Equilibration of Baroclinic Turbulence

The Long Road from QG Theory to the Real World

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Problems

How do baroclinic eddies in the atmosphere and ocean equilibrate? What determines their ultimate magnitude and scale?

- A problem in geostrophic turbulence ('macro turbulence').
- Equilibration theories fall into a few general camps, which may each be idealized limits of what occurs in various parameter regimes.
  - Cascades within quasi-geostrophic turbulence (Rhines, Salmon, Larichev & Held, Simmons and Hoskins, implicitly).
  - Baroclinic adjustment and its relatives (Rhines and Young, Stone and colleagues, Schneider and Walker).
  - Important role of convection (Juckes).
- Each implies different role for beta, friction, stratification, etc.

Discuss and test numerically in idealized model and domains.

Baroclinic Adjustment

Three arguments

- Stone (1982). Essentially proposed that baroclinic eddies were so efficient in transferring heat polewards and upwards that the mean state of the atmosphere would become essentially neutral, or marginally supercritical, to baroclinic instability.

- Somewhat related to the idea that a fluid would homogenize its potential vorticity (Prandtl-Batchelor, Rhines & Young, 1982). Homogenization follows if PV is diffused downgradient, and there are no interior sources. NB: homogenization is not needed for stability. Role of surface temperature gradient?

- Schneider and Walker (2007) proposed a related but different argument, not based on PV homogenization, but still based on PV diffusion in the interior and temperature diffusion at the surface. A mass conservation argument leads to very similar predictions (i.e., marginal supercriticality), but the basis of the argument differs.

Critical Shear

No critical shear in the continuously stratified problem, but in two-layers

$$\Delta U_c = \beta \left(\frac{NH}{f}\right)^2 = \beta L_d^2. \qquad \text{or} \qquad L_{\mathrm{Kuo}}^2 \equiv \left(\frac{\Delta U_c}{\beta}\right) = L_d^2 \tag{1}$$

If Kuo scale is bigger than the deformation scale, then there is instability.

Supercriticality and PV

Criticality: $S = \Delta U / \Delta U_c = (U/\beta L_d^2) = L_{\mathrm{Kuo}}^2 / L_d^2$, where $L_{\mathrm{Kuo}} \equiv \sqrt{\Delta U / \beta}$. Also

$$\frac{\partial Q_1}{\partial y} = \beta + k_d^2 U, \qquad \frac{\partial Q_2}{\partial y} = \beta - k_d^2 U$$

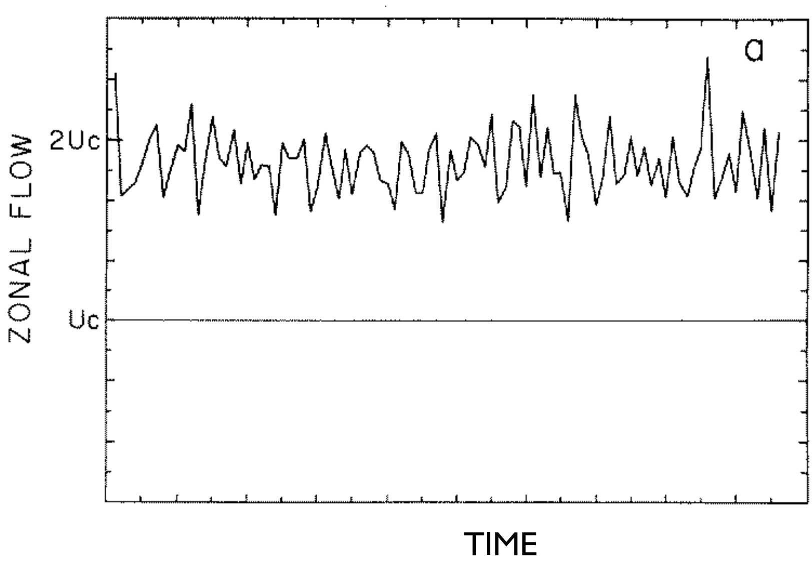
For PV to change sign, get (1). At marginal criticality lower-layer PV gradient is zero.

Comments

- No *a priori* knowledge about *how* the system might evolve toward marginal supercriticality. The shear and the deformation radius can change.
- Deformation scale is imposed in QG models, but is free to evolve in PE models.
- If the Kuo scale equals the Rhines scale, then marginal supercriticality implies no inverse cascade in the barotropic mode.

Numerical Tests

Numerical experiments with quasi-geostrophic models generally do *not* support the idea of baroclinic adjustment.



Vallis 1988

However, the shear *is* the same order of magnitude as the critical shear.

Observations

Isentropic slope

Using thermal wind ($f\partial u/\partial z = \partial b/\partial y$, where b is buoyancy (aka 'temperature'), a critical shear implies an isentropic slope:

$$s = \frac{\partial b/\partial y}{\partial b/\partial z} = \frac{f\partial u/\partial z}{N^2} = H\frac{\beta}{f}$$

at marginal supercriticality. On the sphere β and f are not independent, and this result implies

$$s \sim \frac{H}{a}$$

where a is the radius of the earth. Implications are:

- Isentropic slope is fixed, independent of season.
- If H is the height of the troposphere, an isentrope will move from the equator to the pole over the depth of the troposphere.

This is *very roughly* satisfied in Earth's atmosphere (Stone and Nemet 1996), but variations are large.

Atmosphere vs Ocean

Atmosphere: the eddy scale (few thousand km) is comparable to, and a little larger than, the deformation scale (thousand km). There is no demonstrable inverse cascade (certainly no $-5/3$ range).

— Is this simply because the deformation scale is so large, so no 'room' for it?

Ocean: eddy scale (O(100 km) or a bit more) is significantly larger than the deformation scale (O(10 km) or a bit more). According to Stammer, the eddy scale is the beta scale, and not the deformation scale, implying they scale differently. The beta scale is both bigger than, and has a different latitudinal variation than, the deformation scale. (Rob Scott and Dudley Chelton may have more recent observations and interpretations.)

