# On the Energetics of Turbulent Mixing in Stratified Fluids

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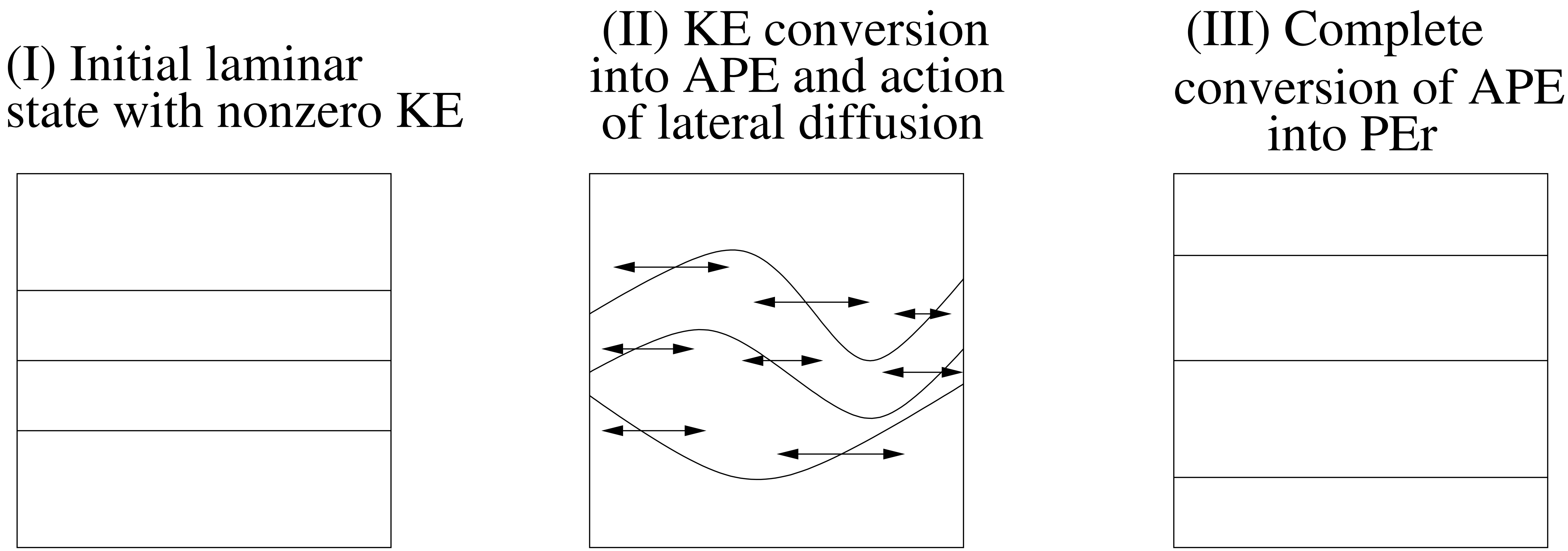
# Goal of the Talk

- To question the classical phenomenology of the diffusive route for kinetic energy dissipation in stratified turbulent flows
- To question the validity and accuracy of two commonly made approximations in the study of stratified turbulence at low Mach numbers:
  - Incompressible assumption (so far proven to be valid for adiabatic reversible motions but never for diabatic irreversible ones, see Ansumali et al, 2005)
  - Linear of the equation of state

# Routes to dissipation in freely decaying stratified turbulence

- Viscous Route: Conversion of Kinetic Energy into Internal Energy IEo
- Diffusive route: Conversion of Kinetic Energy into?
    - mean Gravitational Potential Energy (classical view)
    - Internal Energy as for viscous dissipation (new view)

# Idealized Mechanism for the diffusive dissipation of Kinetic Energy




**Step II:** $\Delta APE = -\Delta KE$ $\Delta PE_r = 0$

**Step III:** $\Delta APE = \Delta KE$ $\Delta PE_r = -\Delta KE$

PE=GPE+IE APE=AGPE+AIE PEr=GPEr+IEr

# Random Shuffling Experiment

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

Initial Stratification

| | | |
|---|---|---|
| 3 | 5 | 9 |
| 6 | 1 | 8 |
| 4 | 2 | 7 |

Random Adiabatic Shuffling

| | | |
|---|---|---|
| 3 | 5 | 9 |
| 6 | 1 | 8 |
| 4 | 2 | 7 |

Lateral Homogenizing

Step II:

$$\Delta APE \approx \Delta AGPE = -\Delta KE$$

$$\Delta AIE \ll \Delta AGPE$$

Step III:

$$\Delta GPE_r \approx -\Delta(IE_r - IE_0) \neq -\Delta KE$$

$$\Delta IE_0 = -\Delta KE$$

# Results

## Adiabatic shuffling

$$\Delta APE = \Delta(AIE + AGPE) = -\Delta KE$$

$$\Delta AIE \ll \Delta AGPE$$

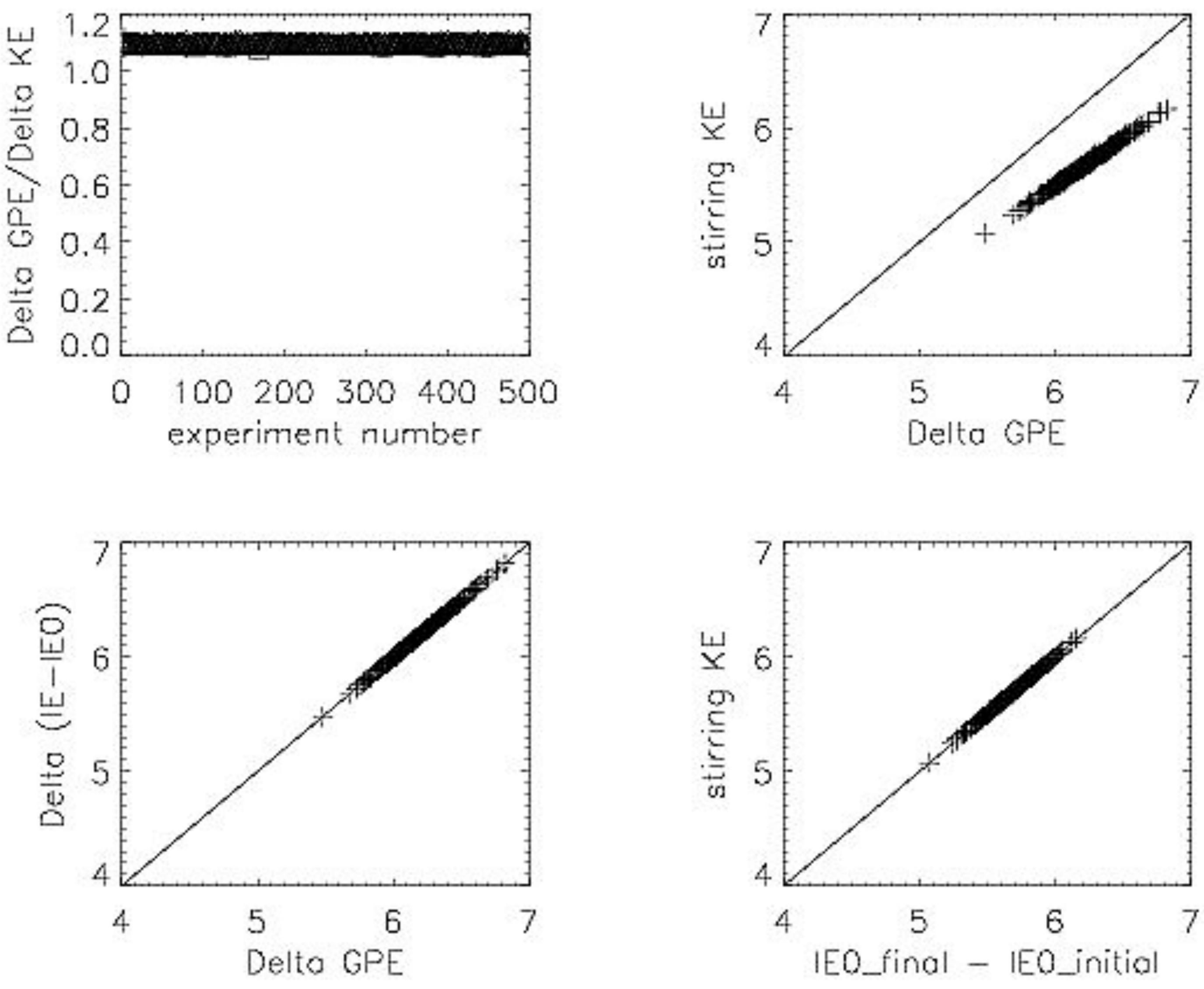
## Lateral Mixing Step

$$\Delta(GPE_r + IE_r) = \Delta KE$$

$$\Delta GPE_r = -\Delta(IE_r - IE_0) \neq \Delta KE$$

$$\Delta IE_0 = \Delta KE$$

# Energetics of the Random Shuffling experiment



# Energetics of Turbulent Mixing: Definitions

Conversion KE/GPE = "Density Flux"

