A Quasi-Normal Scale Elimination (QNSE) theory of turbulent flows with stable stratification

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Introduction

- The aim is to develop a theory that systematically includes anisotropic turbulence and internal waves
- The difficulty: turbulence is a nonlinear, multi-scale, stochastic phenomenon. Analytical theories exist for simplest flows that are locally isotropic and depend on a single dimensionless parameter, Re. Geophysical flows are anisotropic with waves
- Reynolds averaging does not differentiate between scales and does not discern contributions from different processes. Reynolds stress closures employ the concept of "invariant modeling" and are not flexible enough
- **Spectral approach is more suitable**

The Quasi-Normal Scale Elimination (QNSE) theory of turbulence with stratification

We consider fully 3D turbulent flow with imposed vertical temperature gradient $d\Theta/dz$.
Governing equations in Boussinesq approximation:

momentum
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} - \alpha g T \hat{e}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathrm{f}_0$$

temperature
$$\frac{\partial T}{\partial t} + (\mathbf{u}\nabla)T + \frac{d\Theta}{dz}\mathrm{u}_3 = \kappa_0 \nabla^2 T$$

continuity
$$\nabla \mathbf{u} = 0$$

In linear approximation this system supports gravity waves with Brunt-Vaisala frequency $N \equiv \left(\alpha g \frac{d\Theta}{dz}\right)^{1/2}$

Fourier-transformed velocity and temperature equations

Eliminate pressure using continuity equation; obtain momentum equation in a self-contained form using formal solution to the temperature equation:

$$u_\alpha(\hat{k}) = G_{\alpha\beta}(\hat{k})\left[\underbrace{\varphi^T_\beta(\hat{k}) - \frac{i}{2}P_{\beta\mu\nu}(\hat{k})\int u_\mu(\hat{q})u_\nu(\hat{k}-\hat{q})\frac{\mathrm{d}\hat{\mathrm{q}}}{(2\pi)^4}}_{f_\beta(\hat{k})}\right]$$

$$T(\hat{k}) = G_T(\hat{k})\left(-\frac{d\Theta}{dz}u_3(\hat{k}) \underbrace{- ik_\alpha\int u_\alpha(\hat{q})T(\hat{k}-\hat{q})\frac{\mathrm{d}\hat{\mathrm{q}}}{(2\pi)^4}}_{f_T(\hat{k})}\right)$$

Velocity Green function becomes tensorial:

$$G_{\alpha\beta}(\mathbf{k},\omega) = G(\mathbf{k},\omega)[\delta_{\alpha\beta} + A(\mathbf{k},\omega)P_{\alpha 3}(\mathbf{k})\delta_{\beta 3}]$$

complex poles => waves!

where $G(\mathbf{k},\omega) = [-i\omega + \nu_h(k)k_h^2 + \nu_z(k)k_3^2]^{-1}$

ν_h, ν_z - horizontal and vertical eddy viscosities, $P_{\alpha\beta}$ - projection operator

$$A(\mathbf{k},\omega) = -\frac{N^2}{\left(-i\omega + \nu k_h^2 + \nu_z k_3^2\right)\left(-i\omega + \kappa k_h^2 + \kappa_z k_3^2\right) + N^2\sin^2\phi}$$

φ is the angle between **k** and the vertical,

$$G_T(\mathbf{k},\omega) = [-i\omega + \kappa_h(k)k_h^2 + \kappa_z(k)k_3^2]^{-1}$$

is temperature Green function, κ_h and κ_z are horizontal and vertical eddy diffusivities

Quasi-Normal Scale Elimination Model (QNSE)

As in other quasi-normal models, we seek the solution in the form

$$u_i(\mathbf{k},\omega) = G_{ij}(\mathbf{k},\omega)\, f_j(\mathbf{k},\omega) \qquad \text{(LM)}$$

$$T(\mathbf{k},\omega) = G_T(\mathbf{k},\omega)\left(f_T(\mathbf{k},\omega) - \frac{d\Theta}{dz} u_3(\mathbf{k},\omega)\right) \qquad \text{(LT)}$$

In other words: mapping onto quasi-Gaussian fields governed by the Langevin equations

$f_j(\mathbf{k}, \omega)$ is a stochastic force representing **stirring of a given velocity mode by all other modes;** postulated as quasi-Gaussian, solenoidal, homogeneous in space and time:

$$\langle f_\alpha(\omega,\boldsymbol{k})\, f_\beta(\omega',\boldsymbol{k}')\rangle \propto \varepsilon\, k^{-3} P_{\alpha\beta}(\boldsymbol{k})\,\delta(\omega+\omega')\,\delta(\boldsymbol{k}+\boldsymbol{k}')$$

Effective viscosities and diffusivities in Green functions describe **damping of a mode by nonlinear interactions with all other modes.**

Due to anisotropy, viscosities and diffusivities are different in the vertical and horizontal directions.

