right) = 0.$$

for any $t_0 > 0$.

Idea of the proof

- $$\nu \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\nabla\omega^\nu(s+t_0)\|^2\, ds \le$$
  $$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \gamma\|\omega^\nu(s+t_0)\|^2 - (g,\omega^\nu(s+t_0))ds.$$
  for any $t_0 > 0$.
- Because the function $-\gamma\|\omega\|^2_{L^2(\mathbb{R}^2)} + \langle g,\omega\rangle$ is continuous on $clO^+(t_0,\{\omega_0\})$, we can choose a generalized limit such that

$$\underset{T\to\infty}{\mathrm{LIM}} \frac{1}{T}\int_0^T \left[-\gamma\|\omega^\nu(s+t_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g,\omega^\nu(s+t_0)\rangle\right] ds$$
$$= \limsup_{t\to\infty} \frac{1}{t}\int_0^t -\gamma\|\omega^\nu(s+t_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g,\omega^\nu(s+t_0)\rangle ds.$$

Idea of the proof

- Because $clO^+(t_0, \{\omega_0\})$ is weakly compact in $L^2(\mathbb{R}^2)$, and because $\text{LIM}_{T\to\infty}$ is a bounded linear function in this space, by the Kakutani-Riesz representation theorem on compact spaces, we can find a weakly Borel probability measure in $L^2(\mathbb{R}^2)$, representing the last expression

$$\int_{L^2(\mathbb{R}^2)} \left[-\gamma \|\omega\|^2_{L^2(\mathbb{R}^2)} + \langle g, \omega\rangle\right] d\mu^\nu(\omega) =$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t -\gamma \|\omega^\nu(s+t_0)\|^2_{L^2(\mathbb{R}^2)} + \langle g, \omega^\nu(s+t_0)\rangle ds.$$

- We can prove that the supports of the measures μ^ν are uniformly bounded in L^2, which is sufficient to obtain convergent subsequences, as $\nu \to 0$, by the Prokhorov's Theorem

Then, using Prokhorov's Theorem, and harmonic analysis estimates, we can prove that the measures μ^ν converge to a meausure μ^0, that is a renormalized statistical solution of the Euler equations. This measure satisfies the balance of enstrophy:

$$\int_{L^2(\mathbb{R}^2)} -\gamma \|\omega\|^2_{L^2(\mathbb{R}^2)} + \langle g, \omega\rangle d\mu^0(\omega) = 0$$

And we essentially obtain:

$$\nu \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|\nabla\omega^\nu(s+t_0)\|\, ds \leq$$
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Thank you!