Customized Approximation with Radial Basis Functions*

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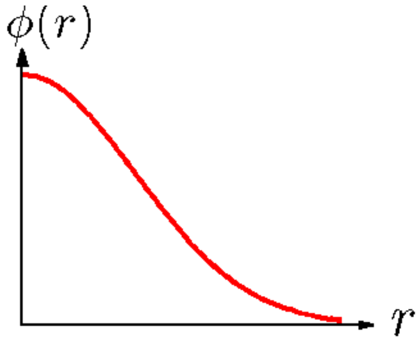
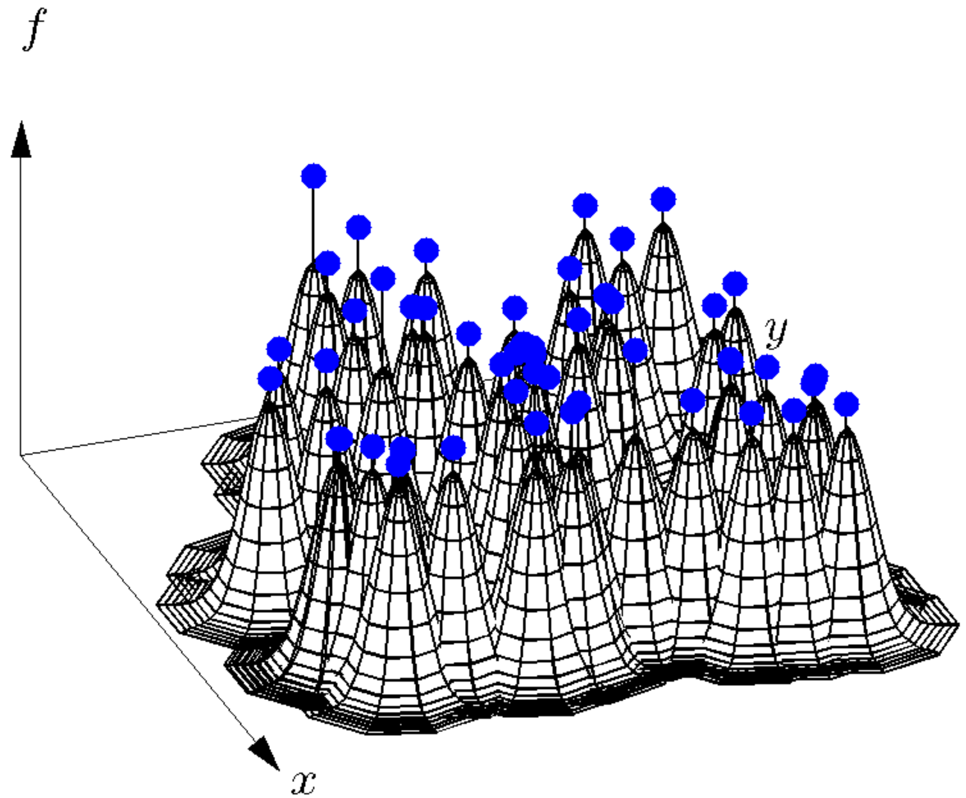
Outline

- Very quick review of Radial Basis Functions (RBFs) interpolation
- Customizing RBF approximation for vector fields:
  - Developed by Narcowich and Ward (1994)
  - Divergence-free vector fields
    - fluid flows, (static) magnetic fields
  - Curl-free vector fields
    - gravity fields, (static) electric fields
- RBF approximation of vector fields tangent to the surface of the sphere:
  - Surface divergence-free approximation
  - Surface curl-free approximation
  - Helmholtz-Hodge decomposition
- Geophysical applications

Scalar RBF interpolation

Key idea: linear combination of translates and rotations of a single radial function:





1-D: $\phi(|x-x_j|)$ > 1-D: $\phi(\|\boldsymbol{x}-\boldsymbol{x}_j\|_2)$

Interpolant: $s(\boldsymbol{x})=\sum_{j=1}^{N}\beta_j\phi(\|\boldsymbol{x}-\boldsymbol{x}_j\|),\;\; s(\boldsymbol{x}_k)=f_k,\; k=1,\ldots,N$

Expansion coeffcients:

$$\begin{bmatrix}\phi(\|\boldsymbol{x}_1-\boldsymbol{x}_1\|) & \phi(\|\boldsymbol{x}_1-\boldsymbol{x}_2\|) & \cdots & \phi(\|\boldsymbol{x}_1-\boldsymbol{x}_N\|)\\ \phi(\|\boldsymbol{x}_2-\boldsymbol{x}_1\|) & \phi(\|\boldsymbol{x}_2-\boldsymbol{x}_2\|) & \cdots & \phi(\|\boldsymbol{x}_2-\boldsymbol{x}_N\|)\\ \vdots & \vdots & \ddots & \vdots\\ \phi(\|\boldsymbol{x}_N-\boldsymbol{x}_1\|) & \phi(\|\boldsymbol{x}_N-\boldsymbol{x}_2\|) & \cdots & \phi(\|\boldsymbol{x}_N-\boldsymbol{x}_N\|)\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{bmatrix}=\begin{bmatrix}f_1\\ f_2\\ \vdots\\ f_N\end{bmatrix},$$

Scalar RBF interpolation

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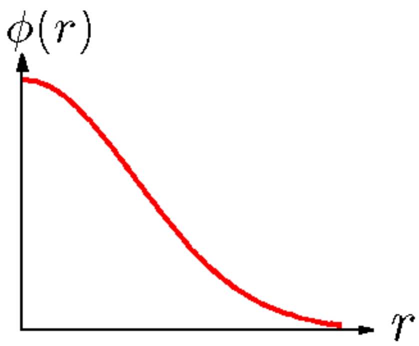