Possibilities

- Same equilibration mechanisms hold in atmosphere and ocean, but deformation radius is so large in atmosphere that no inverse cascade is possible.
- Baroclinic adjustment holds in atmosphere but not in ocean. In ocean there is some form of inverse cascade.

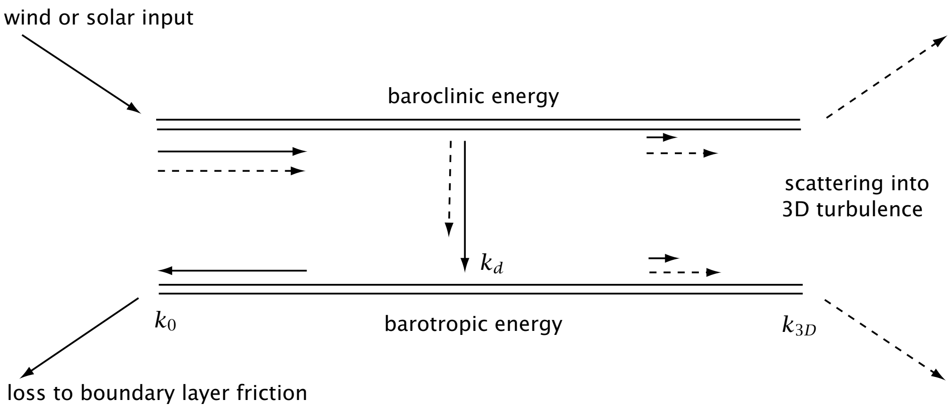
— Ocean is different because it is not eddy dominated?

— Ocean is different because it is on the f-plane?

— Ocean is different because the forcing is different?

Quasi-geostrophic turbulence

Using QG theory and phenomenology we can build a picture of baroclinic turbulence in a simple, homogeneous case. (Salmon)



Two layer QG equations are, in modal form:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + J\left(\tau, (\nabla^2 - k_d^2)\tau\right) + \beta\frac{\partial\psi}{\partial x} = D[\psi],$$

$$\frac{\partial}{\partial t}(\nabla^2 - k_d^2)\tau + J(\tau, \nabla^2\psi) + J\left(\psi, (\nabla^2 - k_d^2)\tau\right) + \beta\frac{\partial\tau}{\partial x} = D[\tau].$$

Quasi-geostrophic Turbulence

Approximations for large scales (with $\beta = 0$ for now):

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) = -J(\tau, \nabla^2\tau) + D[\psi],$$
$$\frac{\partial \tau}{\partial t} + J(\psi, \tau - Uy) = -k_d^{-2} D[\tau].$$

So that:

- Forward cascade of baroclinic energy.
- At deformation scales the barotropic mode is forced by the baroclinic mode, and a cascade to large scales is expected.
- The cascade halts at the beta (Rhines) scale, or a frictional scale.
- If the beta scale is no larger than the deformation scale, then there will be no inverse cascade — the flow will be marginally supercritical in some sense.

Scaling

Barotropic and baroclinic energy spectrum wavenumber (Larichev and Held 1995)

$$\mathcal{E}_\psi(k) = \mathcal{K}_1 \varepsilon^{2/3} k^{-5/3}, \qquad \mathcal{E}_\tau(k) = \mathcal{K}_2 \varepsilon^{1/3} k^{-5/3},$$

so that, if energy is equipartioned between barotropic and baroclinic, at large scales k_0

$$k_0^2 \psi^2 \sim k_d^2 \tau^2$$

Then, if $\tau \sim U/k_0$ then $\psi \sim k_d U/k_0^2$ and the barotropic rms velocity scales like:

$$\boxed{U_{rms} \sim U \frac{k_d}{k_0}}$$

The barotropic rms velocity scales like the mean shear, multiplied by the ratio of the deformation scale to the inverse cascade halting scale.
Atmosphere: $k_0 \sim k_d$
Ocean: $k_d \gg k_0$. Eddy KE much larger than the mean KE.

Rhines Scale

Rhines scale is $k_\beta = \sqrt{\beta/U_{rms}}$. So that length of inverse cascade is, roughly,

$$\boxed{\frac{k_d}{k_\beta} = \frac{k_d^2}{k_{\text{kuo}}^2} = \frac{k_d^2 \Delta U}{\beta} = S}$$

Consequences

- Barotropic energy:

$$U_{rms} \sim \frac{k_d^2 U^2}{\beta}$$

So eddy KE increases rapidly with the mean shear, and with the deformation wavenumber.

- If k_d is big (small deformation radius) there is a long inverse cascade, and lots of eddy KE.

- If $k_d \sim k_{\text{kuo}}$ then there is no inverse cascade (supercriticality gives an inverse cascade).

- Eddy diffusivity:

$$\kappa \sim \frac{U^3 k_d^3}{\beta^2}$$

increases with shear, decreases with β.

Bottom line: Potential for large inverse cascade if deformation radius is smaller than the halting scale (i.e., the beta scale or the frictional scale). That is, if supercriticality is large, so if:

$$S = \frac{k_d^2 \Delta U}{\beta} \gg 1.$$

These are all 'external' parameters in a QG model.

Numerical Tests

Numerical tests with a QG model are in *qualitative*, but *not quantitative*, agreement with these predictions.

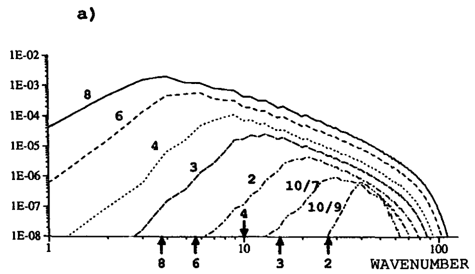
They show:

1. Eddy KE increasing rapidly with imposed mean shear and deformation wavenumber,
2. Eddy scale larger than the deformation scale for large shear and small β.
3. Scalings are not quantitatively satisfied.