$$W = \int_V \rho g w \, dV$$

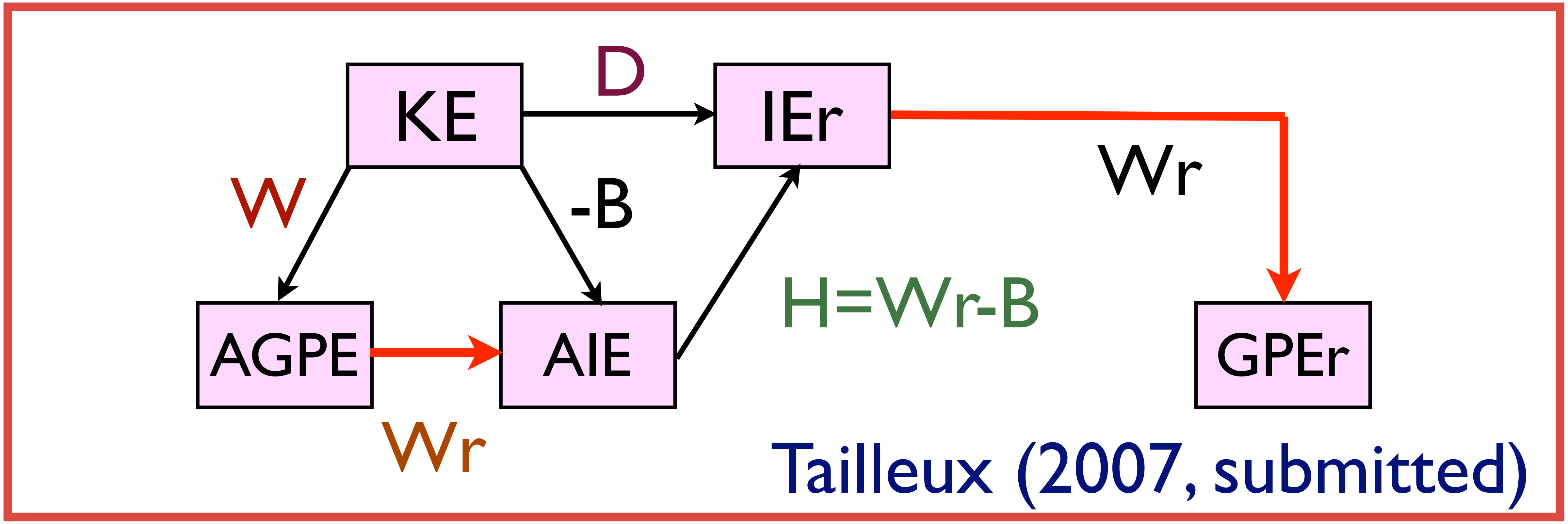
Conversion IE/KE = Work of Expansion/Contraction

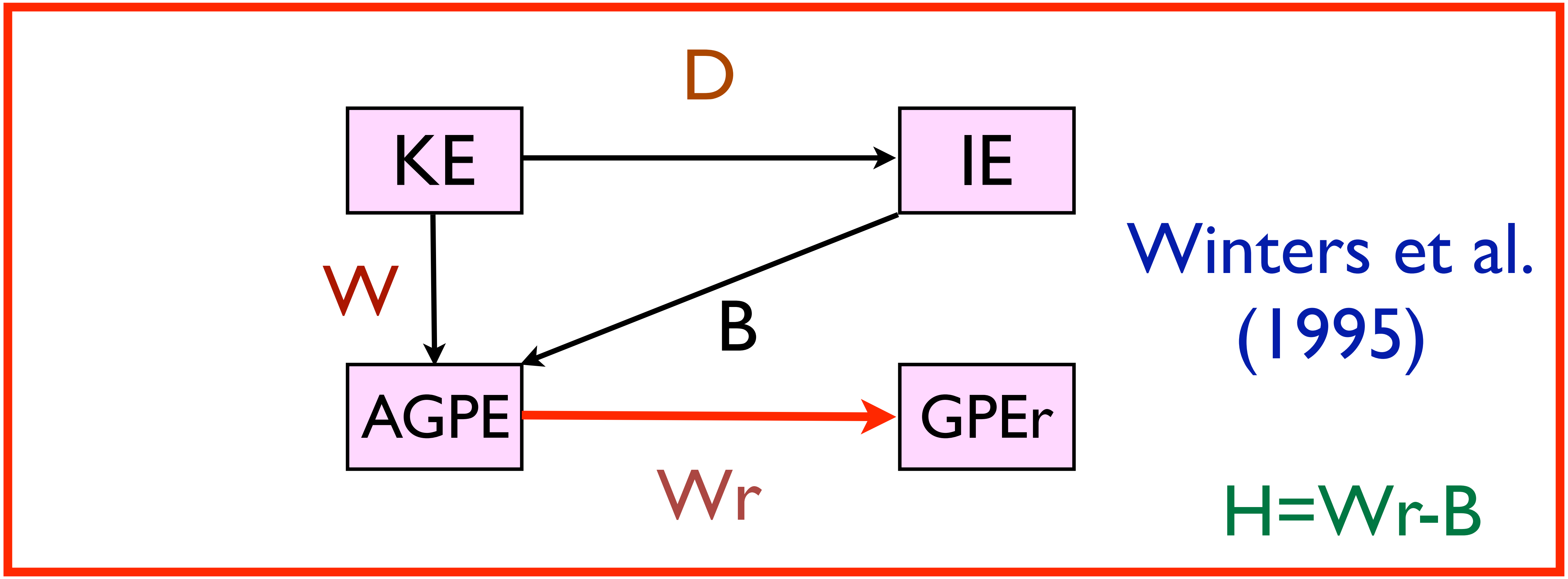
$$B = \int_V P \frac{Dv}{Dt} (\rho dV)$$