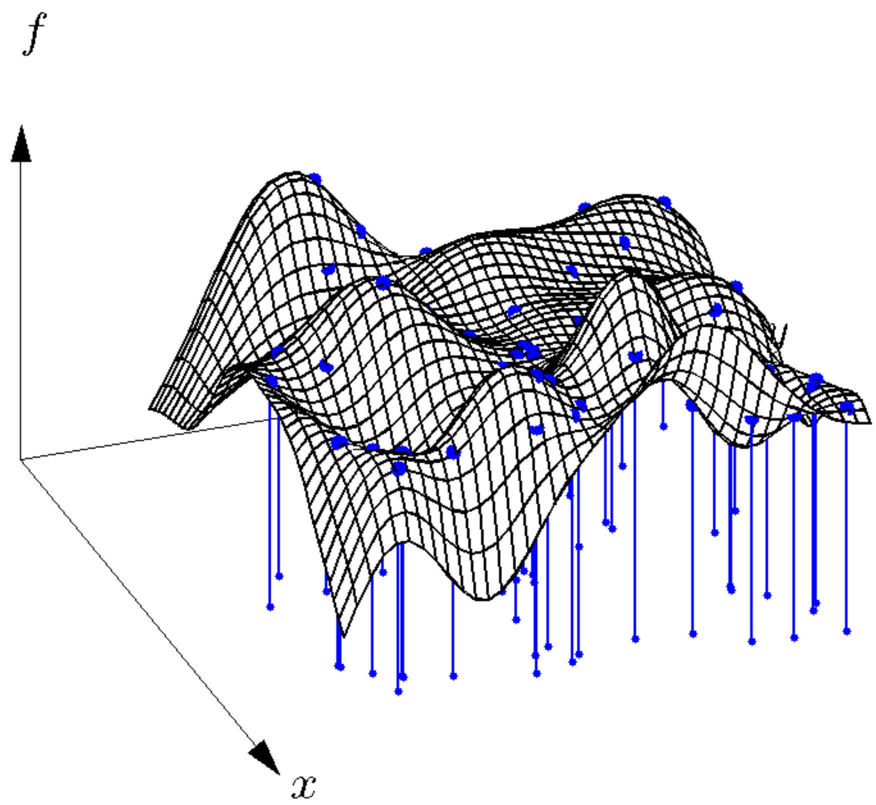

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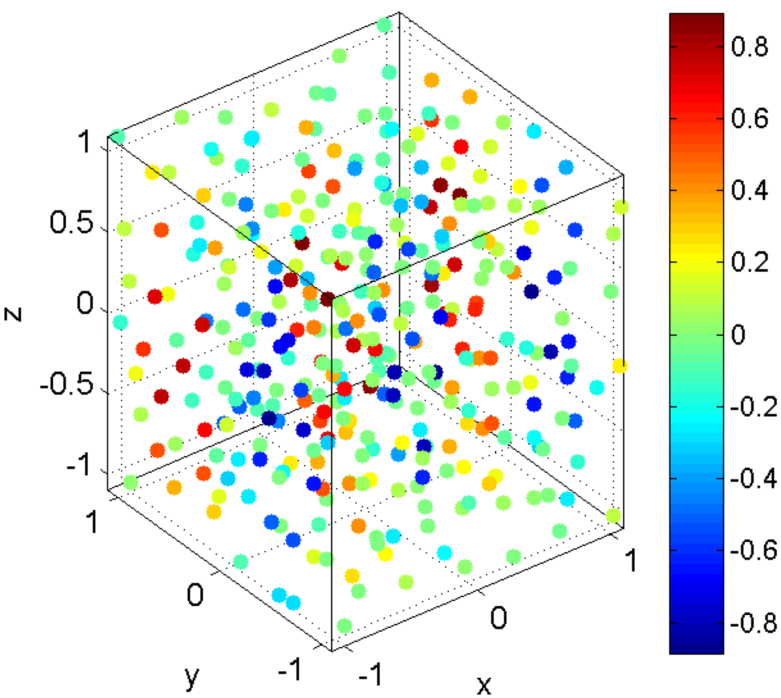
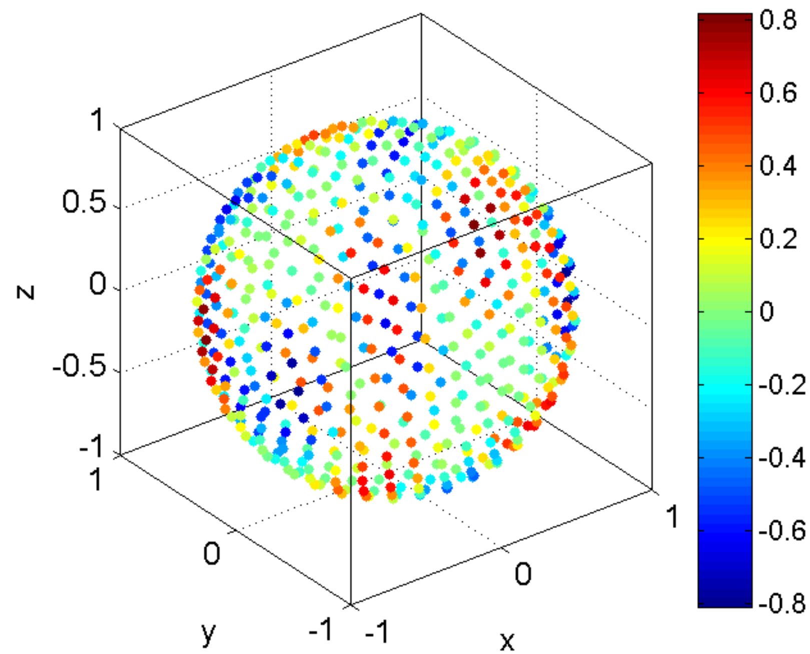
Guaranteed positive-definite for appropriate $\phi(r)$

RBFs for spherical geometries

Interpolation on $\mathbb{R}^3$



Interpolation on $\mathbb{S}^2 \subset \mathbb{R}^3$



- Scalar RBF interpolant does not change:

$$s(\boldsymbol{x}) = \sum_{j=1}^{N} \beta_j \phi(\|\boldsymbol{x} - \boldsymbol{x}_j\|), \quad s(\boldsymbol{x}_k) = f_k,\ k = 1, \ldots, N$$

- Divergence-free and curl-free RBF interpolants do!

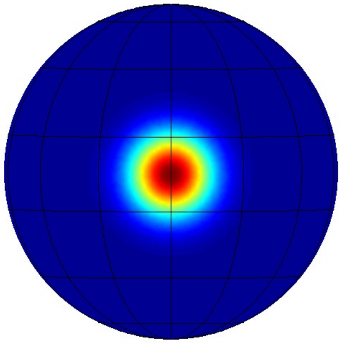
(Surface) Div, Grad, Curl, and all that

	Spherical Coords.	Cartesian Coords.
Point:	$(\lambda, \theta, 1)$	(x, y, z)
Unit vectors:	$\hat{\mathbf{i}}$ = longitudinal $\hat{\mathbf{j}}$ = latitudinal $\hat{\mathbf{k}}$ = radial	$\hat{\mathbf{i}} = x$-direction $\hat{\mathbf{j}} = y$-direction $\hat{\mathbf{k}} = z$-direction
Unit tangent vectors:	$\hat{\mathbf{i}},\ \hat{\mathbf{j}}$	$\boldsymbol{\zeta} = \frac{1}{\sqrt{1-z^2}}\begin{bmatrix}-y\\ x\\ 0\end{bmatrix},\ \boldsymbol{\mu} = \frac{1}{\sqrt{1-z^2}}\begin{bmatrix}-zx\\ -zy\\ 1-z^2\end{bmatrix}$
Unit normal vector:	$\hat{\mathbf{k}}$	$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Gradient of scalar g:	$\mathbf{u}_s = \nabla_s\, g = \frac{1}{\cos\theta}\frac{\partial g}{\partial \lambda}\hat{\mathbf{i}} + \frac{\partial g}{\partial \theta}\hat{\mathbf{j}}$	$\mathbf{u}_c = P_{\mathbf{x}}(\nabla_c\, g) = P_{\mathbf{x}}\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right)$
Surface divergence of $\mathbf{u}$:	$\nabla_s \cdot \mathbf{u}_s = \nabla_s^T \mathbf{u}_s = \frac{1}{\cos\theta}\frac{\partial u_s}{\partial \lambda} + \frac{\partial v_s}{\partial \theta}$	$(\nabla_c P_{\mathbf{x}})^T \mathbf{u}_c = \nabla_c^T (P_{\mathbf{x}} \mathbf{u}_c)$
Curl of a scalar f:	$\mathbf{u}_s = \hat{\mathbf{k}} \times (\nabla_s f) = -\frac{\partial f}{\partial \theta}\hat{\mathbf{i}} + \frac{1}{\cos\theta}\frac{\partial f}{\partial \lambda}\hat{\mathbf{j}}$	$\mathbf{u}_c = \mathbf{x} \times (P_{\mathbf{x}} \nabla_c f) = Q_{\mathbf{x}} P_{\mathbf{x}} (\nabla_c f) = Q_{\mathbf{x}} (\nabla_c f)$
Surface curl of a vector $\mathbf{u}$:	$\hat{\mathbf{k}} \cdot (\nabla_s \times \mathbf{u}_s) = -\nabla_s^T (\hat{\mathbf{k}} \times \mathbf{u}_s)$	$(Q_{\mathbf{x}} \nabla_c)^T \mathbf{u}_c = \nabla_c^T (Q_{\mathbf{x}}^T \mathbf{u}_c) = -\nabla_c^T (Q_{\mathbf{x}} \mathbf{u}_c)$