Our goal is to calculate the mapping parameters, $\nu_h, \nu_z\ \kappa_h, \kappa_z$

Quasi-Normal Scale Elimination (QNSE) method

Central problem is **treatment of nonlinearity**. Perturbative solution based on expansion parameter Re **is strongly divergent**

General idea: Re is O(1) for smallest scales of motion =>

- Derive formal solution for these small scales
- Using the assumption of Quasi-Gaussianity perform averaging over infinitesimal band of small scales. Compute corrections to "effective" or "eddy" viscosity and heat diffusivity. Viscosity increases; **Re for the next band remains small**

$$\Delta G^{-1}_{\alpha\beta}(\omega,k,k_3) = P_{\alpha\beta\vartheta}(\mathbf{k})\int^{>} P_{\nu\sigma\beta}(\mathbf{k}-\mathbf{q})U_{\mu\sigma}(\hat{q})G_{\vartheta\nu}(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4}$$

$$\Delta G^{-1}_{T}(\omega,k,k_3) = k_\alpha k_\beta \int^{>} U_{\alpha\beta}(\hat{q})G_{T}(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4}$$

$$\text{where}\, U_{\mu\sigma}(\hat{q}) = 2Dq^{-3}G_{\alpha\mu}(\hat{q})G^{*}_{\beta\sigma}(\hat{q})P_{\alpha\beta}(\mathbf{q})$$

- Repeat the above procedure for next band of smallest scales.

Theoretical results

We obtain a coupled system of 4 differential equations for all corrections. The system is solved analytically for weak and numerically for arbitrary stratification yielding expressions for scale-dependent horizontal and vertical eddy viscosities and eddy diffusivities.

Partial scale elimination yields a subgrid-scale model for LES; complete scale elimination yields eddy viscosities and eddy diffusivities for RANS (Reynolds-average Navier-Stokes) models.

Case of weak stable stratification

Expansion in powers of spectral Froude number, $\mathcal{F}^{-1} \approx (k_O / k)^{2/3}$

$$\nu_h/\nu_n = 1 + 0.095\ \mathcal{F}^{-2}$$

$$\nu_z/\nu_n = 1 - 0.31\ \mathcal{F}^{-2}$$

$$\kappa_h/\nu_n = \alpha + 0.054\ \mathcal{F}^{-2}$$

$$\kappa_z/\nu_n = \alpha - 0.4\ \mathcal{F}^{-2}$$

$k_O = (N^3/\epsilon)^{1/2}$ is the Ozmidov wavenumber

$\alpha = Pr_t^{-1} = \nu_n/\kappa_n = 0.72$ is the inverse turbulent Prandtl number in neutral flows

