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Roll clouds in the jet stream over Saudi Arabia/Red Sea
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What is a vortical mode, what is a wave mode, what are vortical-wave mode interactions?

When are vortical-wave mode interactions important?

Can we use this framework to understand balanced and unbalanced components of geophysical flows?
Rotating stratified fluid flow is a starting point to understand atmosphere-ocean phenomena.

In the linear limit, the governing equations possess:

- a so-called ‘vortical’ linear eigenmode
- wave eigenmodes

The simplest ‘reduced models’ (e.g. QG models)

- keep nonlinear interactions between vortical modes
- neglect wave mode interactions.

This strategy works to describe large-scale flows on short time scales, e.g. short-term weather prediction.
Overview continued:

However, wave mode interactions contribute in many physical situations, e.g. flow over topography, and influence large-scale coherent flows on long times e.g. they can generate jets, vortices and layers.

We introduce a framework to construct an understanding of all wave-vortical mode interactions.

New, non-perturbative models are derived 'intermediate' between QG and the full governing equations.
I. Analytical Properties of the Governing Equations
   - Solution as a superposition of linear eigenmodes
   - Reduced models

II. Example Numerical Results Depending Crucially on Wave Mode Interactions
   - large-scale forcing, small-scale forcing, decay

III. Derivation of new PDE reduced models to understand wave-vortical mode interactions

IV. Numerical results for new PDE models
The rotating Boussinesq equations

Conservation laws for vertically stratified flow rotating about the vertical \( \hat{z} \)-axis:

- **momentum:** \[ \frac{Du}{Dt} + f\hat{z} \times u = -\nabla \phi - N\theta\hat{z} + \nu \nabla^2 u \]

- **mass:** \[ \nabla \cdot u = 0 \]

- **energy:** \[ \frac{D\theta}{Dt} - Nw = \kappa \nabla^2 \theta, \quad \theta = \frac{g}{N\rho_o} \rho' \]

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\[ f = 2\Omega, \quad Ro = \frac{U}{fL} \]

\[ \rho = \rho_o - bz + \rho', \quad \rho' \ll \rho_o, |bz|, \quad N^2 = \frac{gb}{\rho_o}, \quad Fr = \frac{U}{NH} \]
Rossby and Froude numbers in geophysical flows

Pedlosky (1986) estimates:

- $Ro \approx 0.14$ for typical synoptic-scale winds at mid-latitudes
  $U \approx 10 \text{ m s}^{-1}, \ L \approx 1000 \text{ km}$
- $Ro \approx 0.07$ in the western Atlantic
  $U \approx 5 \text{ cm s}^{-1}, \ L \approx 100 \text{ km}$

Typical values are $N/f \approx 100$ in the stratosphere
and $N/f \approx 10$ in the oceans.

Flows with $N/f \approx L/H \implies Fr \approx 0.1$ (Burger number unity).
Solutions in the unforced, linear, inviscid limit

\[
[u, \theta]^T(x, t; k) = \phi(k) \exp \left[ i \left( k \cdot x - \sigma(k) t \right) \right] + \text{c.c.}
\]

with eigenmodes \( \phi(k) \) and eigenvalues \( \sigma(k) \).

- Wave modes \( \phi_+(k) \) and \( \phi_-(k) \) with
  \[
  \sigma_{\pm}(k) = \pm \left( N^2 k_h^2 + f^2 k_z^2 \right)^{1/2} k
  \]

- A non-wave (vortical or geostrophic) mode \( \phi_0(k) \) with
  \[
  \sigma_0(k) = 0
  \]
Slow wave modes (as important as slow vortical modes!)

- Rotation-dominated flows

\[ \sigma_{\pm}(k) \approx \pm \frac{f k z}{k} \]

slow when \( k z = 0 \), e.g. vortical columns.

- Stratification-dominated flows

\[ \sigma_{\pm}(k) \approx \pm \frac{N k h}{k} \]

slow when \( k h = 0 \), e.g. horizontal shear layers (VSHF)
Eigenmode representation for nonlinear flows

Since $\phi_s(k)$, $s = \pm, 0$ form an orthogonal basis

$$[u, \theta]^T(x, t) = \sum_k \sum_s b_s(t; k) \phi_s(k) \exp \left[ i \left( k \cdot x - \sigma_s(k)t \right) \right]$$

and the equations become

$$\frac{\partial}{\partial t} b_{sk} = \sum_{\Delta} \sum_{s_p, s_q} C_{kpq}^{s_k s_p s_q} b_{sp}^* b_{sq}^* \exp \left[ i \left( \sigma_{sk} + \sigma_{sp} + \sigma_{sq} \right) t \right]$$