where $P_{\mathbf{x}} = I - \mathbf{x}\mathbf{x}^T = \begin{bmatrix}1-x^2 & -xy & -xz\\ -xy & 1-y^2 & -yz\\ -xz & -yz & 1-z^2\end{bmatrix}$ and $Q_{\mathbf{x}} = \begin{bmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{bmatrix}$

Surface divergence-free RBFs on the sphere

- Developed by Narcowich, Ward, and Wright (2007)



- Use extrinsic (Cartesian) coordinates, $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$.
- Start with a radial function centered $\mathbf{x}_0 \in \mathbb{S}^2$: $\phi(\|\mathbf{x} - \mathbf{x}_0\|)$
- Construct 3-by-3 *matrix-valued* function

$$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0) = -Q_{\mathbf{x}}(\nabla\nabla^T \phi(\|\mathbf{x} - \mathbf{x}_0\|))Q_{\mathbf{x}_0}^T.$$

- If $\mathbf{c} = (c_1, c_2, c_3)^T$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}_0$ then $\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c}$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}$.
- Furthermore,

$$\begin{aligned}\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c} &= \left[-Q_{\mathbf{x}}(\nabla\nabla^T \phi(\|\mathbf{x} - \mathbf{x}_0\|))Q_{\mathbf{x}_0}^T\right]\mathbf{c} \\ &= Q_{\mathbf{x}}\nabla\left[\nabla^T\left(\phi(\|\mathbf{x} - \mathbf{x}_0\|)Q_{\mathbf{x}_0}\mathbf{c}\right)\right] \\ &= Q_{\mathbf{x}}(\nabla f).\end{aligned}$$

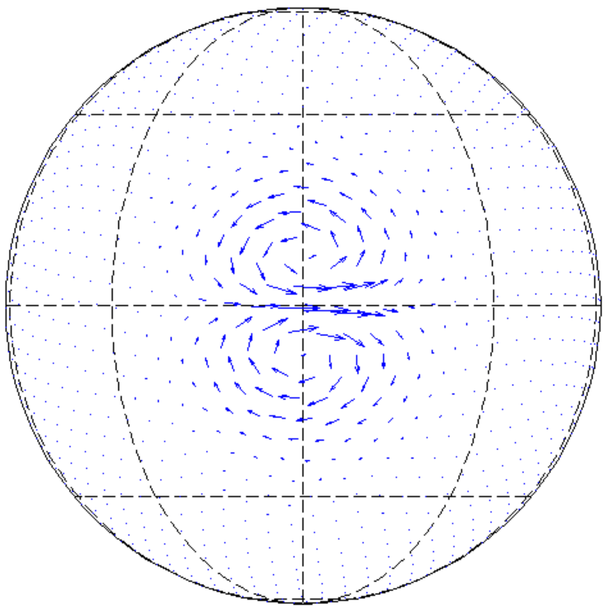
  Thus, $\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c}$ is surface divergence-free.

- Idea can be extended to other smooth manifolds.

Surface div-free RBF interpolation

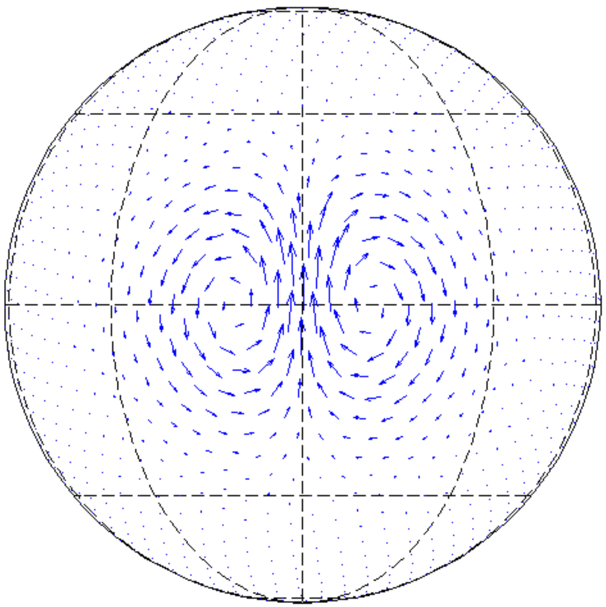
- Illustration of new basis (orthographic projection):

Zonal basis



$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\boldsymbol{\zeta}_0$

Meridional basis

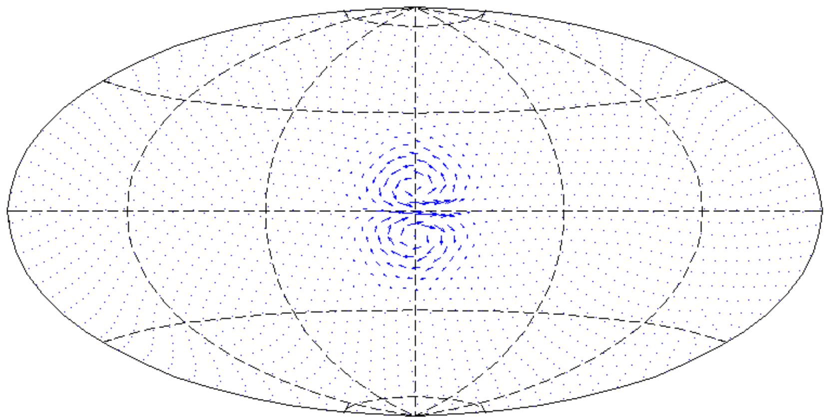


$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\boldsymbol{\mu}_0$

Surface div-free RBF interpolation

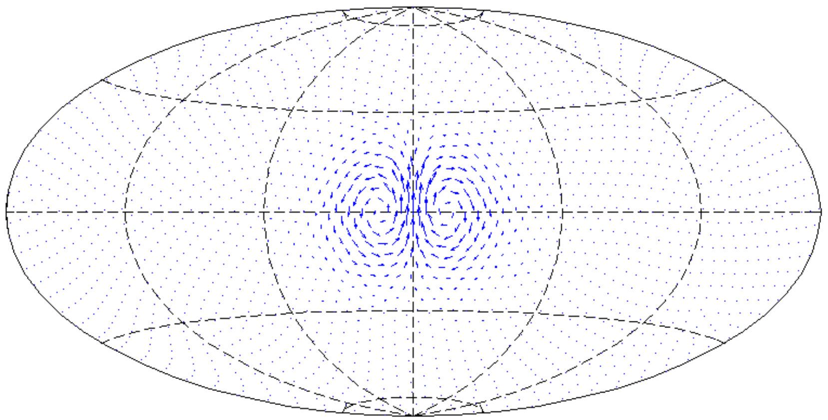
- Illustration of new basis (Hammer-Aitoff projection):