Irreversible Conversion of KE/IE by viscous dissipation

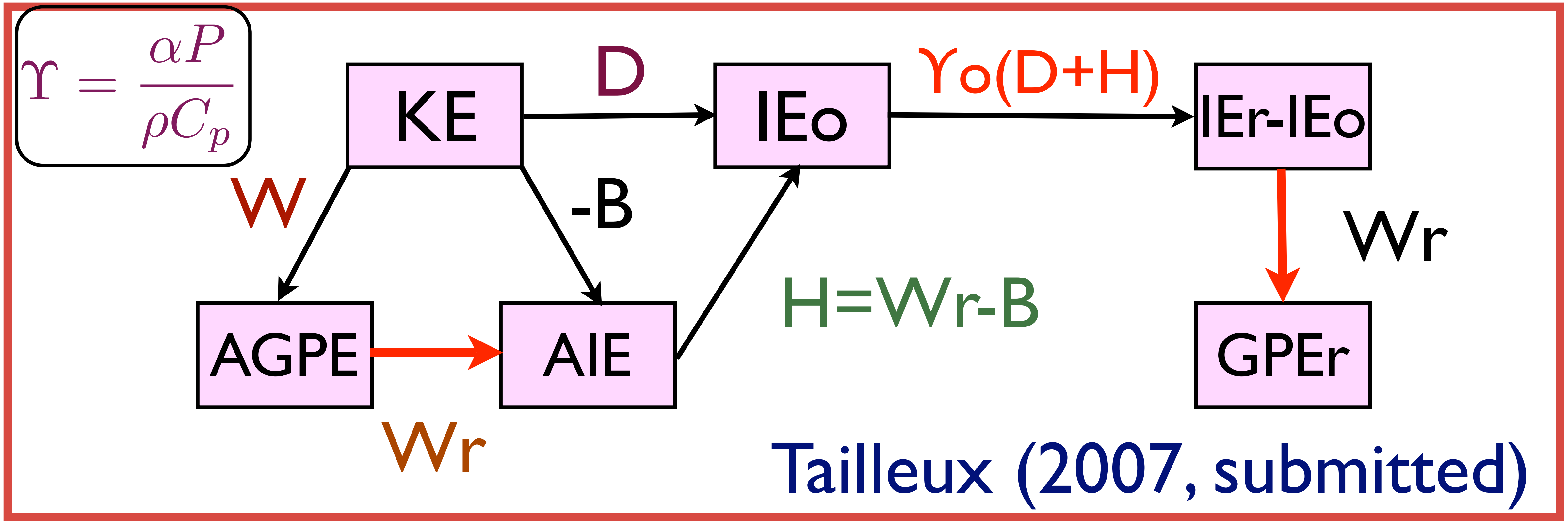
$$D = \int_V \rho \varepsilon \, dV$$

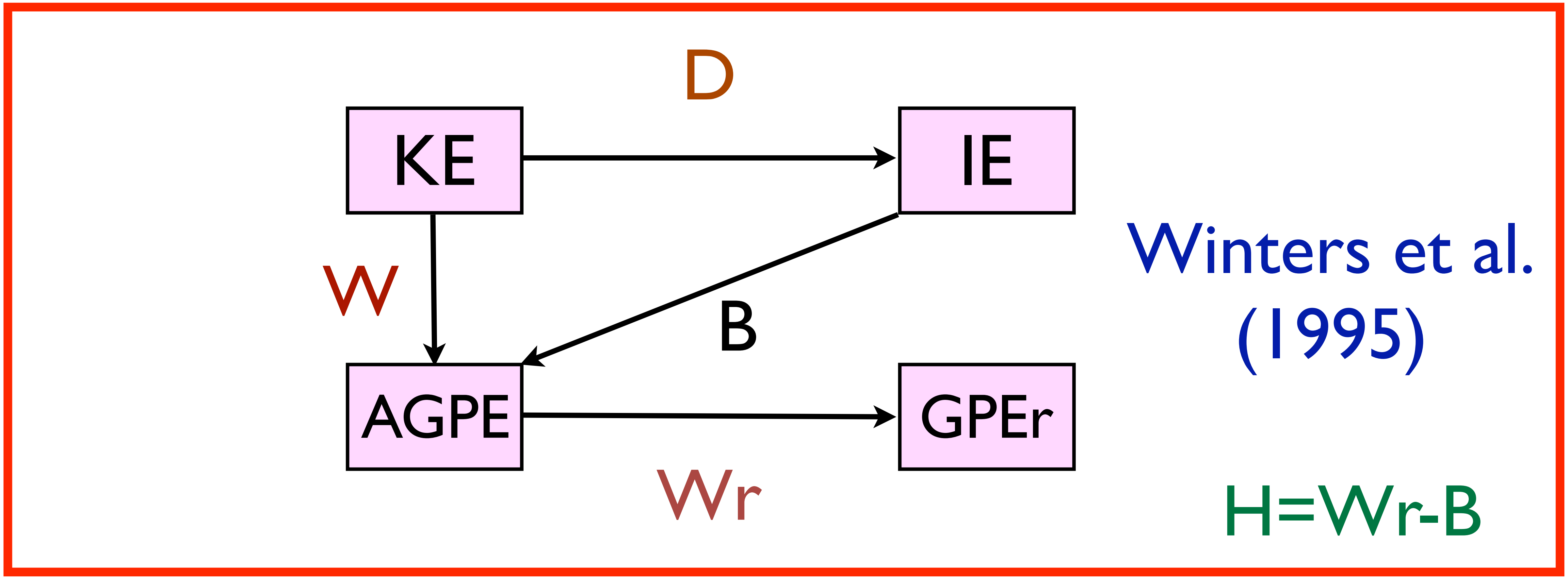
# Exact versus Boussinesq Energetics



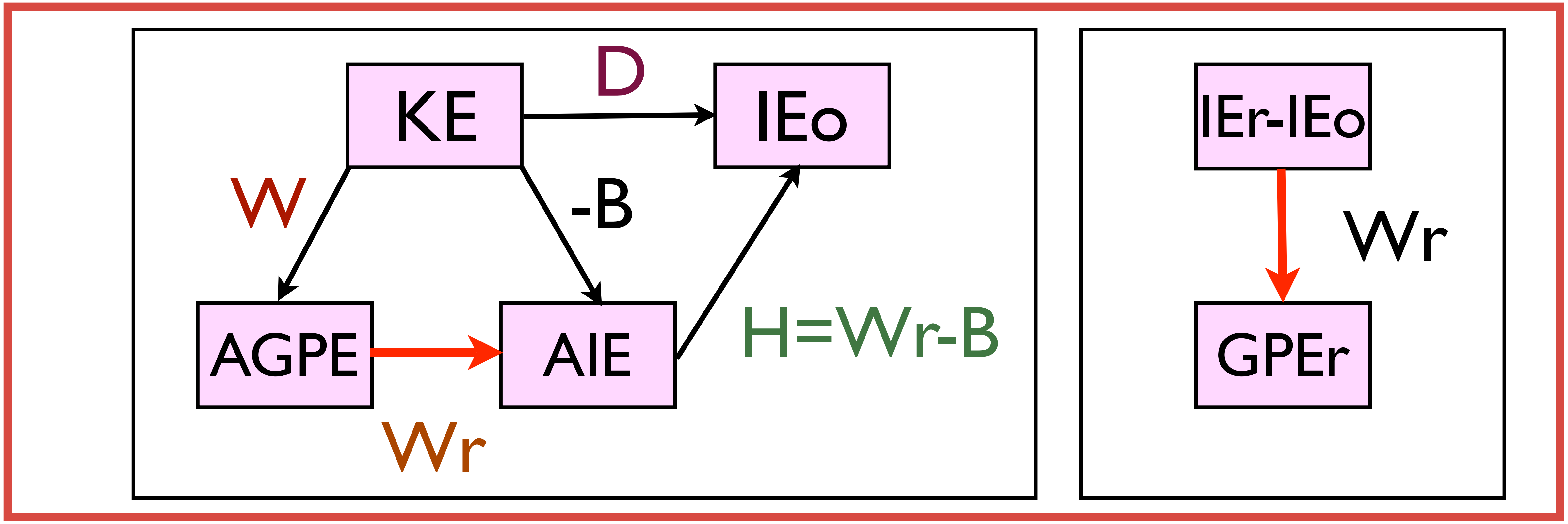



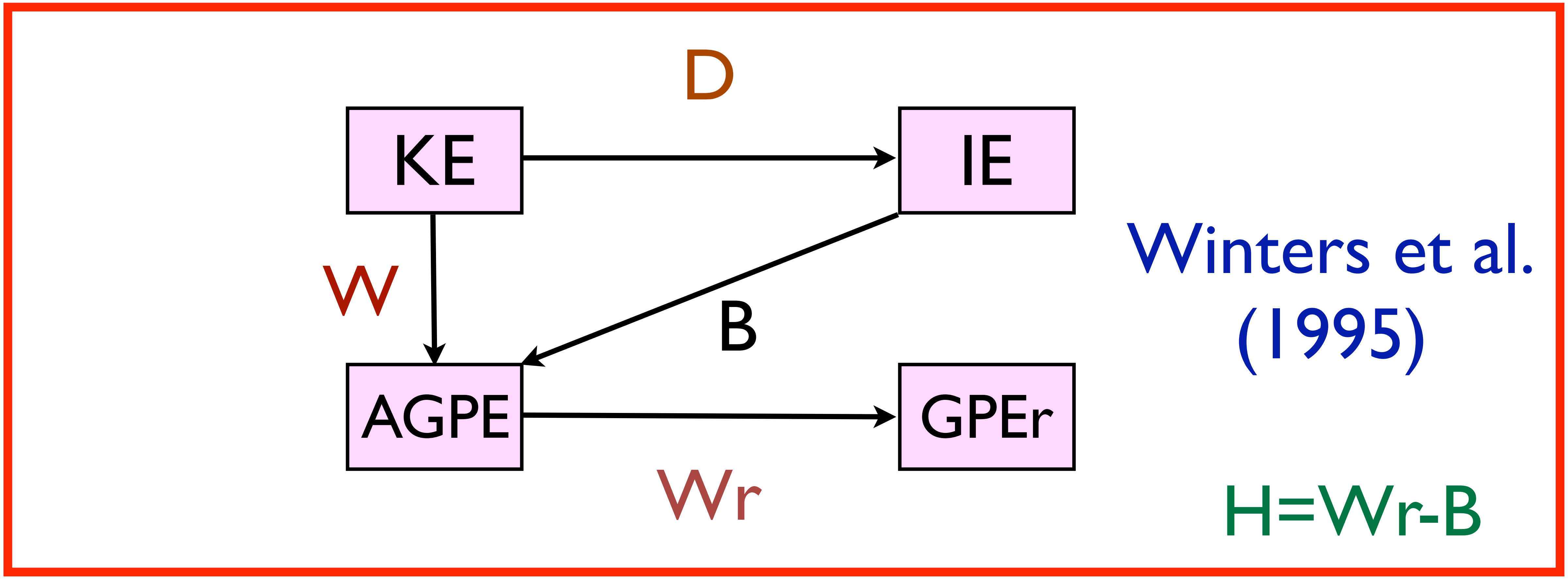
# Exact versus Boussinesq Energetics



# Exact versus Boussinesq Energetics

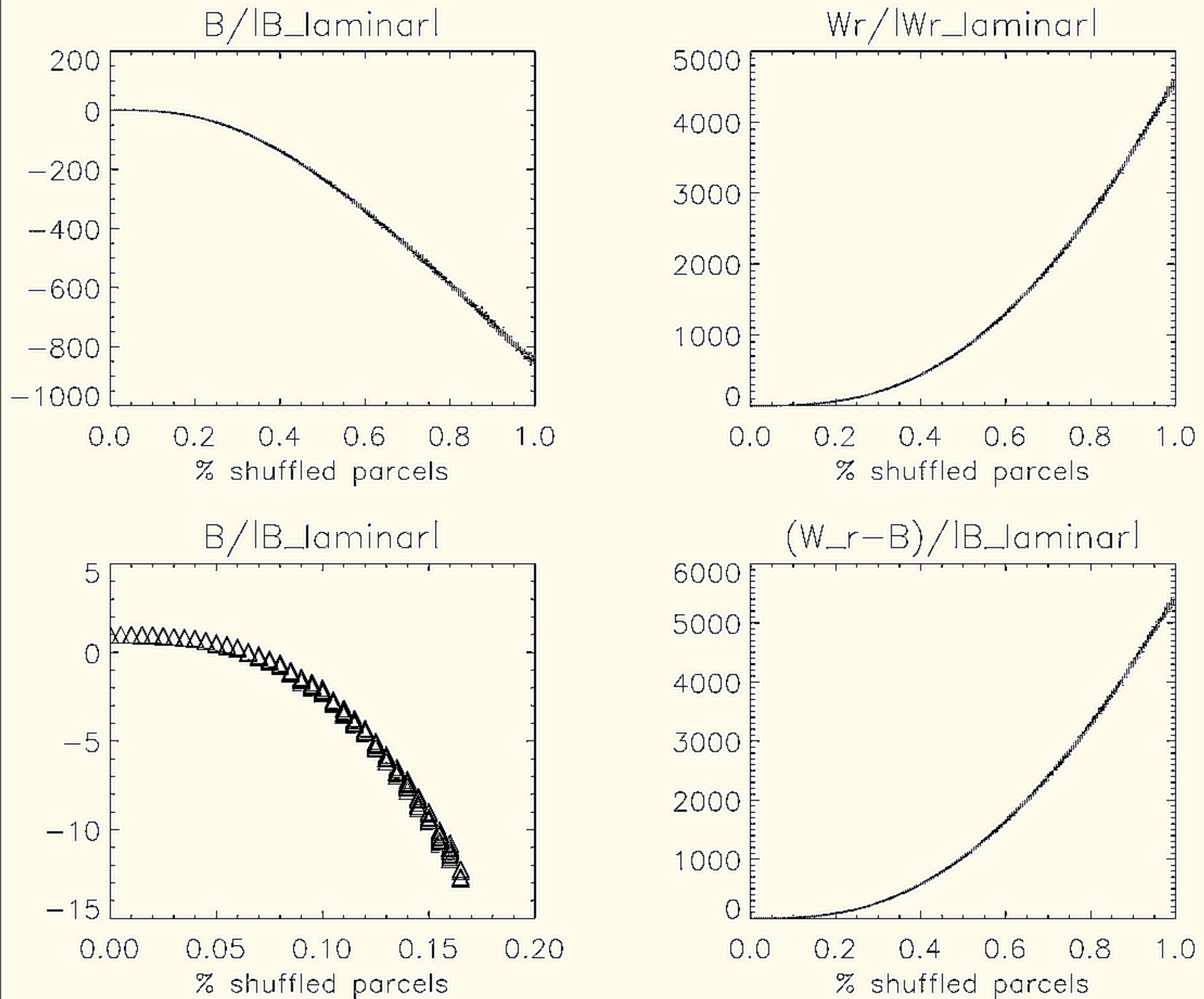


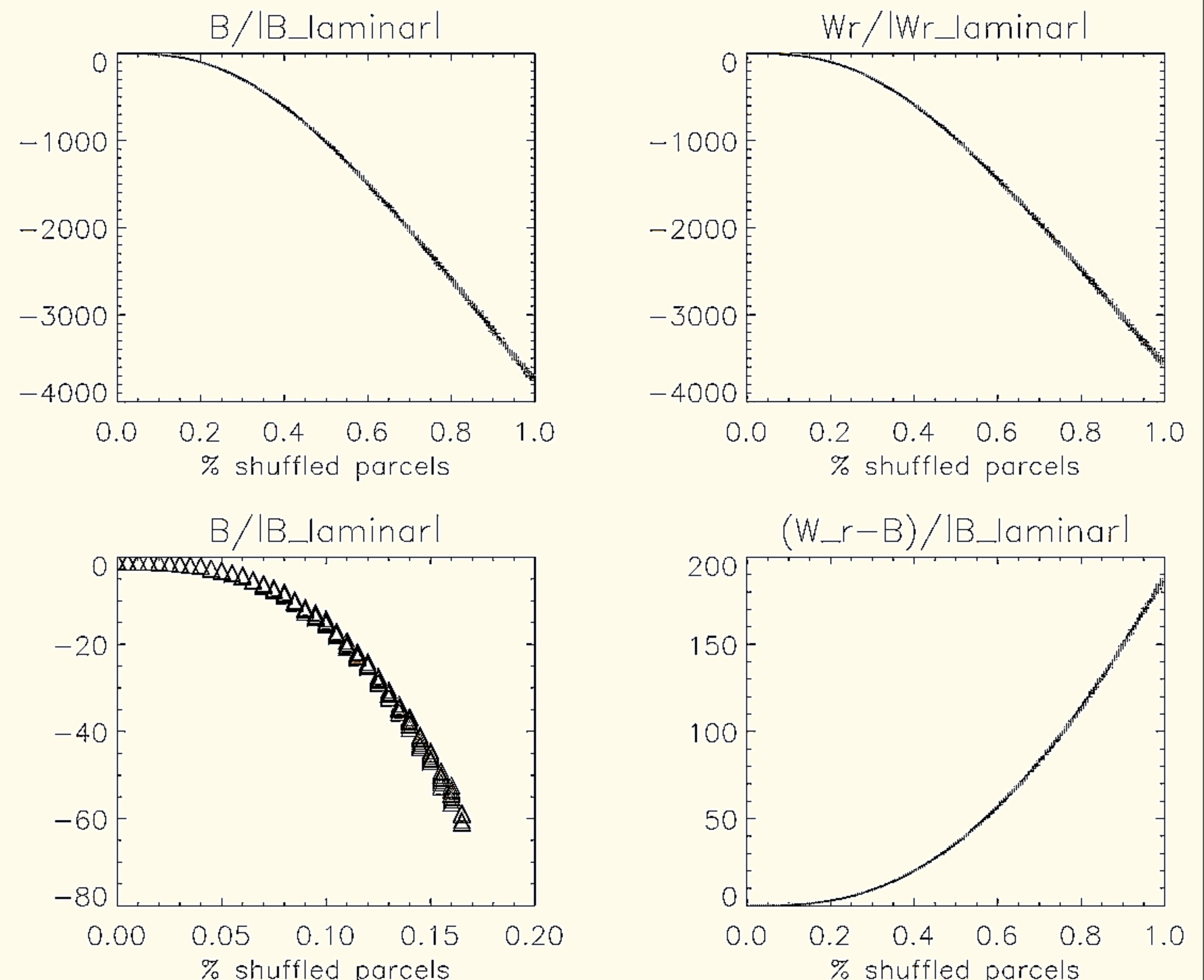
# Results

$$\Upsilon = \frac{\alpha P}{\rho C_p}$$

| Boussinesq | Non Boussinesq |
| --- | --- |
| • B always > 0 and very small (proportional to molecular diffusion) | • B always < 0 as soon as fluid turbulent enough, and proportional to turbulent diffusion |
| • Wr always > 0 and proportional to turbulent diffusion | • Wr > 0 if $d\Upsilon/dz < 0$, but Wr <0 if $d\Upsilon/dz > 0$. Also proportional to turbulent diffusion |
| • Wr-B > 0 and proportional to turbulent diffusion | • Wr-B > 0 and proportional to turbulent diffusion |