In addition:

- Theoretical problems in the use of the Rhines scale as a stopping scale (role of friction?), and, of course, in the Kolmogorov-Kraichnan phenomenology.
- Vis à vis the real world, quasi-geostrophic theory may be questionable, as the system might equilibrate in completely different ways.

Numerical Tests From Held and Larichev (1995)



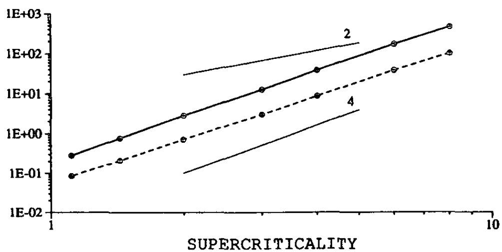


FIG. 2. A log–log plot of $\overline{v_\psi'^2}$, twice the meridional barotropic energy (solid line), and ϵ, the baroclinic eddy energy production rate (dashed line), as functions of the supercriticality ξ. Also shown are slopes consistent with ξ^2 and ξ^4 dependencies.

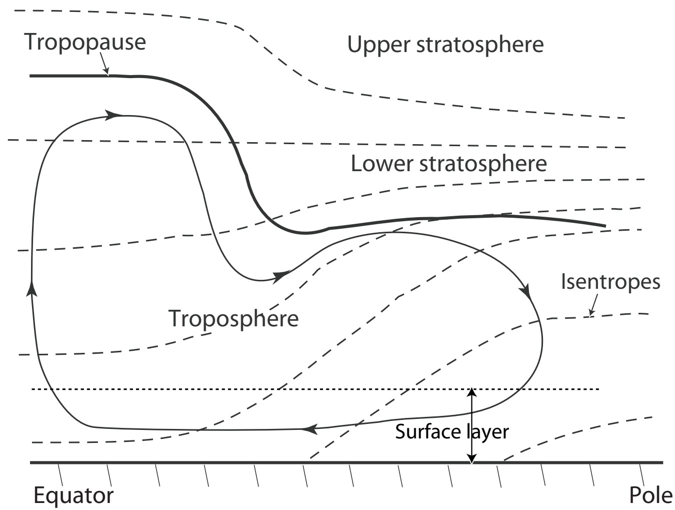
Energy scales and energy levels get larger with supercriticality, as expected. (Arrows indicate Rhines scale.)

Marginal Supercriticality — A Variation on a Theme

Assumptions

Schneider (2004), Schneider and Walker (2007).

1. A ‘two-layer’ atmosphere, comprising a free atmosphere and a surface layer. Net polewards mass flux in free atmosphere, equatorwards in surface layer.
2. Diffusive flux of PV in free atmosphere, and of buoyancy in surface layer, with the same coefficient of eddy diffusivity.
3. Mass conservation of residual flow.



Sketch of overturning circulation

Caricature of Derivation

Statistically steady momentum equation in atmosphere, in TEM (residual) form:

$$-f\overline{v}_a^* = \overline{v'q'} \qquad \Rightarrow \qquad H\overline{v}_a^* = \frac{H\mathcal{K}}{f}\frac{\partial\overline{q}}{\partial y} \approx \frac{H\mathcal{K}\beta}{f}.$$

A similar argument for surface layer gives

$$H_s\overline{v}_s^* = \left(\frac{\mathcal{K}}{\partial\overline{\theta}/\partial z}\right)\frac{\partial\overline{\theta}}{\partial y}.$$

But $H\overline{v}_a^* = H_s\overline{v}_s^*$, so that

$$\frac{\partial_y\overline{\theta}}{\partial_z\overline{\theta}} = \frac{\beta H}{f}$$

or

$$\boxed{S = \frac{f}{\beta H}\frac{\partial_y\overline{\theta}}{\partial_z\overline{\theta}} = \frac{\Delta_y\overline{\theta}}{\Delta_z\overline{\theta}} = 1}.$$

Using thermal wind this implies the supercriticality condition

$$\boxed{S = L_d^2 \sim L_{\text{Kuo}}^2 \qquad \text{or} \qquad S = \frac{L_{\text{Kuo}}^2}{L_d^2} = 1}.$$

Suggests an isentrope goes from surface to troposphere over course of the extratropics.

Tests

Questions

1. Can we obtain an inverse cascade with a primitive equation model, forced at large scales?

    - Or at least, can we make the eddy scale larger than the deformation scale?

2. Is baroclinic adjustment, in any of its forms, a valid model?

Model

Use 2-level, dry, Boussinesq, primitive equation model in a beta-plane channel with idealized thermal forcing.

The β and f parameters can be adjusted *independently*.

Model can in principle allow stratification and/or shear to adjust as needed.

Rayleigh drag plus hyperviscosity.

Compare with QG model with (as far as possible) same parameters, some taken *post facto* from PE simulation.

No convective adjustment scheme.

Model Details

- $L_x = 30\,000\,\text{km}$, $L_y = 18,000\,\text{km}$, $\Delta x = \Delta y = 150\,\text{km}$. $H = 10\,\text{km}$.
- Linear equation of state.
- Simple thermodynamic equation:

$$D\theta/Dt = -(\theta - \theta^*)/\tau$$

where θ^* is stably stratified and has sharp meridional variation in midlatitudes. Specifically, for control experiment,