Zonal basis



$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\boldsymbol{\zeta}_0$

Meridional basis



$\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_0)\boldsymbol{\mu}_0$

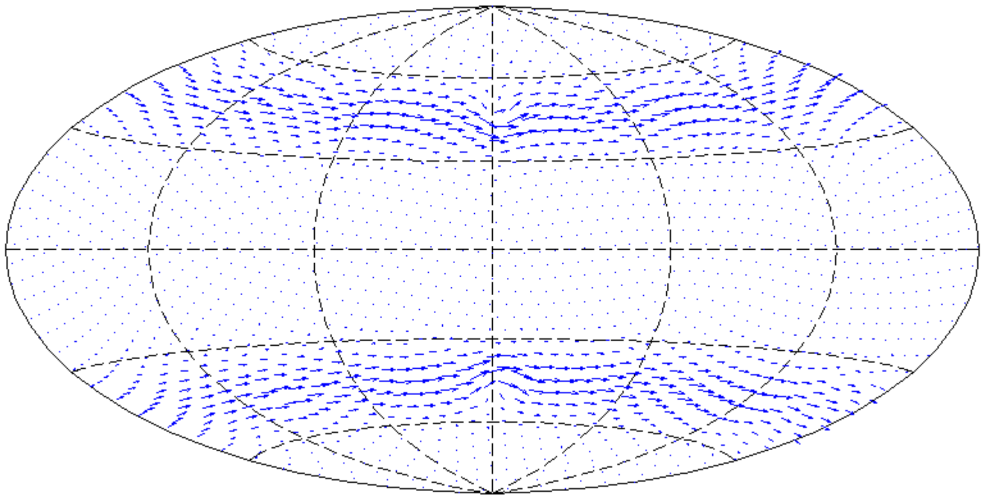
- Construction similar to the scalar RBF interpolant :

1. For each node $\mathbf{x}_j$, center a $\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\mu}_j$ and $\Psi_{\text{div}}(\mathbf{x}, \mathbf{x}_j)\boldsymbol{\zeta}_j$.
2. Linearly combine these *vector-valued* functions to satisfy the interpolation conditions.
3. Interpolant will be surface divergence-free.
4. AND free of any pole singularity.

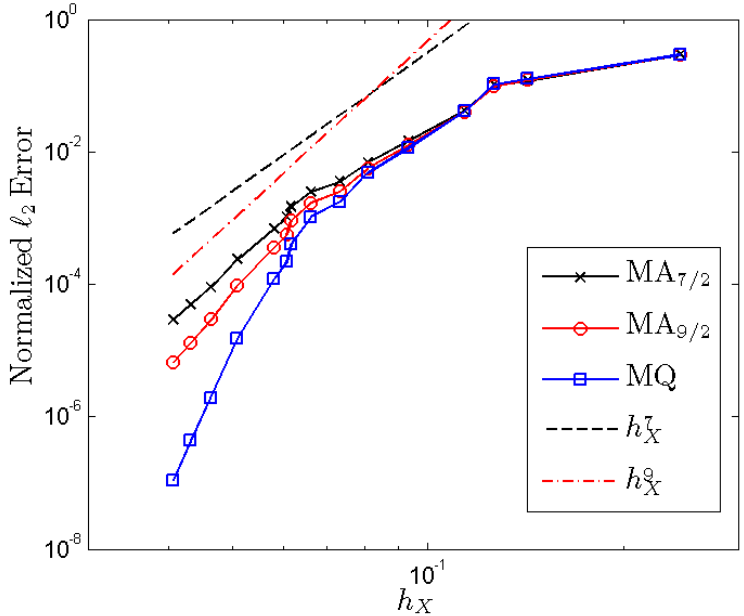
- Slight modification needed to uniquely solve for the interpolation coefficients.

Surface div-free RBF interpolation: examples

- Smooth divergence-free test field sampled at “scattered” nodes on the sphere.



- Error in the RBF reconstructed field vs. node spacing (log-log scale):

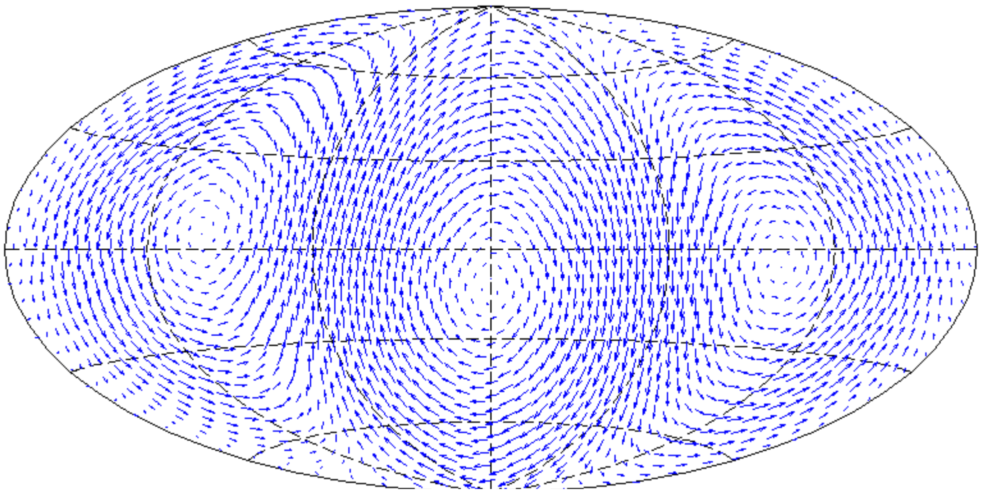


- $\mathrm{MA}_{7/2}$=Matérn $\nu = 7/2$
- $\mathrm{MA}_{9/2}$=Matérn $\nu = 9/2$
- MQ=Multiquadric

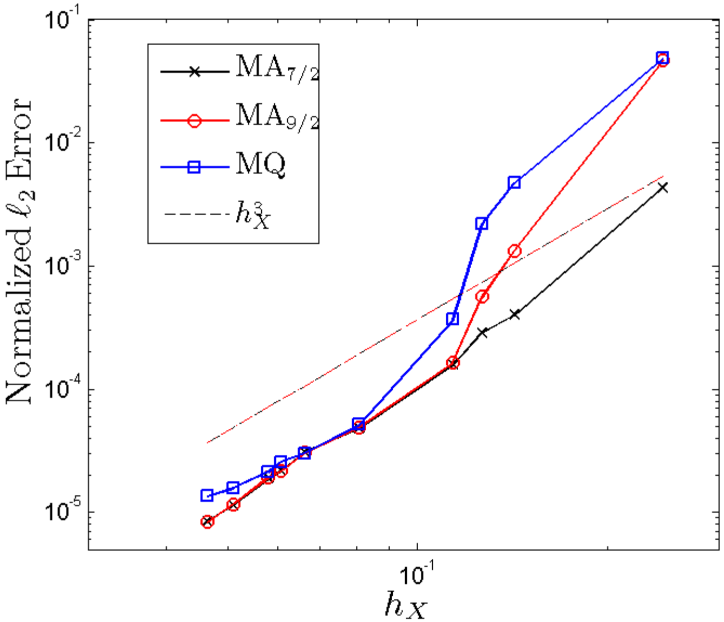
- Dashed and dashed-dotted lines predicted error rates for Matérn (Fuselier *et. al.* 2008.)
- Divergence of recovered field = 0.