# High Pressure Case 1000-1010 dbar




# Low Pressure Case 0-10 dbar

# Where is the Error?

We may have

$$\int_V X dV = \int_V \left[X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots\right] dV$$

such that: $|\varepsilon X_1| \ll X_0, \qquad |\varepsilon^2 X_2| \ll X_1, \cdots$

**It is WRONG, however, to conclude that:**

$$\left|\int_V X_0 dV\right| \gg \left|\int_V \varepsilon X_1 dV\right|, \qquad \left|\int_V \varepsilon X_1 dV\right| \gg \left|\int_V \varepsilon^2 X_2\right|$$

$$B_{bouss} = \int_V P\alpha\kappa\nabla^2 T dV \approx \int_V \left[-\rho_0 g z \alpha_0 \kappa \nabla^2 T + \cdots\right] dV$$

$$\approx \kappa g \left[\langle\rho\rangle_{bottom} - \langle\rho\rangle_{top}\right] + \cdots$$

$$B_{exact} = \int_V P\alpha\kappa\nabla^2 T dV = -\int_V \kappa\nabla T \cdot \nabla(\alpha P) dV$$

$$= -\int_V \kappa\alpha\nabla T \cdot \nabla P \, dV - \int_V \kappa P \frac{\partial\alpha}{\partial T} \|\nabla T\|^2 \, dV$$

$$\approx \kappa g \left[\langle\rho\rangle_{bottom} - \langle\rho\rangle_{top}\right] - \int_V \kappa P \frac{\partial\alpha}{\partial T} \|\nabla T\|^2 dV + \cdots$$

# The coefficient Wr

$$W_r^{bouss} = \int_V \kappa \|\nabla z_r\|^2 \rho_r \alpha_r g \frac{\partial T_r}{\partial z_r} dV_r$$

$$W_r^{exact} \approx \int_V \kappa \|\nabla z_r\|^2 \rho_r \alpha_r g \frac{\partial T_r}{\partial z_r} dV_r - \int_V \kappa \|\nabla z_r\|^2 \rho_r C_{pr} P_r \frac{\partial}{\partial T_r} \left( \frac{\alpha_r}{\rho_r C_{pr}} \right) \left( \frac{\partial T_r}{\partial z_r} \right)^2 dV_r$$

# Conclusions

- Compressibility effects and nonlinearity of the equation of state are leading order effects in stratified turbulence: Coupled dynamics and thermodynamics!
- Boussinesq approximation removes KE diffusively as in reality, but phenomenology is very different
  - very small B>0 versus very large B<0
  - Put dissipated KE into GPE instead of IEo
  - d(GPE)/dt > 0 whereas d(GPE)/dt >0 or <0 in reality

# Some Implications

- Questions foundations for Munk and Wunsch (1998), Wunsch and Ferrari (2004), Huang (2004) whose validity critically relies on a fraction of mechanical energy input being converted into GPE
- Questions foundations for the anti-turbulence theorem of Paparella and Young (2002) whose validity critically relies on B going to zero as molecular diffusivity goes to zero
- Requires re-thinking and re-defining the concept of mixing efficiency

# Want to know more?

Tailleux (2007): On the energetics of stratified turbulent mixing, irreversible thermodynamics, Boussinesq models, and the Ocean Heat Engine Controversy,

submitted to Journal of Fluid Mechanics, presently being extensively revised (NB: available version contains errors)

http://www.met.reading.ac.uk/~remi/publications
email: R.G.J.Tailleux@reading.ac.uk

# Mixing efficiency

Classical Obsolete? Definitions

$$\Gamma = \frac{\text{Net GPE increase}}{\text{Viscous KE dissipation Rate}}$$

$$= \frac{\text{Fraction of KE irreversibly going into GPE}}{\text{Viscous KE dissipation Rate}}$$

Proposed New Definition

$$\Gamma = \frac{\text{Diffusive KE dissipation Rate}}{\text{Viscous KE dissipation Rate}}$$

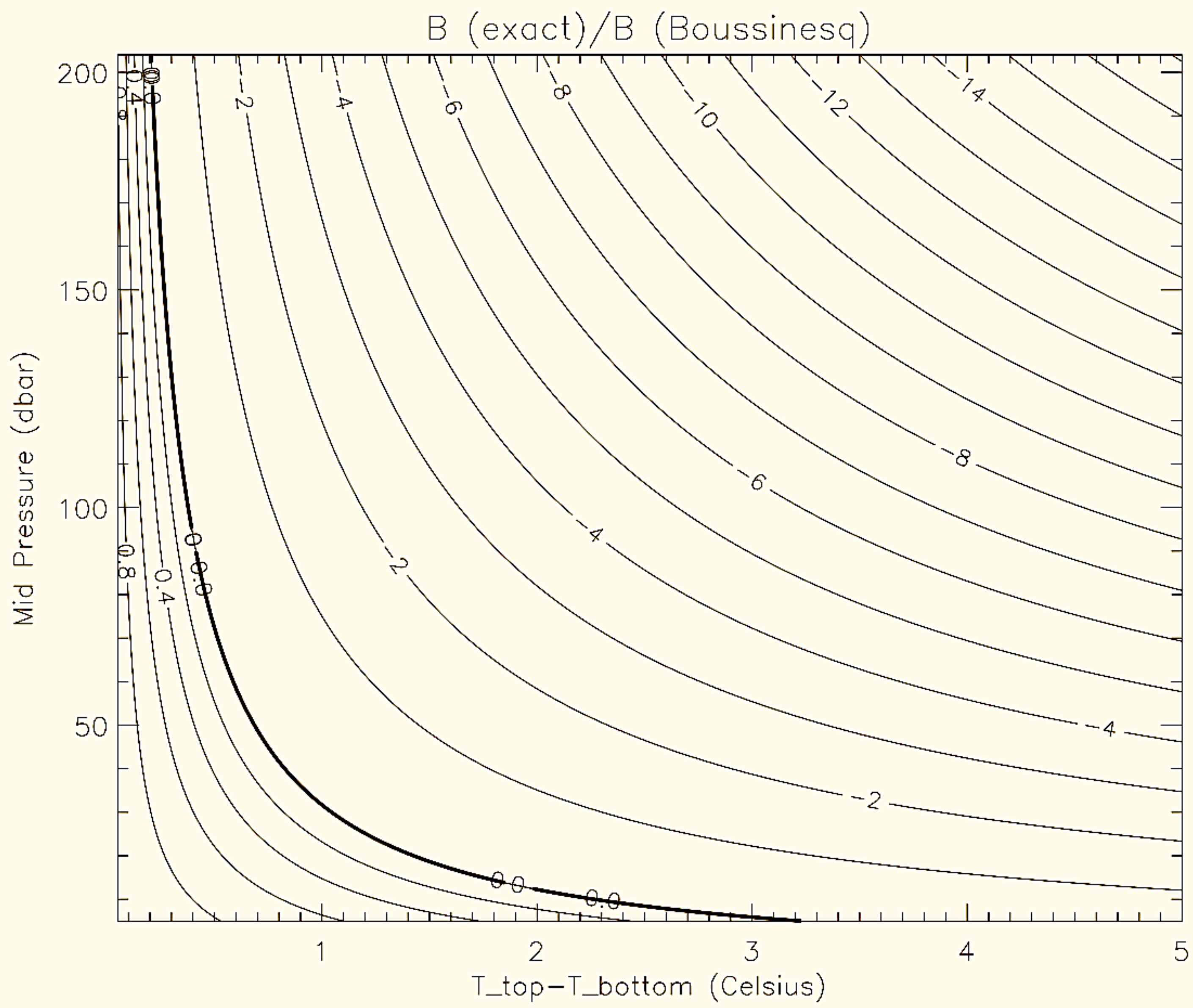
# The work of expansion/contraction B

$$B = \int_V P \frac{Dv}{Dt}\, dm = \int_V \frac{\alpha P}{\rho C_p} \dot{Q}\, dm - \int_V \frac{1}{\rho c_s^2} \frac{DP}{Dt} dV$$

$$B_{bouss} = \int_V \kappa \alpha_0 g \rho_0 \frac{dT}{dz} = \kappa \rho_0 \alpha_0 g \left( \langle T \rangle_{top} - \langle T \rangle_{bottom} \right)$$