ν_n and κ_n are eddy viscosity and eddy diffusivity in neutral flows

$$\nu_n = 0.46\ \epsilon^{1/3} k^{-4/3}$$

Scale- dependent horizontal and vertical eddy viscosities and diffusivities

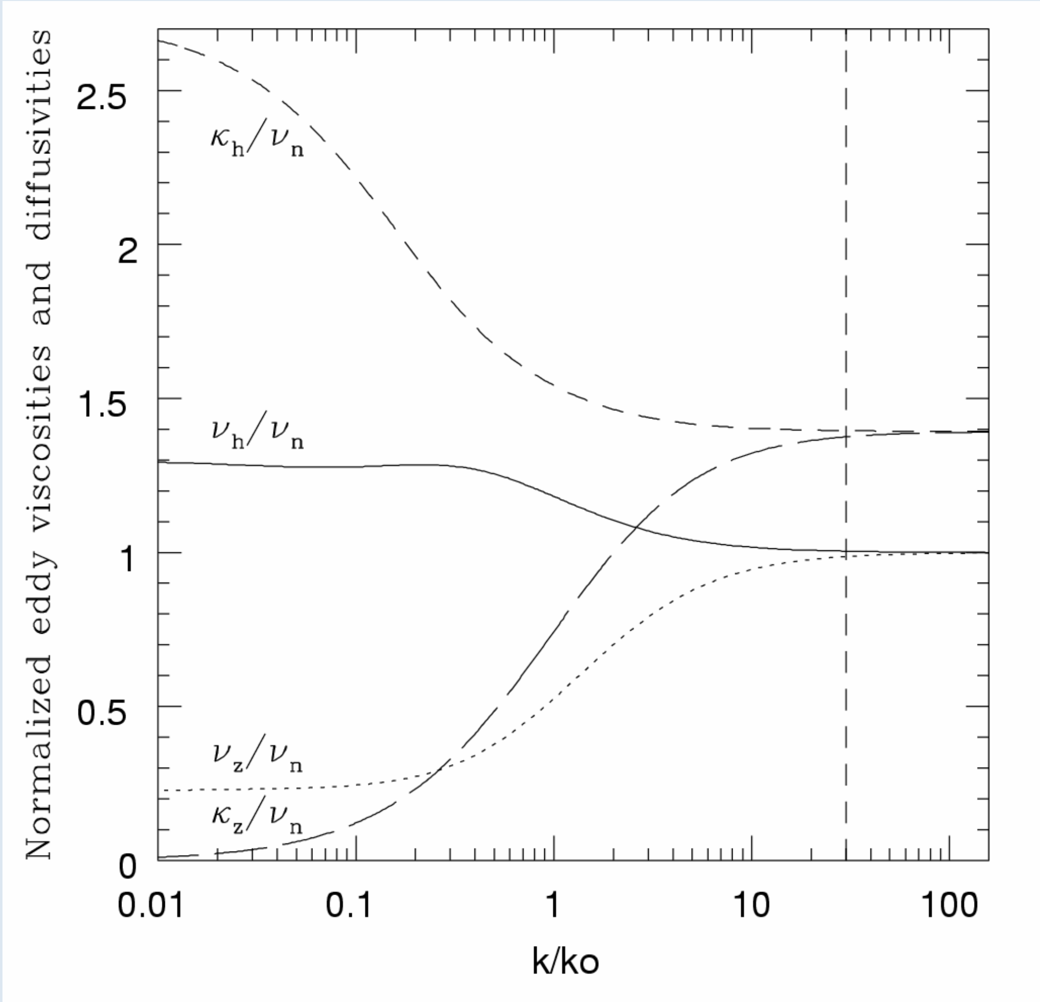


Figure. 1: Normalized horizontal and vertical eddy viscosities and diffusivities as functions of k / k_0. Dashed vertical line indicates the maximum wave number threshold of internal wave generation in the presence of turbulence; $k_O = (N^3/\varepsilon)^{1/2}$

Turbulence spectra

Due to anisotropy traditional 3D energy spectrum provides only limited information. Various 1D spectra are computed analytically for weak stratification:

$$E_1(k_3) = \frac{8}{(2\pi)^4}\int U_{11}(\omega,\mathrm{k})\,\mathrm{d}\omega\,\mathrm{d}k_1\,\mathrm{d}k_2 = 0.626\varepsilon^{2/3}k_3^{-5/3} + 0.214N^2k_3^{-3}$$

Spectrum ~ *k* $^{-3}$ is generated!

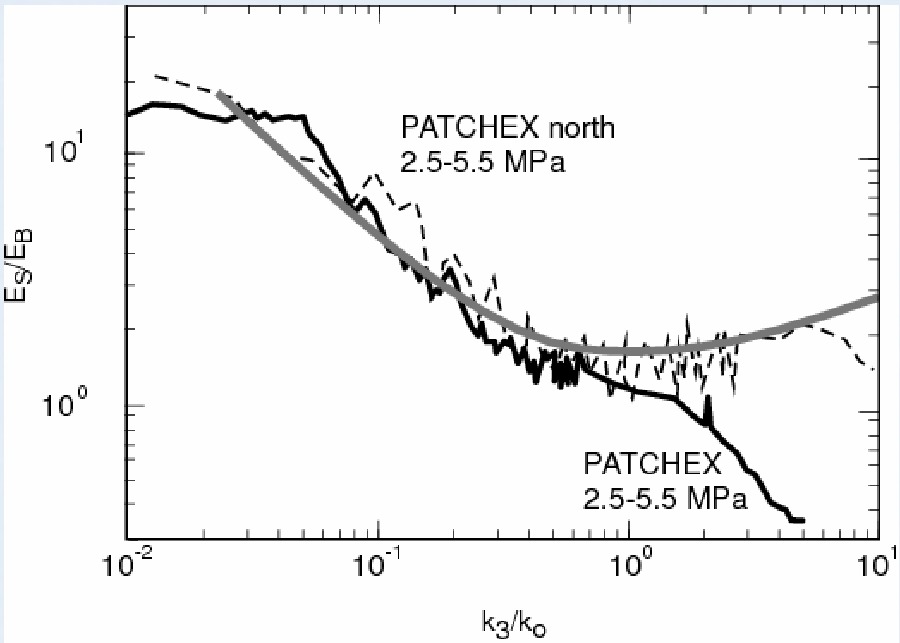
Transition from -5/3 to -3 spectrum at large scales; coefficients are in a good agreement with experimental data and LES (Carnevale et al, JFM, 2001).