Surface div-free RBF interpolation: examples

- Less smooth divergence-free test field sampled at "scattered" nodes on the sphere.



- Error in the RBF reconstructed field vs. node spacing (log-log scale):



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- Divergence of recovered field = 0.

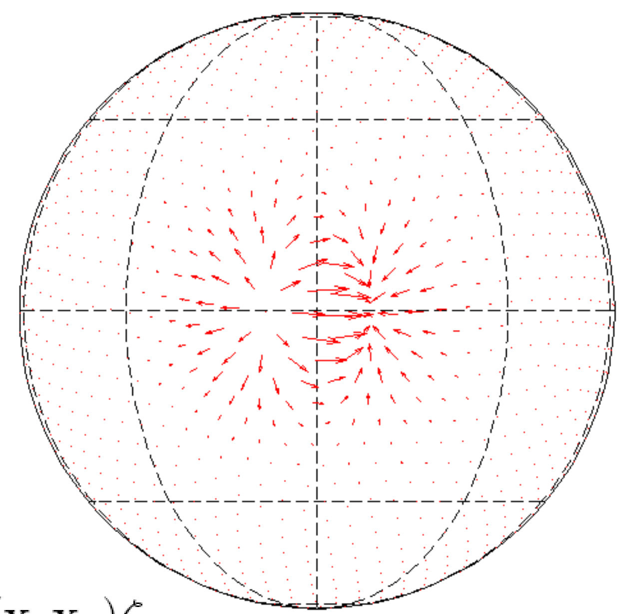
Surface curl-free RBF interpolation

- Surface curl-free basis:
  - Use extrinsic (Cartesian) coordinates, $\mathbf{x} = (x, y, z) \in \mathbb{S}^2$.
  - $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0) = -P_{\mathbf{x}}(\nabla\nabla^T \phi(\|\mathbf{x} - \mathbf{x}_0\|))P_{\mathbf{x}_0}^T$.
  - If $\mathbf{c} = (c_1, c_2, c_3)^T$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}_0$ then $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c}$ is tangent to $\mathbb{S}^2$ at $\mathbf{x}$.
  - Furthermore, $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c} = P_{\mathbf{x}}\nabla \underbrace{\left[-\nabla^T \phi(\|\mathbf{x} - \mathbf{x}_0\|))P_{\mathbf{x}_0}\mathbf{c}\right]}_{g} = P_{\mathbf{x}}(\nabla g)$.

    Thus, $\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0)\mathbf{c}$ is surface curl-free.

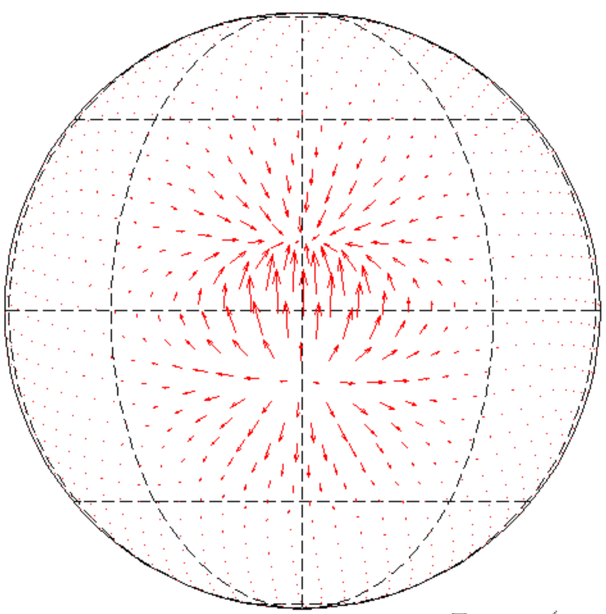
- Illustration of new basis (orthographic projection):

Zonal basis



$\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0)\zeta_0$

Meridional basis



$\Psi_{\text{curl}}(\mathbf{x}, \mathbf{x}_0)\boldsymbol{\mu}_0$

Helmholtz-Hodge Decomposition

- Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

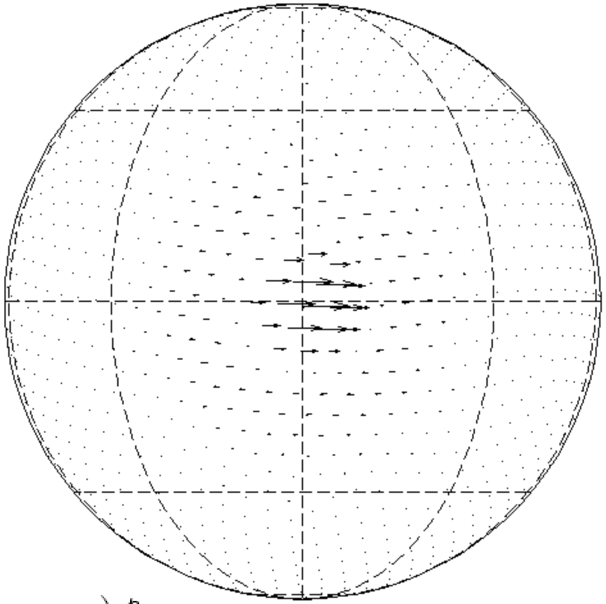
$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_{\mathrm{div}}(\mathbf{x}) + \mathbf{u}_{\mathrm{curl}}(\mathbf{x})$$
$$= Q_{\mathbf{x}}\nabla\psi(\mathbf{x}) + P_{\mathbf{x}}\nabla\chi(\mathbf{x})$$

ψ = stream function and χ = velocity potential

- Matrix-valued Helmholtz-Hodge RBF:

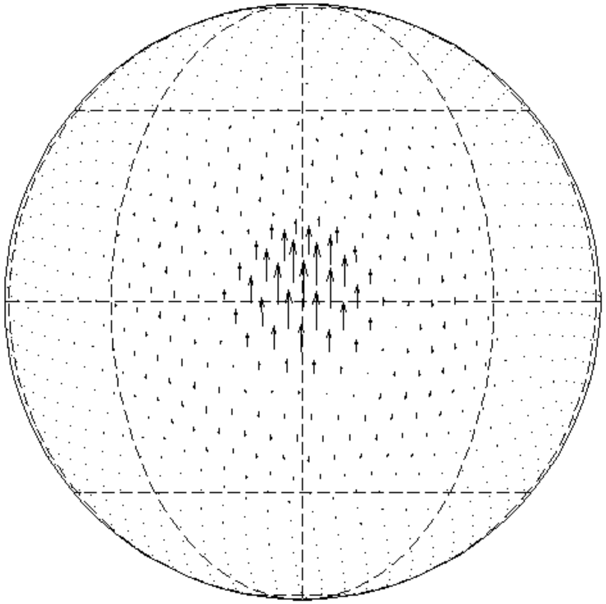
$$\Psi(\mathbf{x},\mathbf{x}_0) = \Psi_{\mathrm{div}}(\mathbf{x},\mathbf{x}_0) + \Psi_{\mathrm{curl}}(\mathbf{x},\mathbf{x}_0)$$

Zonal basis



$\Psi_{\mathrm{curl}}(\mathbf{x},\mathbf{x}_0)\zeta_0$

Meridional basis



$\Psi_{\mathrm{curl}}(\mathbf{x},\mathbf{x}_0)\boldsymbol{\mu}_0$

Helmholtz-Hodge Decomposition

- Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\text{div}}(\mathbf{x}) + \mathbf{u}_{\text{curl}}(\mathbf{x}) \\ &= Q_{\mathbf{x}}\nabla\psi(\mathbf{x}) + P_{\mathbf{x}}\nabla\chi(\mathbf{x}) \end{aligned}$$

$$\psi = \text{stream function and } \chi = \text{velocity potential}$$

- Helmholtz-Hodge RBF interpolant of **f** sampled at $\mathbf{x}_j$:

$$\begin{aligned} \mathbf{s}(\mathbf{x}) &= \sum_{j=1}^{N} \Psi(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j \qquad \text{(where } \mathbf{s}(\mathbf{x}_j) = \mathbf{u}_j,\ j = 1,\ldots,N) \\ &= \sum_{j=1}^{N} \left[\Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j) + \Psi_{\text{curl}}(\mathbf{x},\mathbf{x}_j)\right]\mathbf{c}_j \\ &= \underbrace{\sum_{j=1}^{N} \Psi_{\text{div}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j}_{\approx\,\mathbf{u}_{\text{div}}} + \underbrace{\sum_{j=1}^{N} \Psi_{\text{curl}}(\mathbf{x},\mathbf{x}_j)\mathbf{c}_j}_{\approx\,\mathbf{u}_{\text{curl}}} \end{aligned}$$

- Can get an approximation to the surface divergence-free and surface curl-free components!

Helmholtz-Hodge Decomposition

- Any vector field tangent to the sphere can be *uniquely* decomposed into surface divergence-free and surface curl-free components:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_{\mathrm{div}}(\mathbf{x}) + \mathbf{u}_{\mathrm{curl}}(\mathbf{x}) \\ &= Q_{\mathbf{x}} \nabla \psi(\mathbf{x}) + P_{\mathbf{x}} \nabla \chi(\mathbf{x}) \end{aligned}$$

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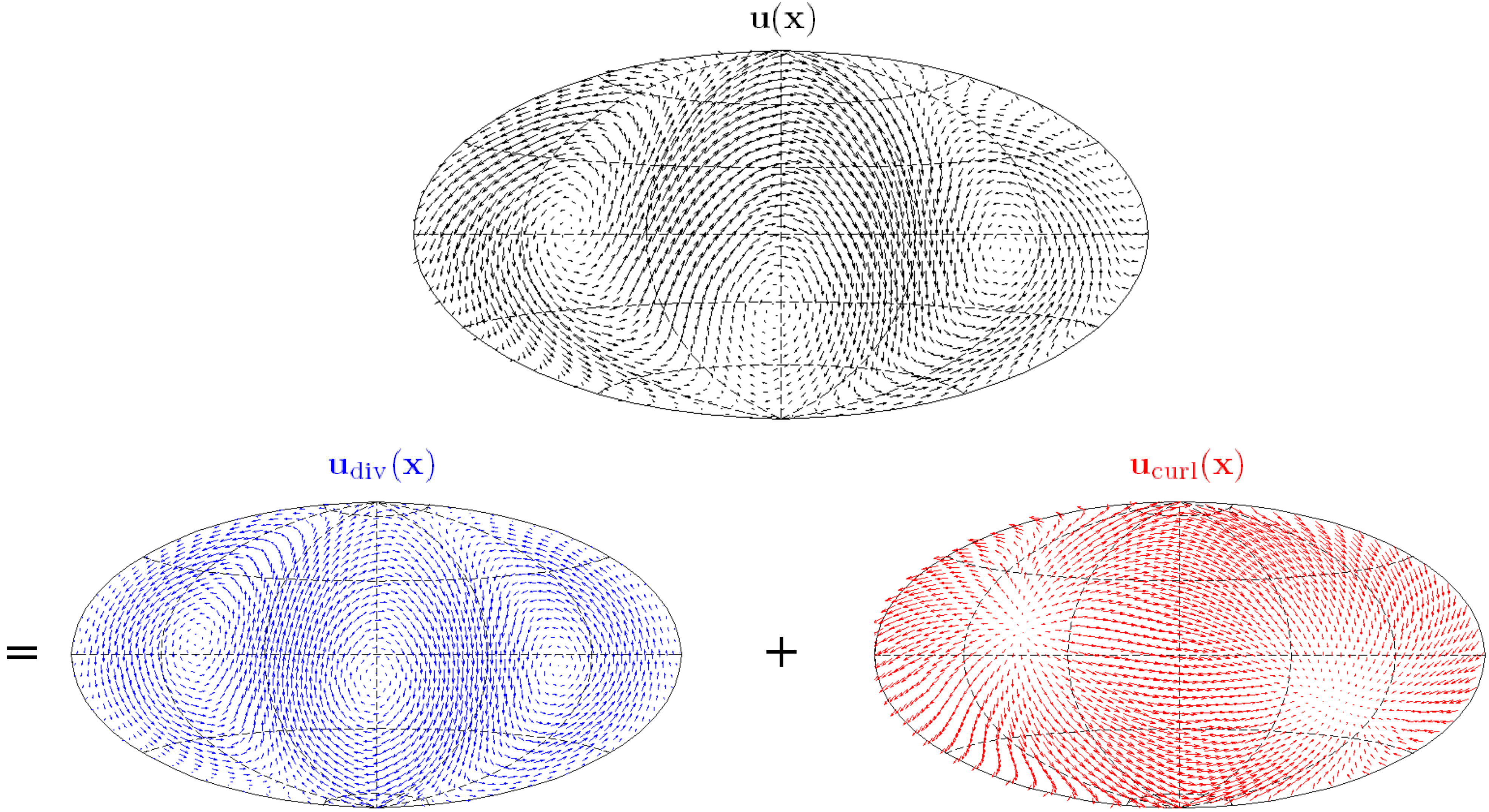
- Can get a stream function and velocity potential for the interpolant.