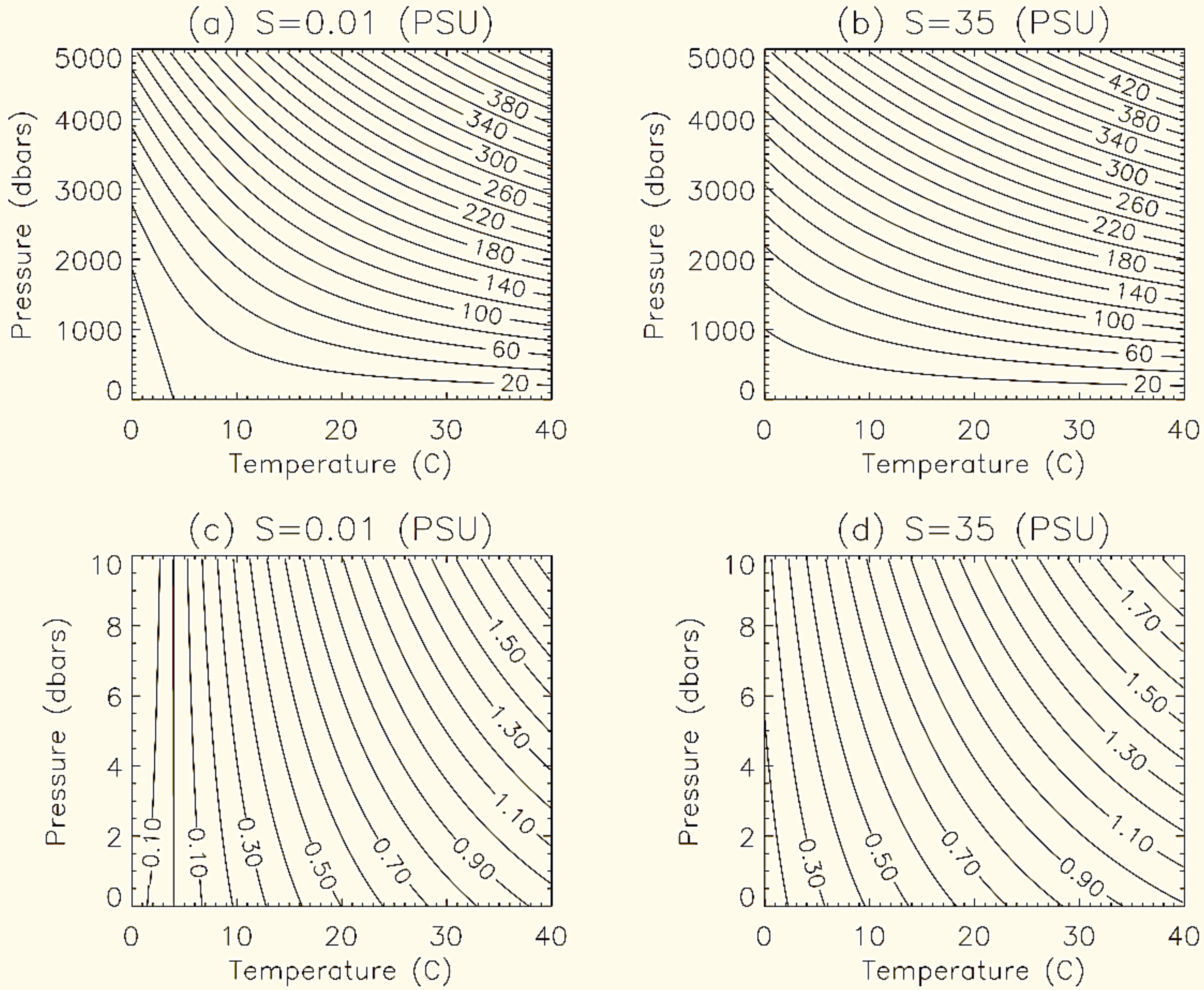
$$B_{exact} \approx \int_V \kappa \alpha \rho g \left( 1 + \frac{\|\nabla_h P\|^2}{2\rho^2 g^2} \right) \frac{\partial T}{\partial z} dV + \int_S \frac{\kappa}{\rho_b g} \frac{\|\nabla_h P_b\|^2}{2} dS$$

$$- \int_V \kappa \rho C_p \frac{\partial}{\partial T} \left( \frac{\alpha P}{\rho C_p} \right) \|\nabla T\|^2 \, dV$$

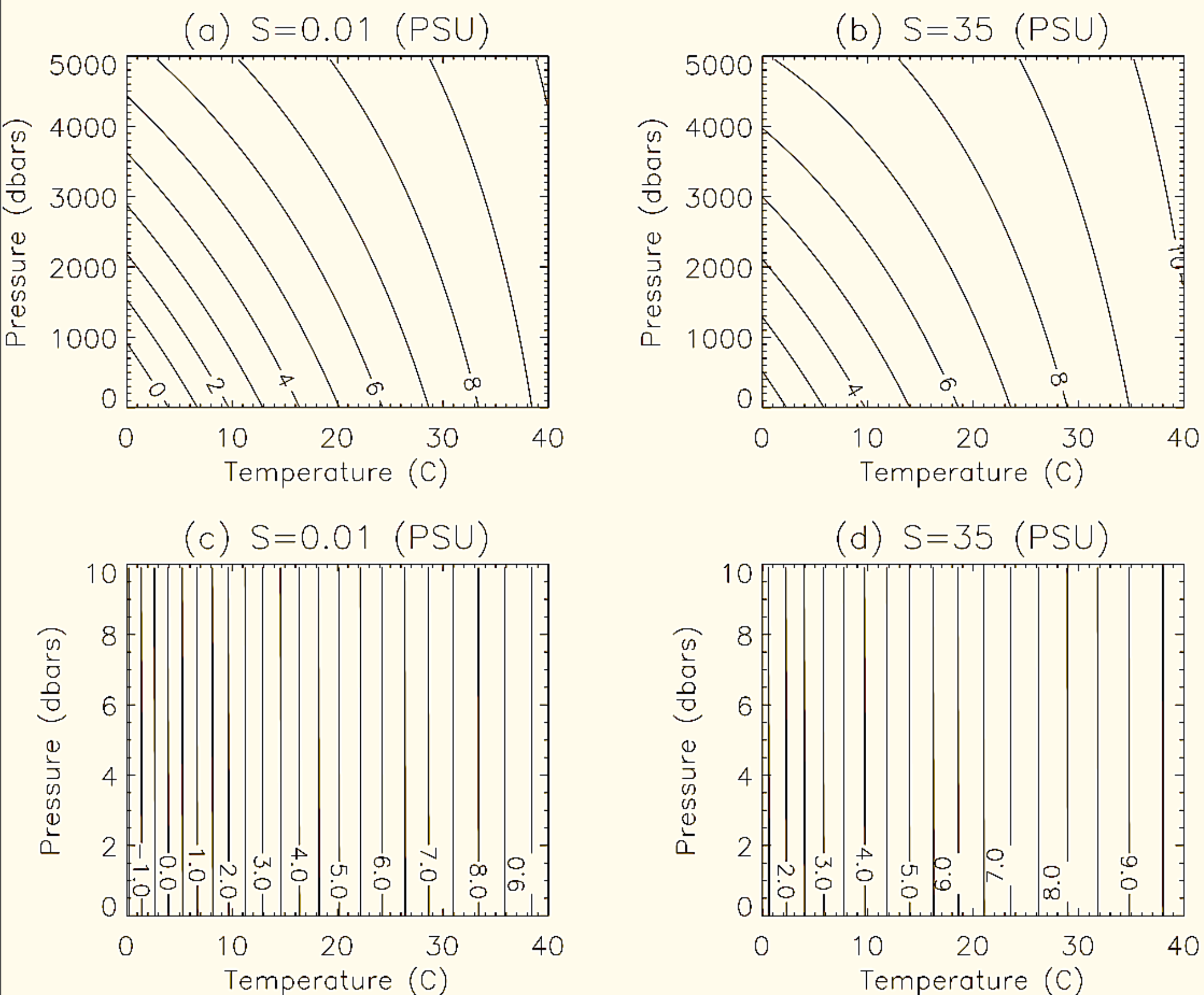


B (exact)/B (Boussinesq)
Mid Pressure (dbar)
T_top-T_bottom (Celsius)
200
150
100
50
1
2
3
4
5
0.0
0.4
0.8
2
4
6
8
10
12
14