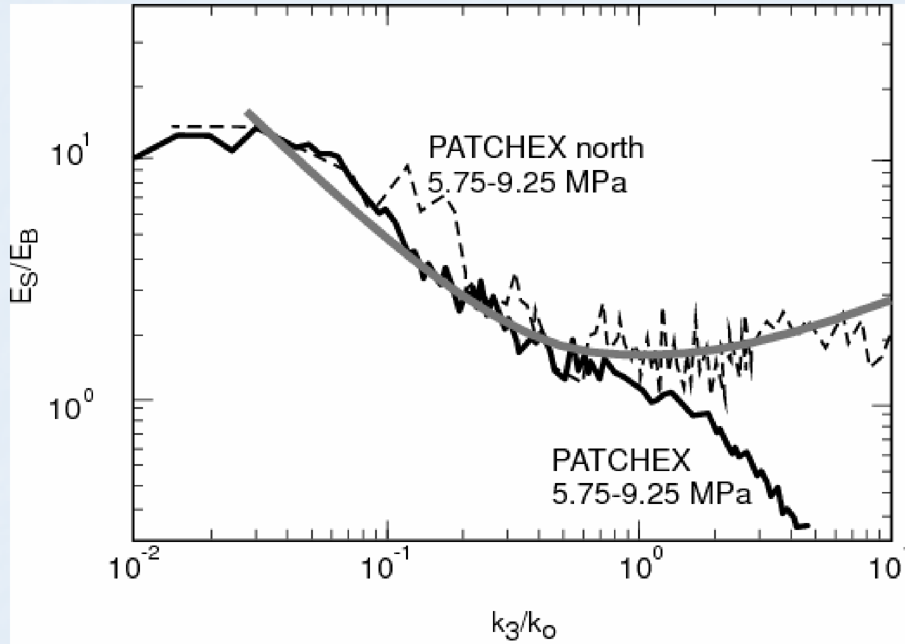
The Gargett et al.(1981) -normalized spectrum of the vertical shear:

$$\frac{E_S}{E_B} = \frac{2k_3^2 E_1(k_3)}{(\epsilon N)^{1/2}} = F(k_3/k_O) = 2 * 0.626(k_3/k_O)^{1/3}\left[1 + 0.34(k_3/k_O)^{-4/3}\right]$$

This scaling presents the normalized vertical shear spectrum as a universal function of (k/k_O). **The QNSE theory provides rigorous theoretical basis for this universal scaling**

Comparison with experimental data





The spectrum of the vertical shear of the horizontal velocity in the ocean; data from Gregg, Winkel, Sanford, JPO (1993). The theoretical prediction is shown by a gray line.
This is the first time that these spectra are derived within an analytical theory.

Another example - Atmospheric spectra

Our theory predicts the vertical spectrum which is in a good agreement with the spectra observed in the stratosphere, troposphere, mesosphere, and thermosphere (our prediction is well approximated by the dashed line).