27 interaction types, including 3-wave interactions

Exact and near resonances dominate: $|\sigma_{sk} + \sigma_{sp} + \sigma_{sq}| \ll 1$. 
Reduced models resulting from restriction of the sum

\[ \frac{\partial}{\partial t} b_{sk} = \sum_{\Delta} \sum_{s_p, s_q} C_{kpq}^{skspq} b_{sp}^* b_{sq}^* \exp \left[ i \left( \sigma_{sk} + \sigma_{sp} + \sigma_{sq} \right) t \right] \]

automatically conserve energy because each triad \((k, p, q)\)

satisfies the detailed balance:

\[ C_{kpq}^{skspq} + C_{pqk}^{spqs} + C_{qkp}^{sqskp} = 0 \]
keeping only slow wave modes with $k_z = 0 \implies$ symmetric 2D flow for $(u,v)$; $w$ is a passive scalar.

Fig. shows (symmetric) 2D decay with large-scale drag.
3D Pure Rotation

all interactions with $|\sigma_{sk} + \sigma_{sp} + \sigma_{sq}| < Ro \implies$ cyclone dominance (but is not a PDE !).

Full

Near resonances (12%)

Smith & Lee 2005
Reduced Models for 3D Boussinesq

Keeping only slow vortical mode interactions $\implies$ the symmetric 3D quasi-geostrophic equation.

In Fourier space

$$\frac{d}{dt} b_0(t; k) = \sum_{\triangle} C_{kpq}^{000} b_0^*(p) b_0^*(q)$$

An inverse transform gives 3DQG

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_H \cdot \nabla \right) q = 0, \quad q = \left( \nabla^2_H + \frac{f^2}{N^2} \frac{\partial^2}{\partial z^2} \right) \psi(x, t)$$

$$\nabla^2_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \mathbf{u}_H = \hat{z} \times \nabla \psi, \quad \theta = -\frac{f}{N} \frac{\partial \psi}{\partial z}$$
Part II. Phenomena not captured by QG

Example 1: Anticyclone dominance in RSW decay

\[ Ro = \frac{U}{fL} = 0.4, \quad Fr = \frac{U}{gH}^{1/2} = 0.25 \]
Example 2: Generation of VSHF; $Bu = fL/(NH)=1; H/L=1/5$

$Ro = U/(fL) = 0.1; Fr = U/(NH) = 0.1$
Example 3: Asymmetry with respect to $f/N = 1$ (QG-like)

\[ f/N = 1/4, 1/5 \]

Bartello 1995, Sukhatme & Smith 2008
Intermediate models add more physics to QG

Improving upon 2DQG: Allen, Barth & Newberger 1990; Spall & McWilliams 1992; Yavneh & McWilliams 1994; Warn, Bokhove, Shepherd & Vallis 1995; Vallis 1996


Previous intermediate models are perturbative in nature
A hierarchy of NEW Intermediate Models

- Derived by adding subsets of wave-vortical mode interactions to QG
- Non-perturbative
- Include near-resonant triads
- Provides a framework for understanding the coupling between balanced and unbalanced flow components
e.g. Kuo, Allen, Polvani 99; Ford, McIntyre, Norton 00; Majda 03
Symbolically

The Full Equations:

\[ \text{Symbolecally} \]

\[ 0 \mid 00 \oplus 0^+ \oplus 0^- \oplus ++ \oplus +-- \oplus -- \quad (1) \]

\[ + \mid 00 \oplus 0^+ \oplus 0^- \oplus ++ \oplus +-- \oplus -- \quad (2) \]

\[ - \mid 00 \oplus 0^+ \oplus 0^- \oplus ++ \oplus +-- \oplus -- \quad (3) \]

QG (vortical mode interactions only):

\[ 0 \mid 00 \]
Two NEW Models

PPG (add to QG interactions involving exactly 1 wave):

\[
\begin{align*}
0 & \mid 00 \oplus 0 + \oplus 0- \\
+ & \mid 00 \quad \text{and} \quad - & \mid 00
\end{align*}
\]  
\( (ppg1) \)

P2G (add to PPG interactions involving exactly 2 waves):

\[
\begin{align*}
0 & \mid 00 \oplus 0 + \oplus 0- \oplus ++ \oplus + - \oplus - - \\
+ & \mid 00 \oplus 0 + \oplus 0- \\
- & \mid 00 \oplus 0 + \oplus 0-
\end{align*}
\]  
\( (p2g1, p2g2, p2g3) \)
QG and PPG Rotating Shallow Water (RSW) Equations

QG: \( \frac{\partial Q}{\partial t} + J(\Psi, Q) = 0 \)