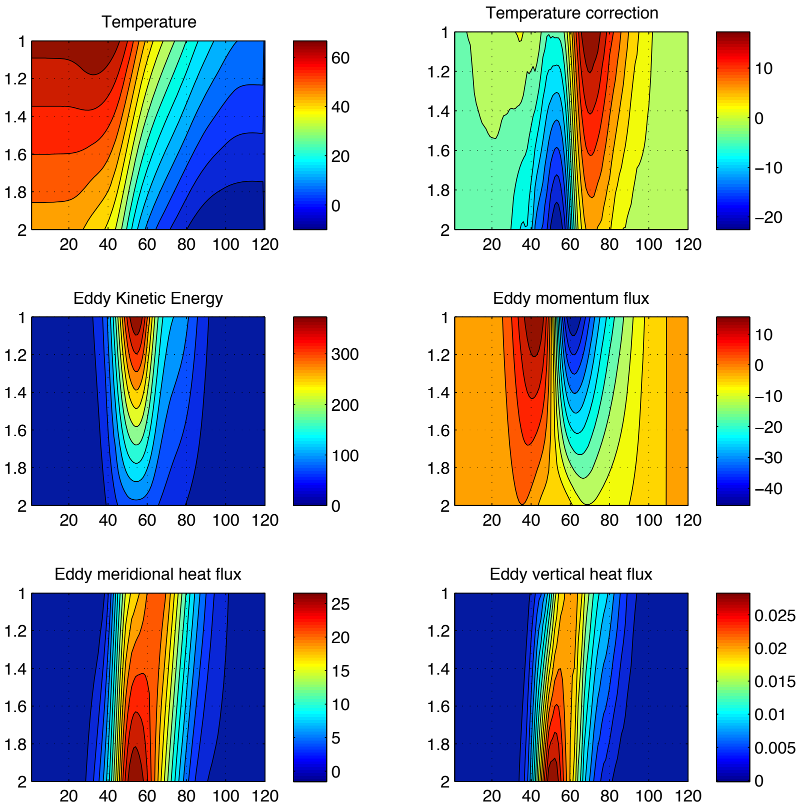
$$\theta^* = \theta_0 + \delta_z\theta \times (0, 1) + \delta_y\theta \tanh((y - LY/2)/\sigma)$$

where $\delta_z\theta = 40\,K$, $\delta_y\theta = 60\,K$, $\tau = 20$ days.

- Rayleigh drag in lower layer and biharmonic horizontal diffusion.
- No exact PV conservation, even in unforced, inviscid, adiabatic case.

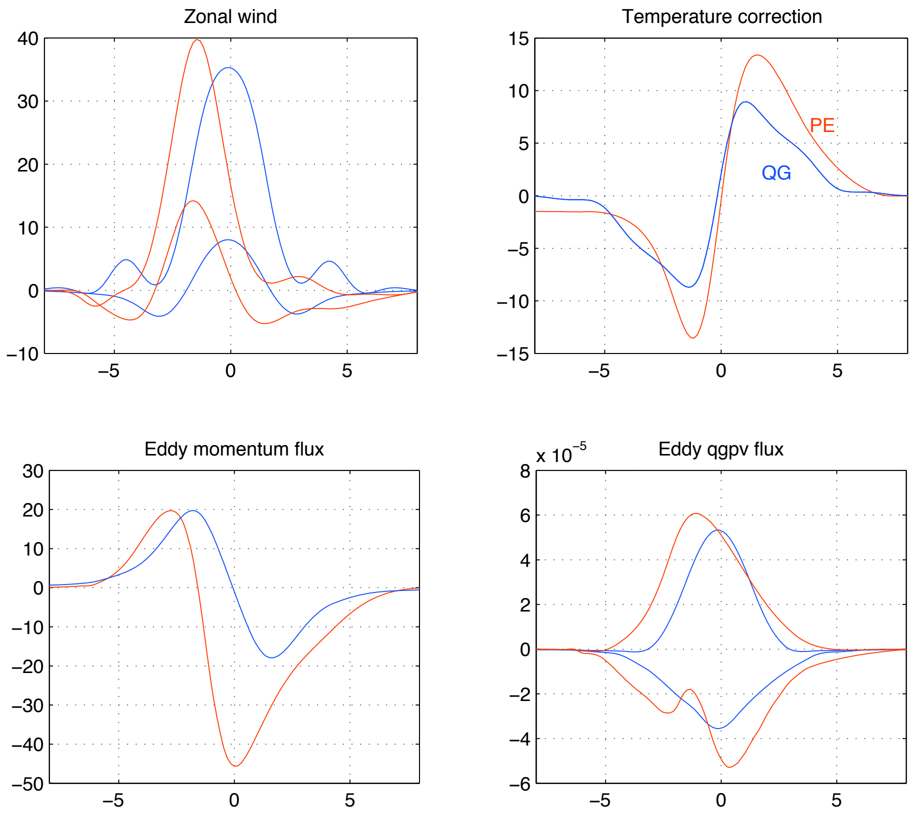
Model becomes baroclinically unstable, grows eddies, equilibrates. Large scale heat transport typically stabilizes the flow to vertical convection; some grid-scale convection away from baroclinic zone.

Mean State, Control Run



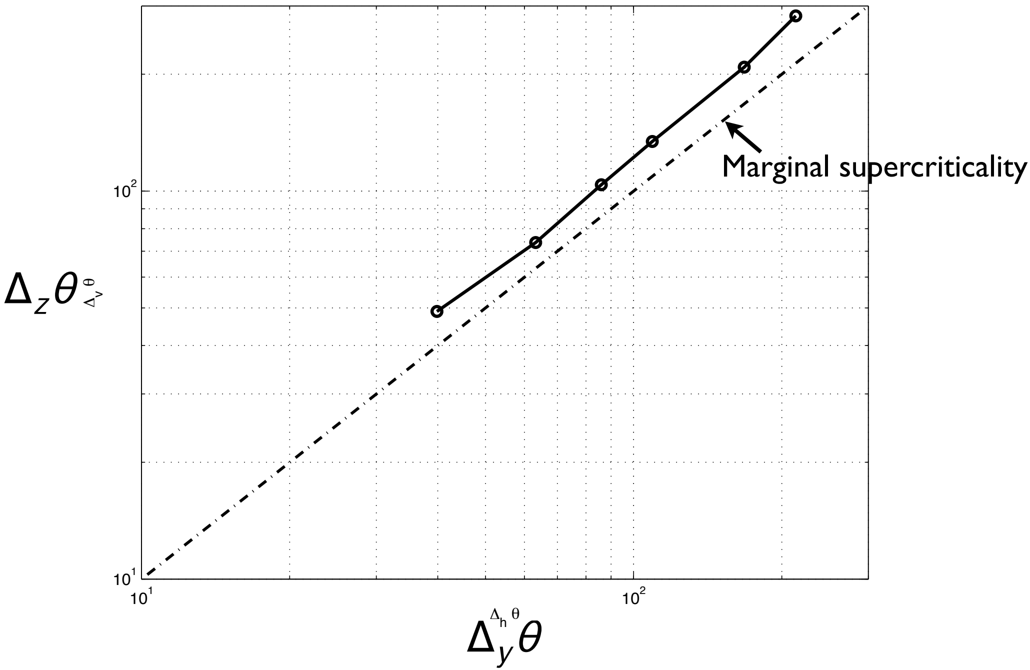
PE - QG comparison

PE: Red. QG: Blue.