Helmholtz-Hodge Decomposition: Example 1

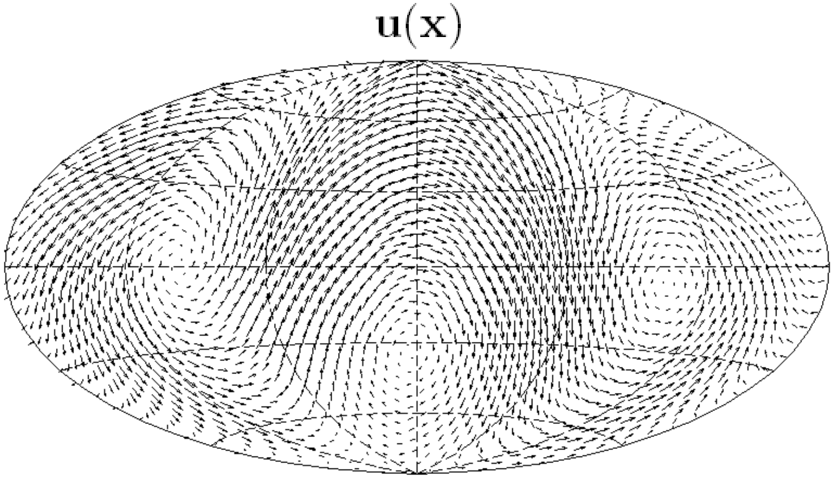
- Tangent vector field **u**(**x**) sampled at "scattered" nodes on the sphere.



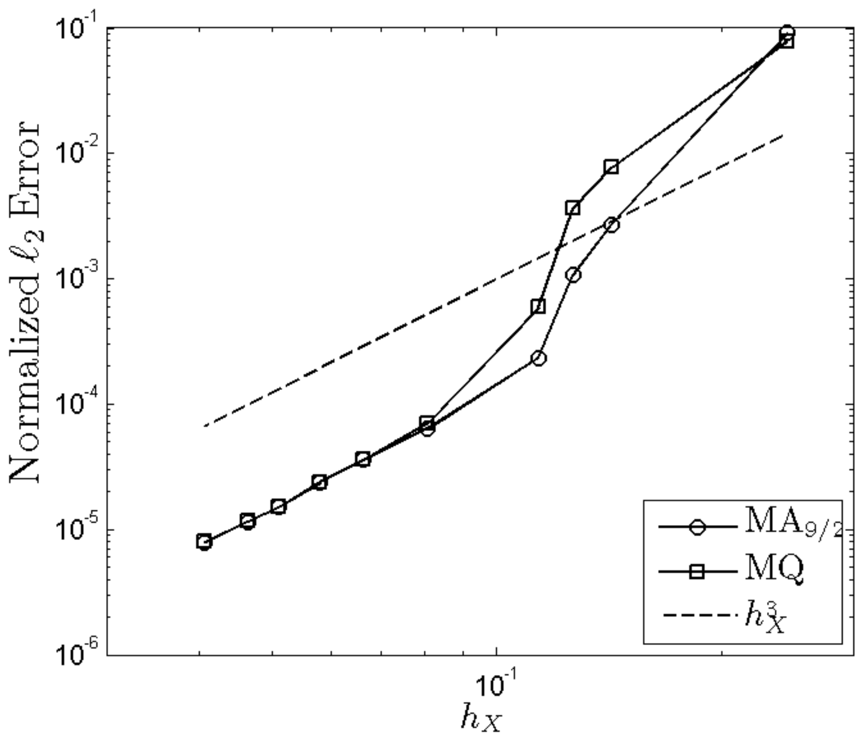
- Only **u** is sampled; $\mathbf{u}_{\mathrm{div}}$ and $\mathbf{u}_{\mathrm{curl}}$ are not known to the interpolant.

Helmholtz-Hodge Decomposition: Example 1

- Tangent vector field $\mathbf{u}(\mathbf{x})$ sampled at "scattered" nodes on the sphere.



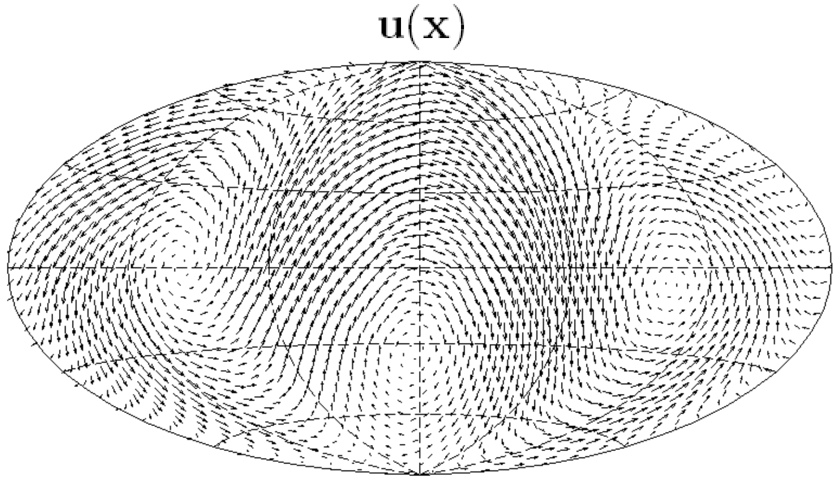
- Error in the RBF reconstructed field vs. node spacing (log-log scale):



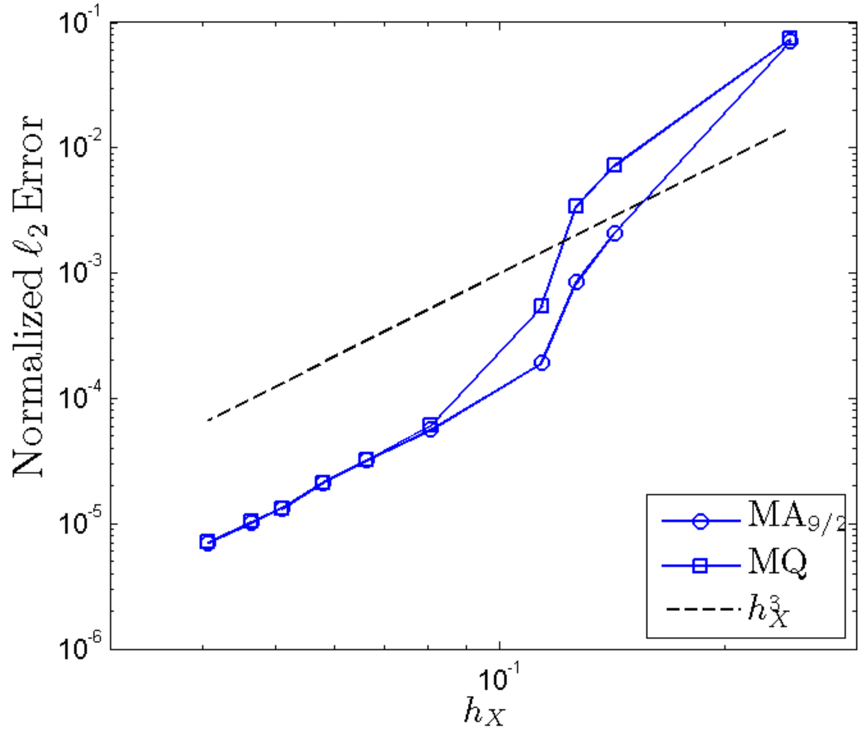
- Dashed line predicted error rate for Matérn (Fuselier and Wright 2008)