PPG:
\[
\frac{\partial \nabla^2 \chi}{\partial t} - \nabla^2 V = 2J\left( \frac{\partial A}{\partial x}, \frac{\partial A}{\partial y} \right), \tag{1}
\]
\[
\frac{\partial Q}{\partial t} + J(\Psi, Q) + \nabla \chi \cdot \nabla Q + \langle u \rangle \frac{\partial Q}{\partial x} + \langle v \rangle \frac{\partial Q}{\partial y} + Q \nabla^2 \chi = 0, \tag{2}
\]
\[
\frac{\partial \nabla^2 V}{\partial t} - c^2 \nabla^4 \chi + f^2 \nabla^2 \chi = fJ(A, Q) \tag{3}
\]

\( Q = (\nabla^2 - \frac{f^2}{gH})\Psi, \quad u = \chi_x - \Psi_y, \quad v = \chi_y + \Psi_y, \quad \nabla^2 \chi = u_x + v_y \)

\( \nabla^2 V = \nabla^2 (f\Psi - gh) \) is a measure of geostrophic imbalance (Vallis 96); also called geostrophic departure (Warn 95); ageostrophic vorticity (Mohebalhojeh & Dritschel 01)

\( A \equiv (f^2 - c^2 \nabla^2)^{-1} c^2 Q \)
RSW decay, Ro=0.4, Fr = 0.25, divergence-free unbalanced i.c.
Centroid in RSW decay; divergence-free unbalanced i.c.

\[
\text{Cent}(k) = \frac{\sum_k k (|u_k|^2 + |v_k|^2)}{\sum_k (|u_k|^2 + |v_k|^2)}
\]

Ro = 0.4, Fr = 0.25

Ro = 0.25, Fr = 0.2
Vorticity skewness in RSW decay; divergence-free unbalanced i.c.

\[ \text{Ro} = 0.4, \text{Fr} = 0.25 \]
Centroid in RSW decay with divergent initial conditions

$Ro = 1, \ Fr = 0.3$

25% divergence-free  5% divergence-free
Skewness in RSW decay with divergent initial conditions

$\textbf{Ro} = 1, \textbf{Fr} = 0.3$

25% divergence-free

5% divergence-free
\[ Ro = \varepsilon^{1/3} \left( \frac{k_f}{\pi} \right)^{2/3} f^{-1} = 0.05, \quad Fr = \left( \varepsilon \pi \right)^{1/3} k_f^{-1/3} \left( gH \right)^{-1/2} = 0.03 \]
Differences between RSW and RB

Unlike for RB, truncations of RSW are not guaranteed to maintain energy as an integral invariant (Warn 86).

RSW intermediate models with both energy conservation and Lagrangian invariance of PV do not perform as well as models conserving only PV (Allen 93).

Model performance is not necessarily linked to conservation properties (Allen 93).
Differences between RSW and RB

- In RSW, two wave modes cannot transfer energy into or out of a vortical mode, i.e.
  \[ C_{kpq}^{0+-} = C_{kpq}^{0++} = C_{kpq}^{0--} = 0 \]
  (true only for exact resonances in RB)

- In RSW the catalytic resonances with
  \((s_k, s_p, s_q) = (\pm, 0, \pm)\) only exchange energy in the same wavenumber shell

- In RSW, there are no exact 3-wave resonances
  (Majda 96)
Rotating Boussinesq Equations

- In RB, two wave modes can transfer energy into or out of a vortical mode, except for exact resonances.

- In RB the catalytic resonances with \((s_k, s_p, s_q) = (\pm, 0, \pm)\) exchange energy between scales (on a cone in wavevector space).

- In RB, there are exact 3-wave resonances.
Rotating Boussinesq Equations

- In RB, two wave modes can transfer energy into or out of a vortical mode, except for exact resonances.

- In RB the catalytic resonances with \((s_k, s_p, s_q) = (\pm, 0, \pm)\) exchange energy between scales (on a cone in wavevector space)

- In RB, there are exact 3-wave resonances

Richer behavior in RB from interactions involving one vortical mode and two wave modes (including catalytic exact/near resonances), and from 3-wave exact/near resonances.
We predict the P2SG model can reproduce asymmetry with respect to $f/N = 1$ in the forward transfer range:

\begin{align*}
0 & | 00 \oplus ++ \oplus + - \oplus -- \\
+ & | 0 + \oplus 0 - \\
- & | 0 + \oplus 0 -
\end{align*}
Example 3: Asymmetry with respect to $f/N = 1$ (QG-like)

- $f/N = 1/4, 1/5$
- $f/N = 4, 5$
Conclusions/Discussion

- A nonperturbative method to derive a hierarchy of models including wave-vortical interactions

- Provides a framework for complete understanding of balanced and imbalanced flow components

- Interactions involving exactly one gravity wave responsible for anticyclone dominance in RSW

- Stay tuned for results on RB