Criticality

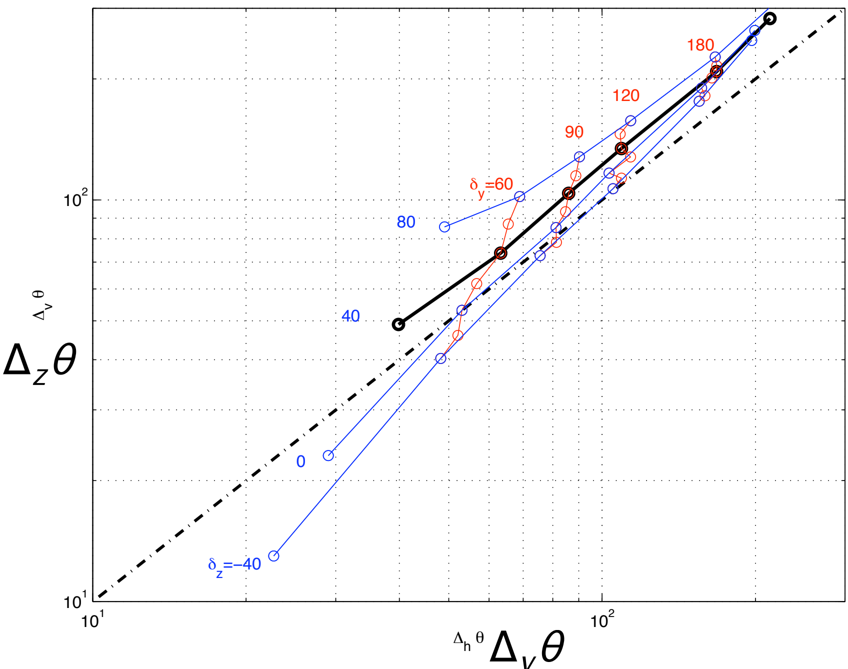
Vary horizontal temperature gradient in control



For a range of forcing parameters, flow is marginally subcritical (but likely a little supercritical some of the time)

Criticality

Vary horizontal and vertical temperature gradient (of forcing) in control



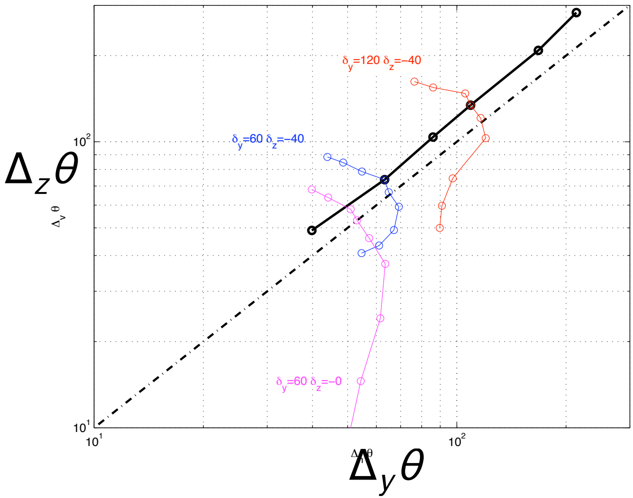
Red, change $\delta_z\theta$

Blue, change $\delta_y\theta$

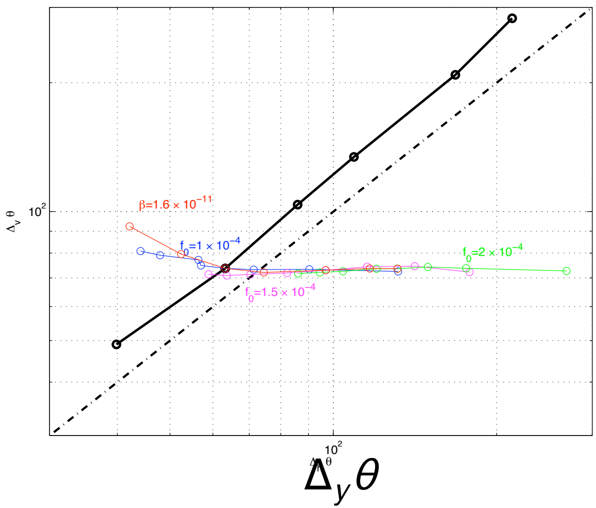
Varying vertical temperature gradient leads to variations in criticality