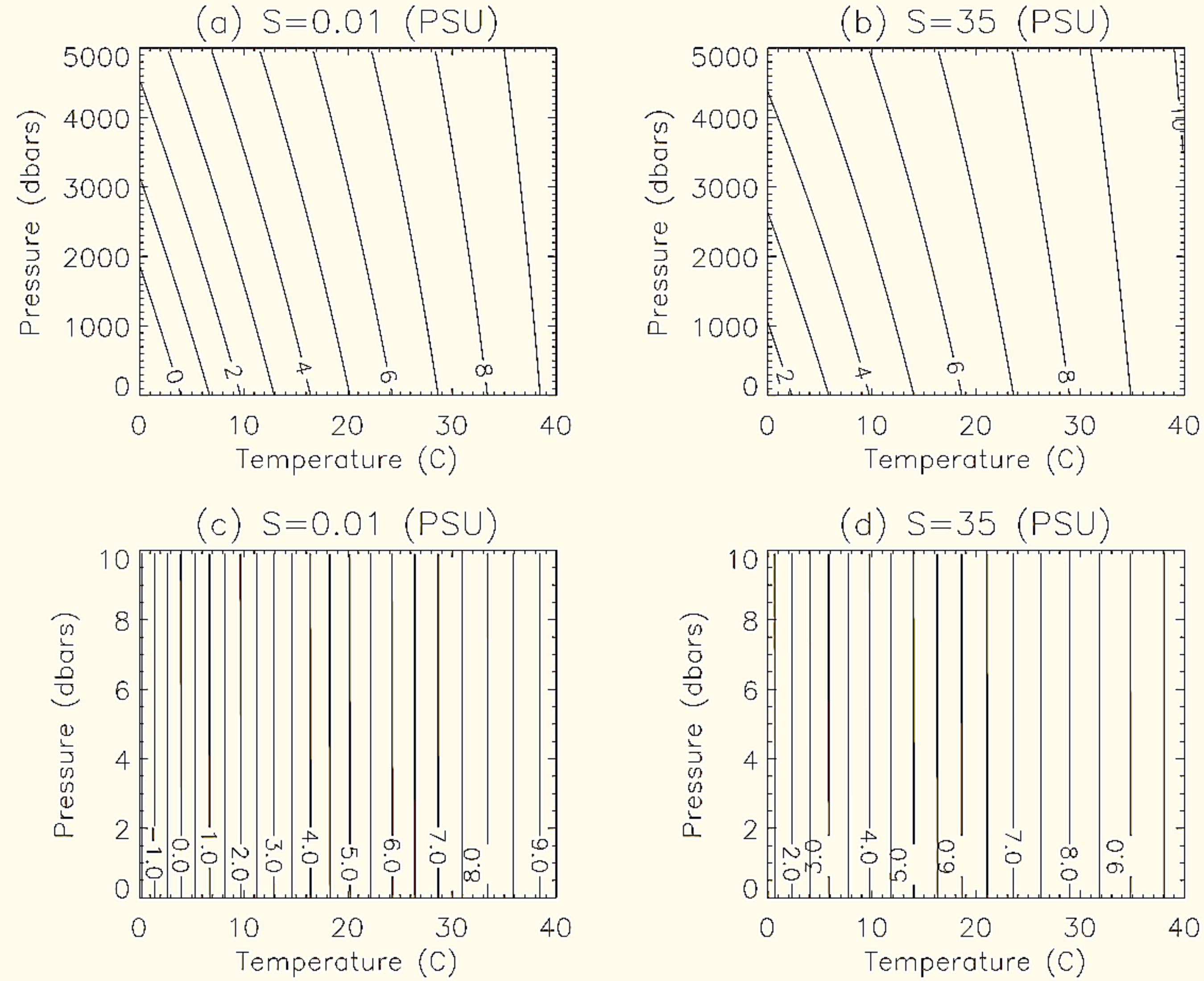
$$\boxed{\frac{\alpha P}{\rho C_p} \times 10^5}$$



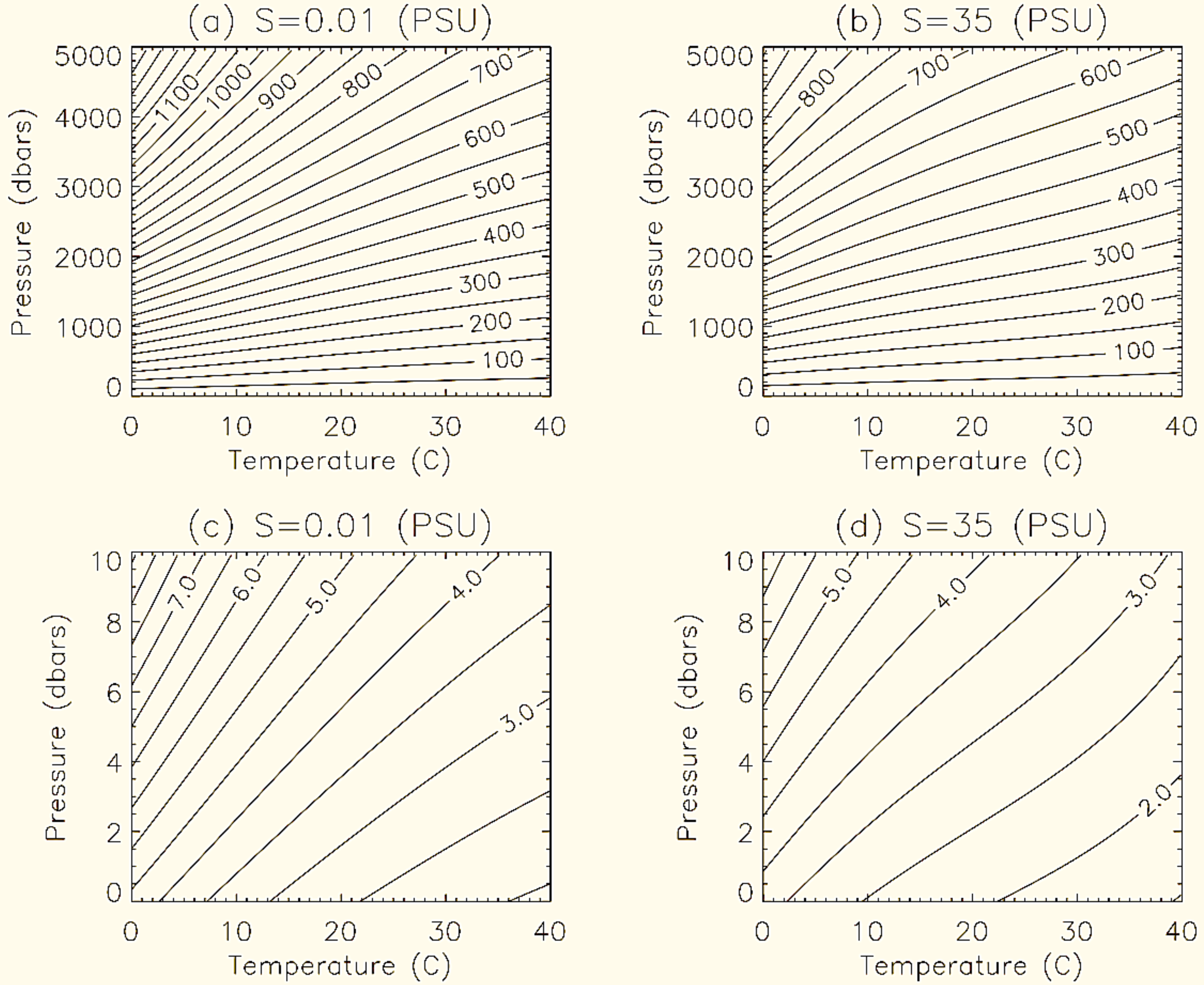
$$\frac{\partial}{\partial P}\left(\frac{\alpha P}{\rho C_p}\right) \times 10^{11}$$