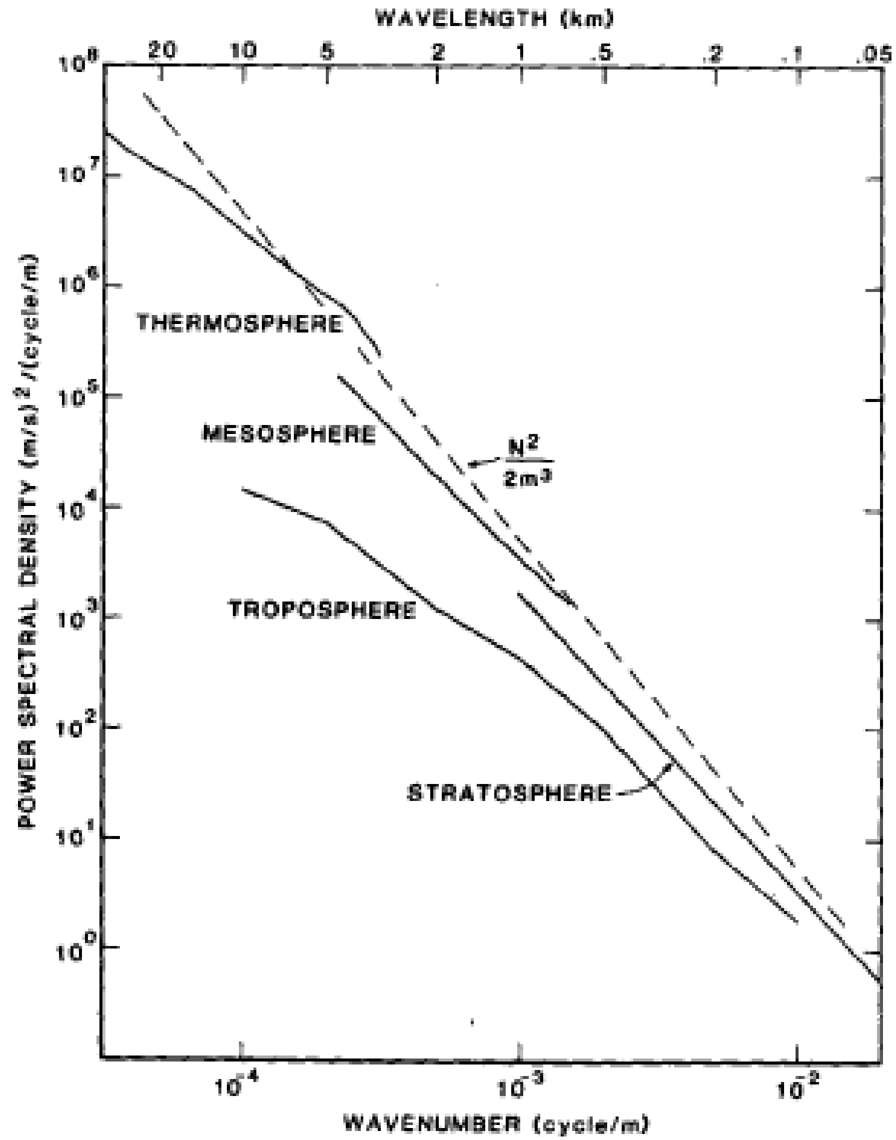


FIG. 1. Spectra of horizontal velocity versus vertical wavenumber as a function of altitude.

S. Smith et al., JAS, 1987

Other 1D spectra

$$E_3(k_1) = \frac{8}{(2\pi)^4}\int U_{33}(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}k_2\,\mathrm{d}k_3 = 0.626\,\varepsilon^{2/3}k^{-5/3} - 0.704\,N^2k_3^{-3}$$

$$E_3(k_3) = \frac{8}{(2\pi)^4}\int U_{33}(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}k_1\,\mathrm{d}k_2 = 0.47\varepsilon^{2/3}k^{-5/3} - 0.143N^2k_3^{-3}$$

Recall the vertical spectrum of horizontal velocity,

$$E_1(k_3) = \frac{8}{(2\pi)^4}\int U_{11}(\omega,\mathbf{k})d\omega dk_1 dk_2 = 0.626\epsilon^{2/3}k_3^{-5/3} + 0.214N^2k_3^{-3}$$

The anisotropization manifests itself as energy increase in the horizontal velocity components at the expense of their vertical counterpart

Temperature spectra

$$E_T(k_1) = C_\theta \varepsilon_\theta \varepsilon^{-1/3} k_1^{-5/3} \left[1 + C_1 \left(\frac{k_1}{k_O} \right)^{-4/3} \right] + C_2 \left(\frac{d\Theta}{dz} \right)^2 k_1^{-3}$$

$$E_T(k_3) = C_\theta \varepsilon_\theta \varepsilon^{-1/3} k_3^{-5/3} \left[1 + C_3 \left(\frac{k_3}{k_O} \right)^{-4/3} \right] + C_4 \left(\frac{d\Theta}{dz} \right)^2 k_3^{-3}$$

where $C_\theta = 0.62 - \text{Corrsin constant}, \quad C_1 = 0.068, \quad C_2 = 0.27,$

$C_3 = 0.41, \quad C_4 = 0.13$

Assuming that for relatively strong stratification $\varepsilon_\theta = 2\Gamma\varepsilon(d\Theta/dz)^2 / N^2$ where $\Gamma \sim 0.3$ is the *mixing efficiency*, one gets,

$$E_T(k_{1,3}) \approx C_\theta \varepsilon_\theta \varepsilon^{-1/3} k_{1,3}^{-5/3} + 0.3 \left(\frac{d\Theta}{dz} \right)^2 k_{1,3}^{-3}$$

RANS modeling

Invoking energy balance equation, the eddy coefficients are recast in terms of Richardson number $Ri = N^2/S^2$ or Froud number $Fr = \varepsilon / NK$