Criticality

Change τ

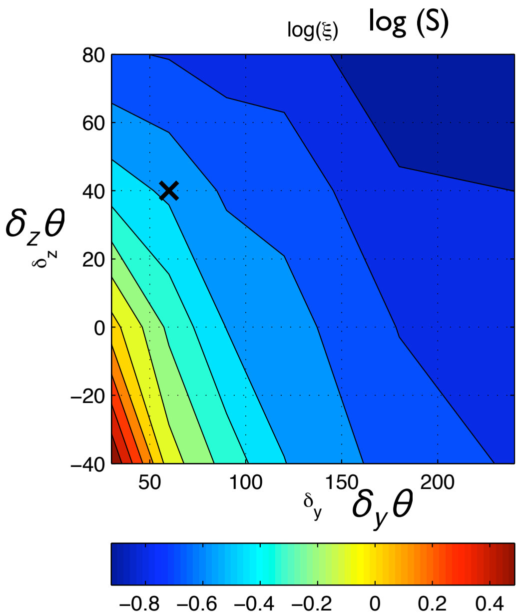


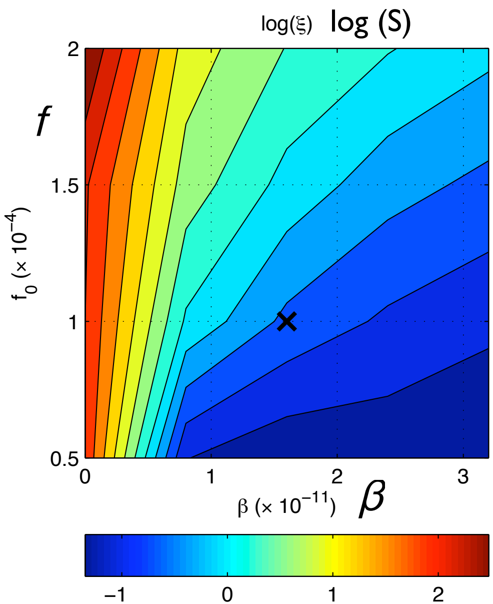
Change f_0 and β.



$\Delta_y\theta$ increases with f and decreases with β.

Contours of Criticality



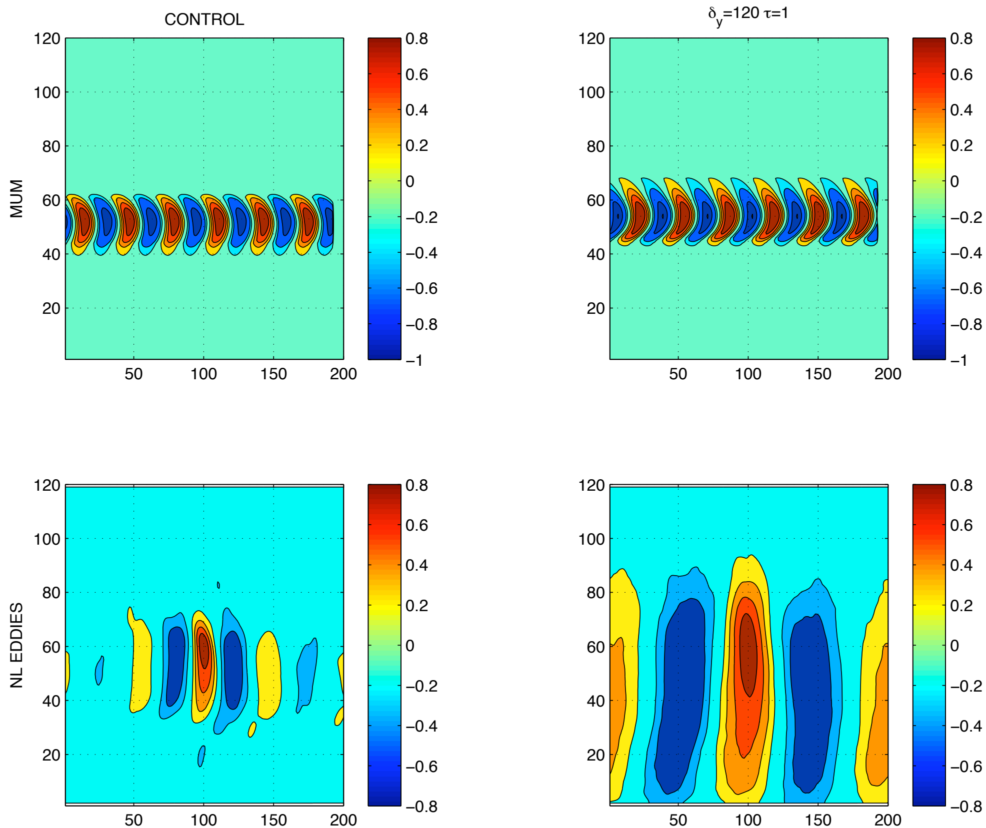


Red is supercritical, blue is subcritical

Scales of Motion

Linear Instability

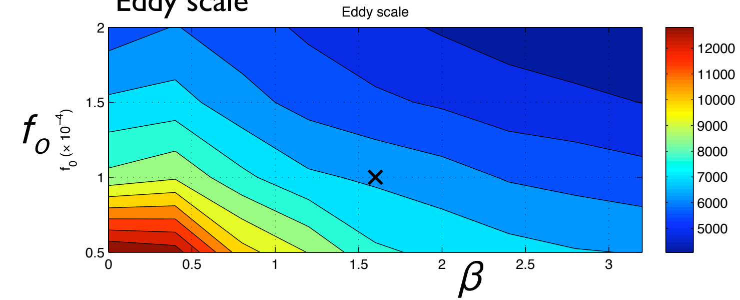
Equilibrated scale



Eddies are bigger than the scale of the instability

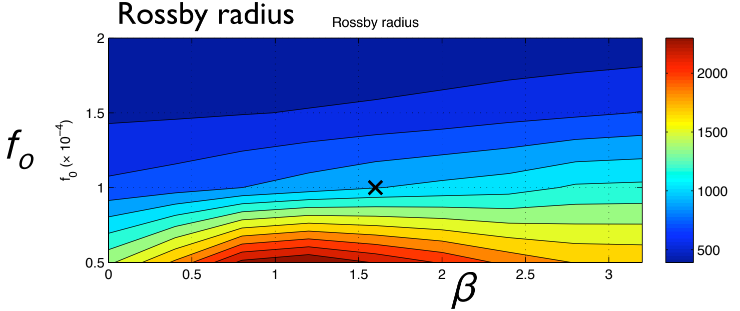
Scales of Motion

Eddy scale



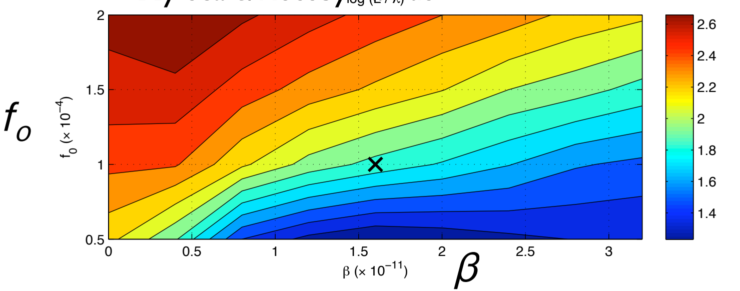
Eddy scales decrease as beta and f_0 increase.