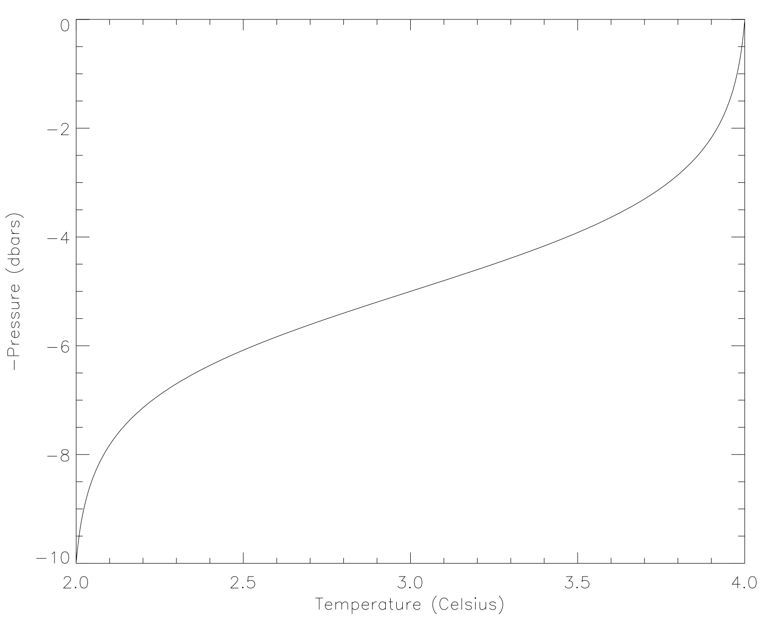



$$\frac{\alpha}{\rho C_p} \times 10^{11}$$

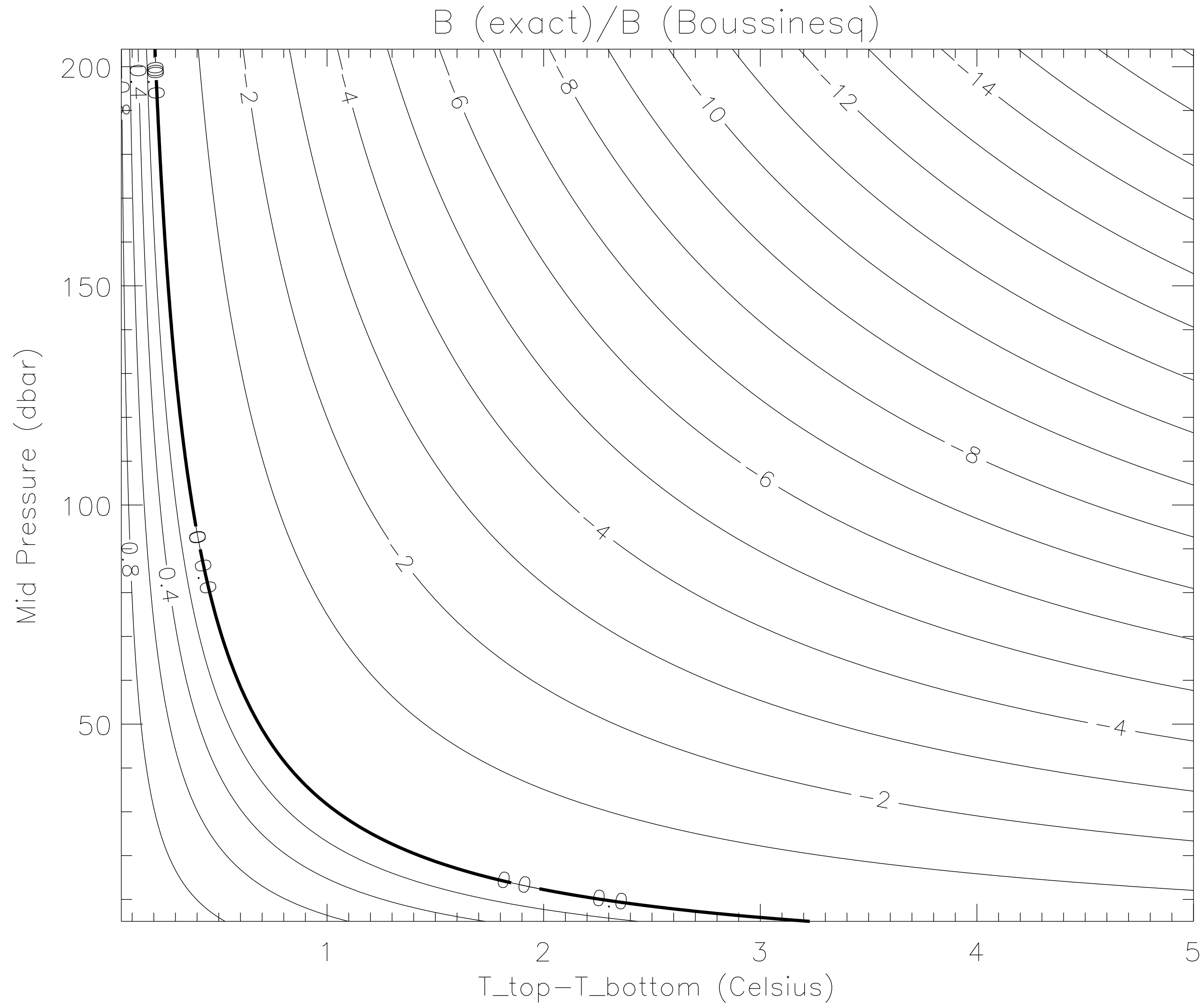


$$\boxed{\frac{\partial}{\partial T}\left(\frac{\alpha P}{\rho C_p}\right) \times 10^{11}}$$

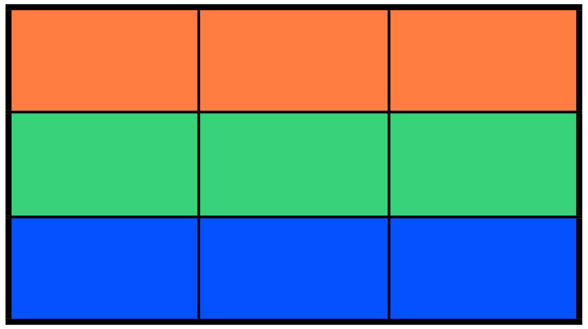




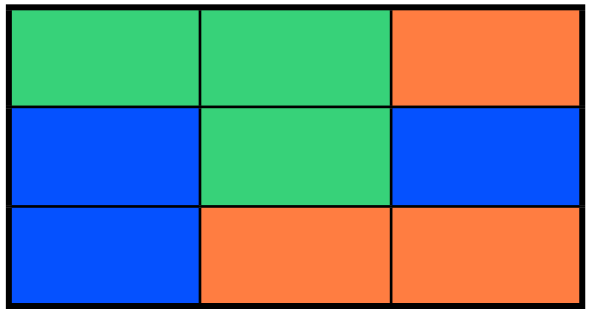
0
-2
-4
-6
-8
-10
2.0
2.5
3.0
3.5
4.0
-Pressure (dbars)
Temperature (Celsius)

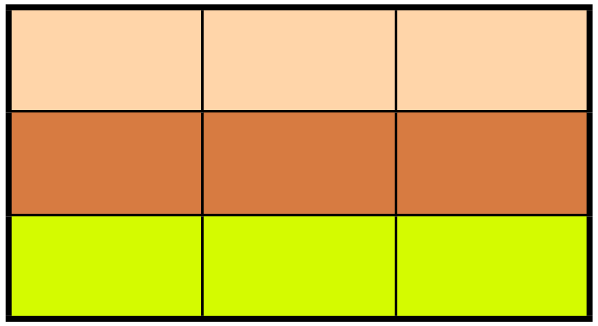
B (exact)/B (Boussinesq)
Mid Pressure (dbar)
T_top-T_bottom (Celsius)
200
150
100
50
1
2
3
4
5
0.0
0.4
0.8
2
4
6
8
10
12
14



1. Start with a piece of stratified fluid at rest
AGPE=AIE=0



2. Stir the fluid randomly by adding some mechanical energy KE
APE = AGPE+AIE = KE



3. Homogenize the temperature of the parcels horizontally
ΔAPE = -KE
ΔGPE_r + ΔIE_r = KE