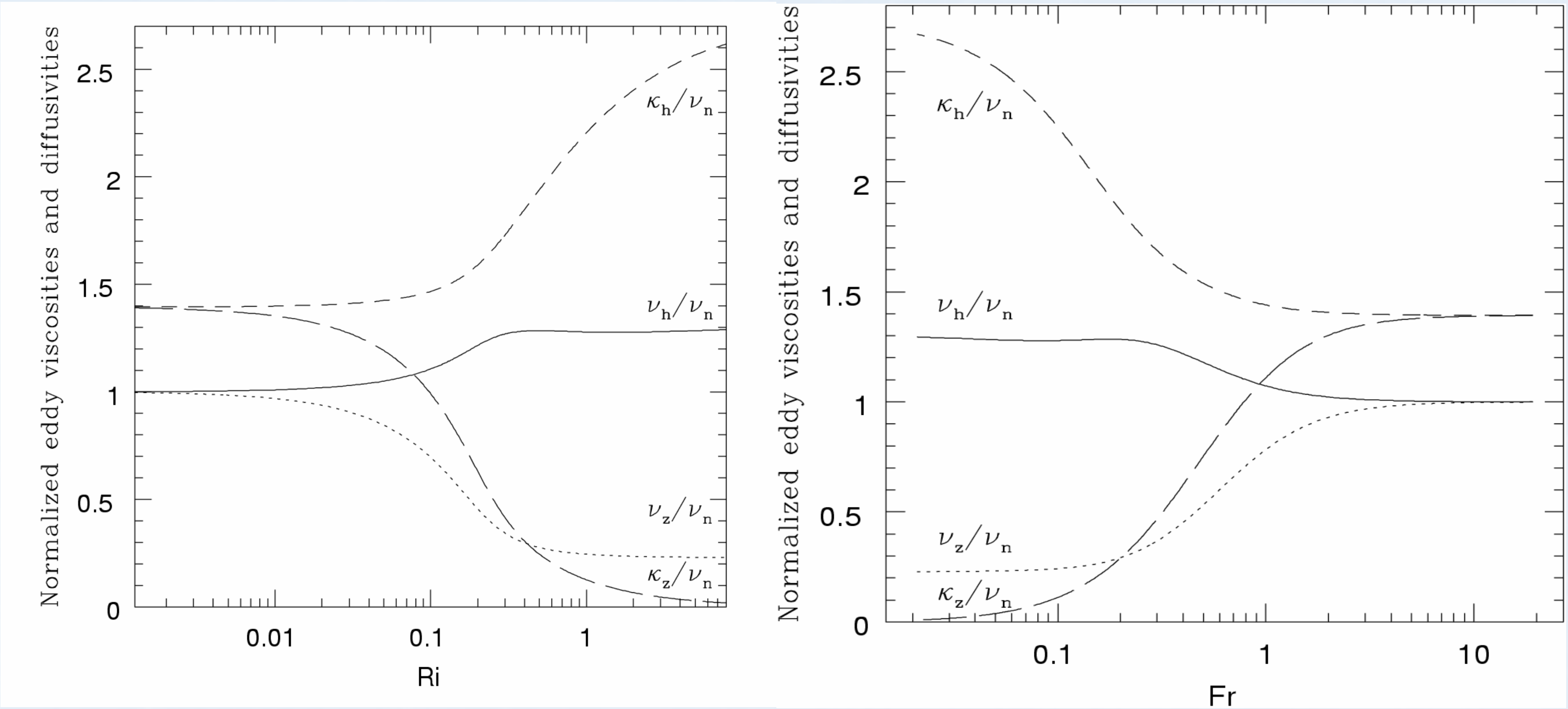
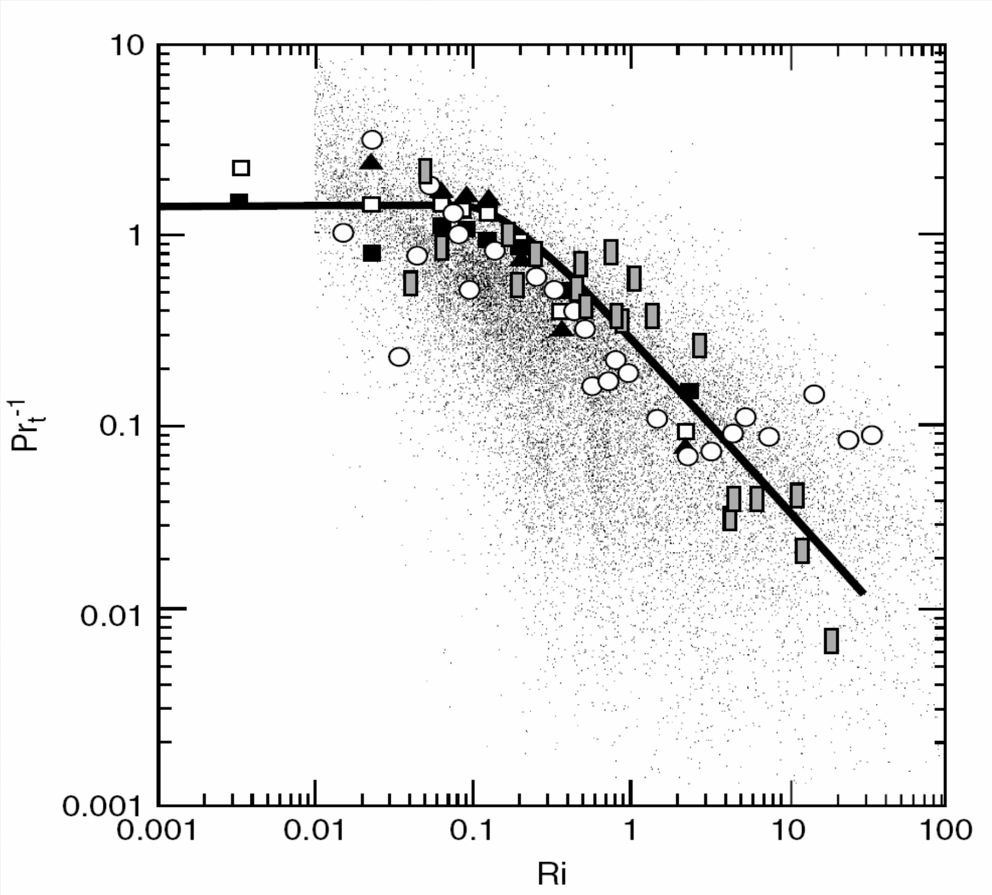