Rossby radius



Rossby radius is flat with beta, decreases as f_0 increases.

Eddy scale/Rossby radius

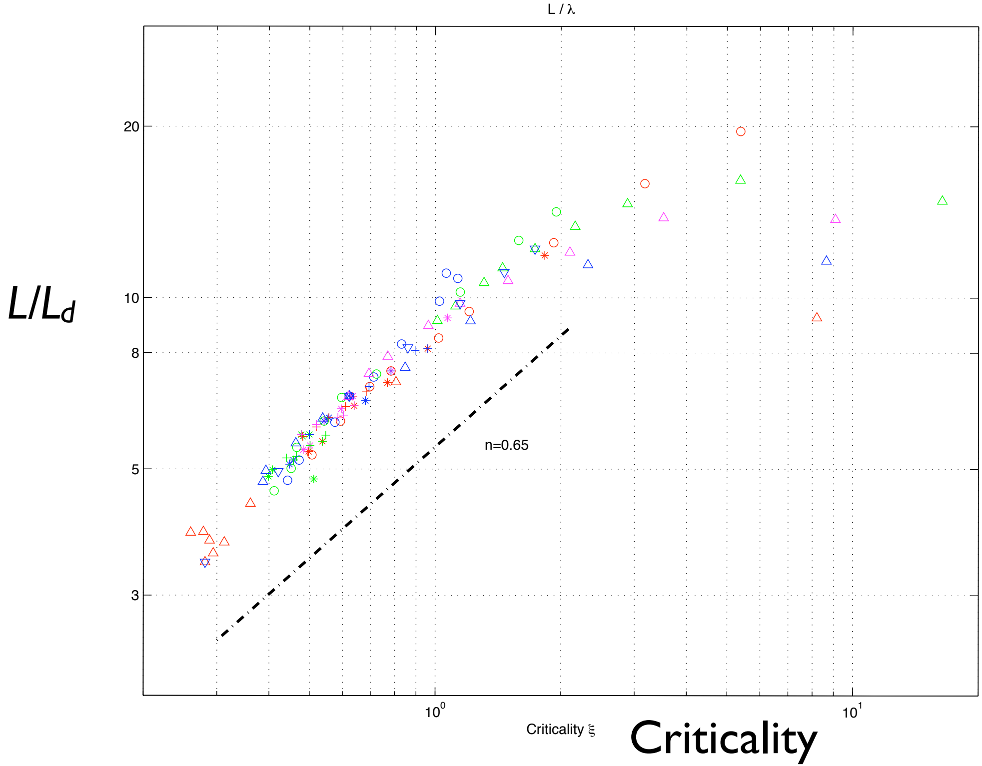


Ratio increases as beta falls and as f_0 increases.

Eddy scales vs criticality

Summary of many experiments. Ratio of eddy scale to deformation scale as a function of criticality.

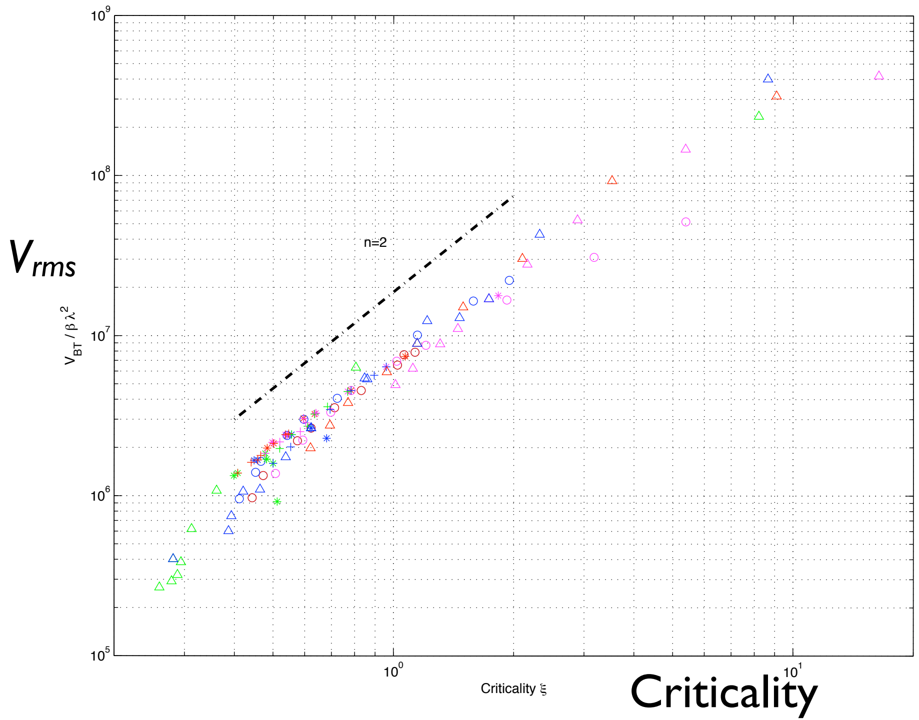
Suggests there can be an inverse cascade. As QG suggests, a bigger inverse cascade for more supercritical flow.



Eddy KE vs criticality

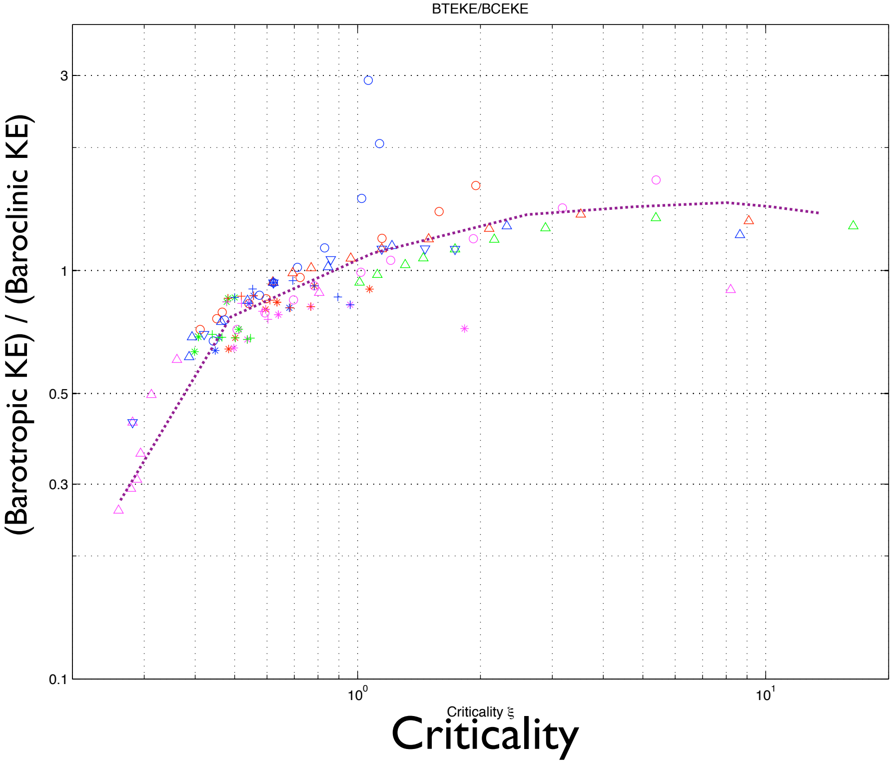
Barotropic velocity scale as a function of criticality.

Increasing EKE as criticality increases.

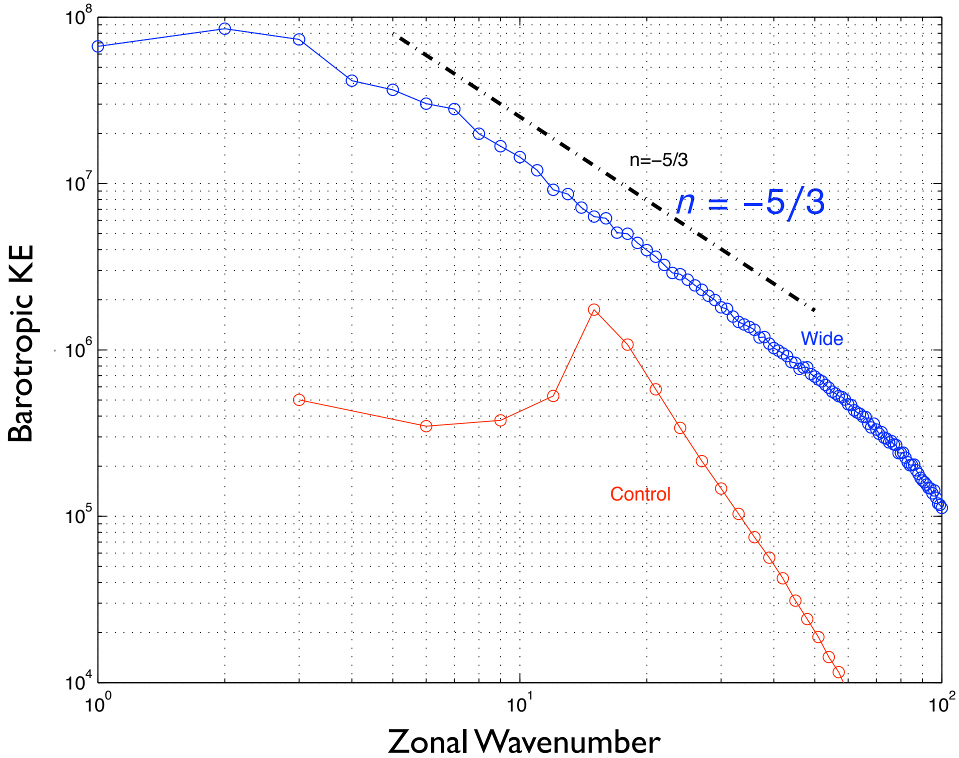


Barotropization?

Eddy flow is more barotropic with increasing criticality, but this seems to saturate.
Possible effects of friction and domain size,



Energy Spectra



Obtain a -5/3 barotropic spectra (inverse cascade) in some but not all simulations.

Conclusions

The nature of the flow changes continuously with the parameters, and there are parameter ranges for which the flow is supercritical and a form of inverse cascade exists.

- The model can evolve to a supercritical, turbulent, state, especially for small β.
- The eddy scale increases with the supercriticality, and is larger than the instability scale.
  - Suggests presence of an inverse cascade, and a $-5/3$ cascade can (sometimes) be obtained.
- Nonetheless, for finite values of β, there is a range of parameters for which changing the meridional gradient of the forcing (the radiative equilibrium temperature) *does not significantly change the supercriticality.*
- Quasi-geostrophic models seem useful, and give qualitatively good predictions, *provided the stratification is diagnosed from the PE model, or observations*. We have no good theory for stratification. (Some combination of upwards heat transport in baroclinic waves and convection.)
- The difference in importance of the β effect may explain why the ocean, but not the atmosphere, appears supercritical.
  - Alternatively, the ocean is not eddy dominated to the extent that the atmosphere is, and stratification may not be determined by baroclinic turbulence.