Figure 2: Normalized eddy viscosities and diffusivities as functions of *Ri* and *Fr.*

- For *Ri*>0.1, both vertical viscosity and diffusivity decrease, with the diffusivity decreasing faster than the viscosity (“residual” mixing due to effect of IGW?)
- Horizontal mixing increases with Ri. The model accounts for flow anisotropy.
- The crossover from neutral to stratified flow regime is replicated.

Comparison with data: Pr_t as a function of Ri



Inverse Prandtl number, κ_z/ν_z, as a function of *Ri*.

Solid line - theoretical prediction by QNSE theory Sukoriansky *et al*. (2006). Black and white squares and black tringles - data from Halley Base, Antarctica collected in 1986 (Yague *et al*., 2001). White circles show data by Monti *et al*. (2002) and grey rectangles are data by Strang and Fernando (2001). Small dots show data from Halley Base (British Antarctic Survey 2003–2004)

Dispersion relation for internal waves with turbulence

Complex poles yield the secular equation $\det\left[G^{-1}_{\alpha\beta}(\omega,\mathrm{k})\right]=0$

Waves exist if the solution of the secular equation has a real part. Identifying this real part with the wave frequency ω we obtain the dispersion relation

$$\omega^2 = N^2 \sin^2\theta \left\{ 1 - \left(\frac{k}{k_O}\right)^{4/3} \left[\frac{\left(\frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n}\right)\cos^2\theta + \left(\frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n}\right)\sin^2\theta}{4\sin\theta} \right]^2 \right\}$$

The limit of strong stratification => classical dispersion relation for linear waves, **$\omega = N \sin\theta$.** Turbulence dominates at small scales. Criterion for wave generation is $\omega^2 \geq 0$ giving

$$k_t = k_O \left| \frac{4\sin\theta}{\left(\frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n}\right)\cos^2\theta + \left(\frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n}\right)\sin^2\theta} \right|^{3/2} \simeq 32 k_O |\sin\theta|^{3/2}$$

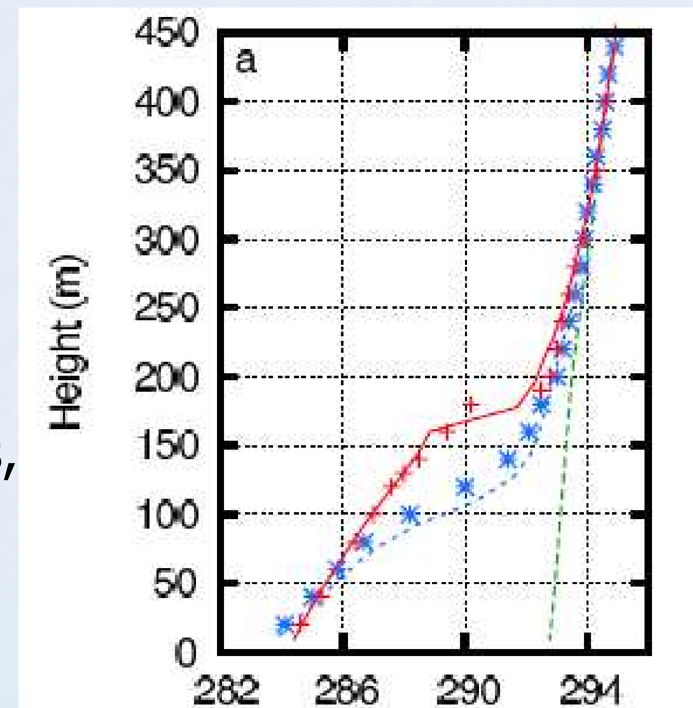
Validation of the QNSE theory in modeling of atmospheric boundary layers and numerical weather prediction

- Validation was conducted for models in both K-ε and K-ℓ format
- Data from numerous observation campaigns was employed

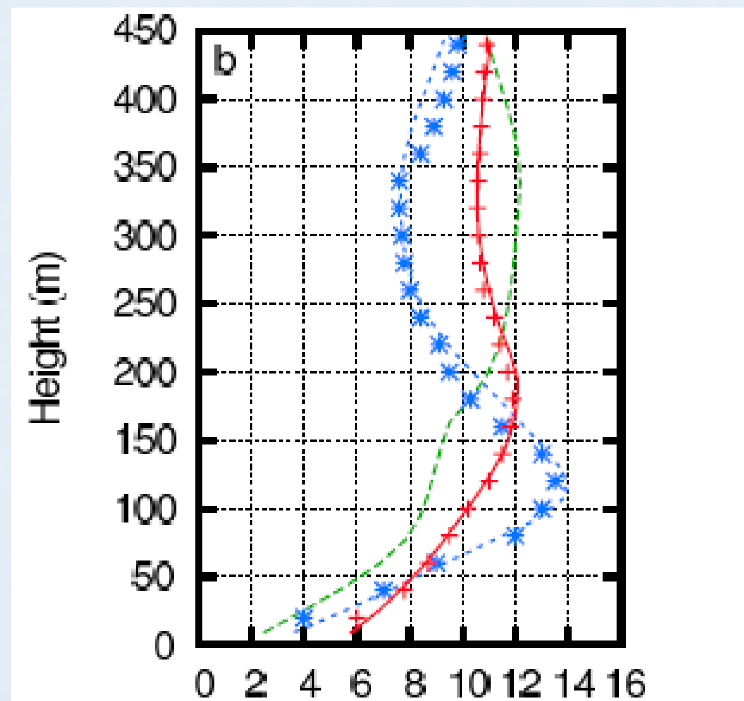
CASES99

Data-asterisks,

model - lines

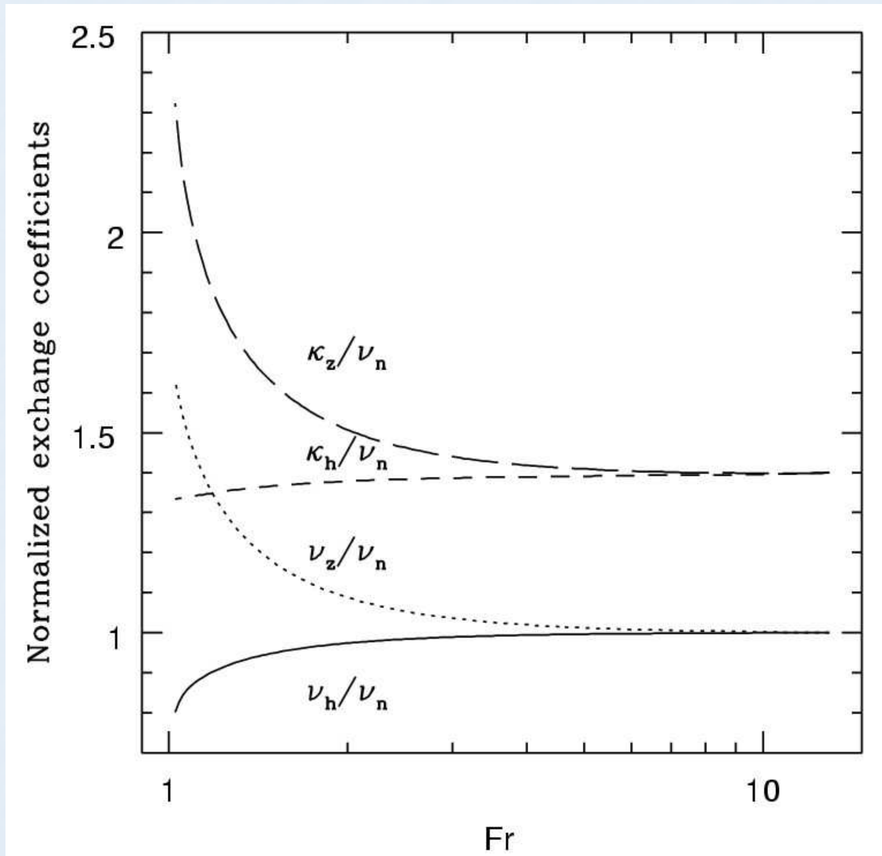


Potential temperature



Wind speed

Unstable stratification (Convection)



Conclusions

- Derivation of the QNSE model of turbulence is maximally proximate to first principles
- Theory explicitly resolves horizontal-vertical anisotropy
- Accounts for the combined effect of turbulence and waves
- Predicts correct behavior of Pr_t as a function of Ri
- Anticipates the absence of the critical Ri
- Yields modification of the classical dispersion relation for internal waves that accounts for turbulence
- Yields analytical expressions for various 1D and 3D spectra; captures transition from the -5/3 to the $N^2 k_z^{-3}$ vertical spectrum of the horizontal velocity and recovers Gargett et al. scaling
- Provides subgridscale closures for both LES and RANS
- The QNSE theory has been implemented in K-ε and K-ℓ models of stratified ABL
- Good agreement with CASES-99 and other data sets has been found for cases selected for the GABLES1 and GABLS 2 experiment
- The new stability functions improve predictive skills of HIRLAM in +24h and +48h weather forecasts
- Is being incorporated in WRF (Weather Research and Forecasting) – a new model developed at NCAR.

Acknowledgments

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