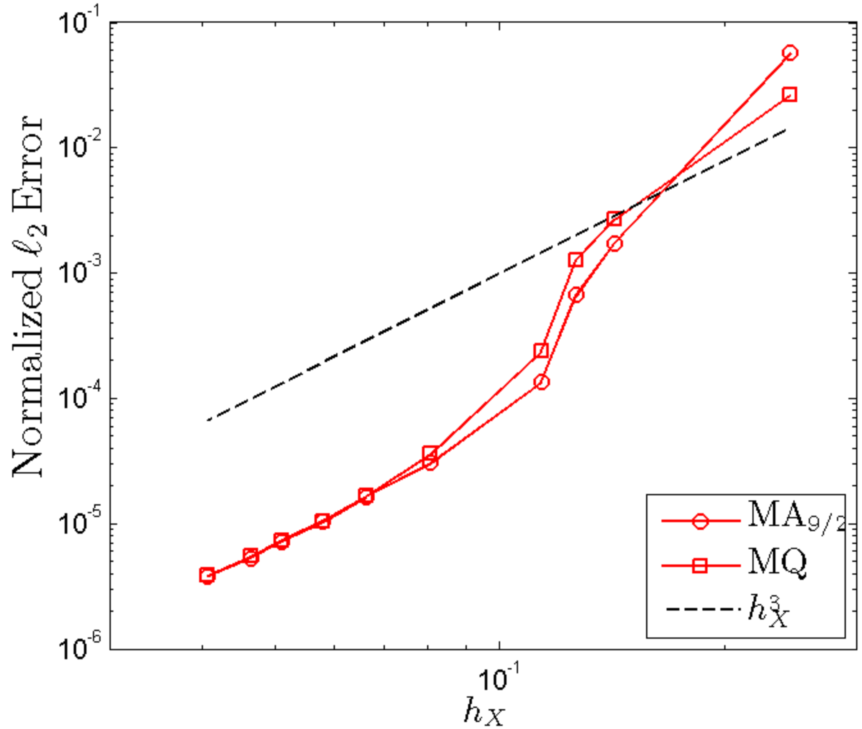
Helmholtz-Hodge Decomposition: Example 1

- Tangent vector field **u**(**x**) sampled at "scattered" nodes on the sphere.



- Error in the RBF reconstruction of the surface div-free and curl-free parts vs. node spacing:

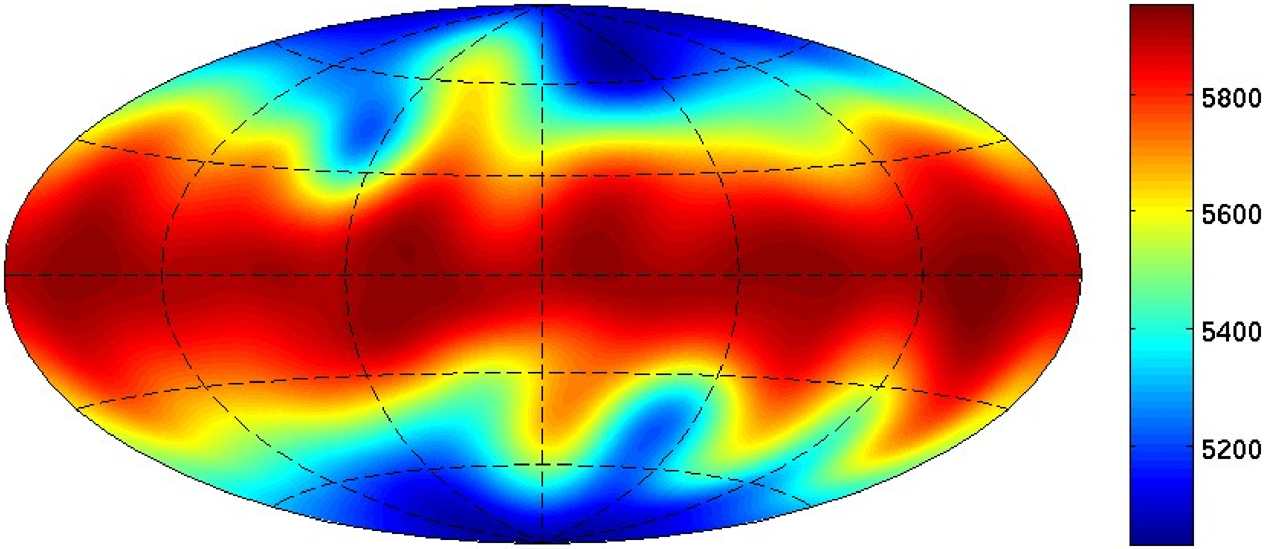




Helmholtz-Hodge Decomposition: Example 2

- Test case 5 (flow over an isolated mountain) from Williamson *et. al.* JCP (1992).

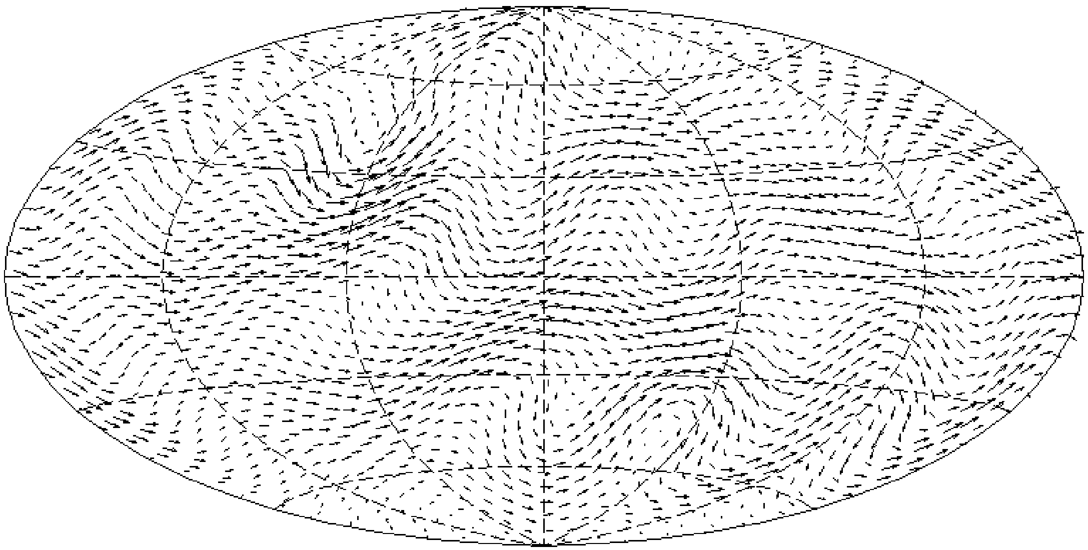
Height field t=15 days



Solution details

- GME SWM (Majewski *et. al.* MWR 2002)
- Icosahedral grid point model (92162 grid points).

Velocity field **u** t=15 days



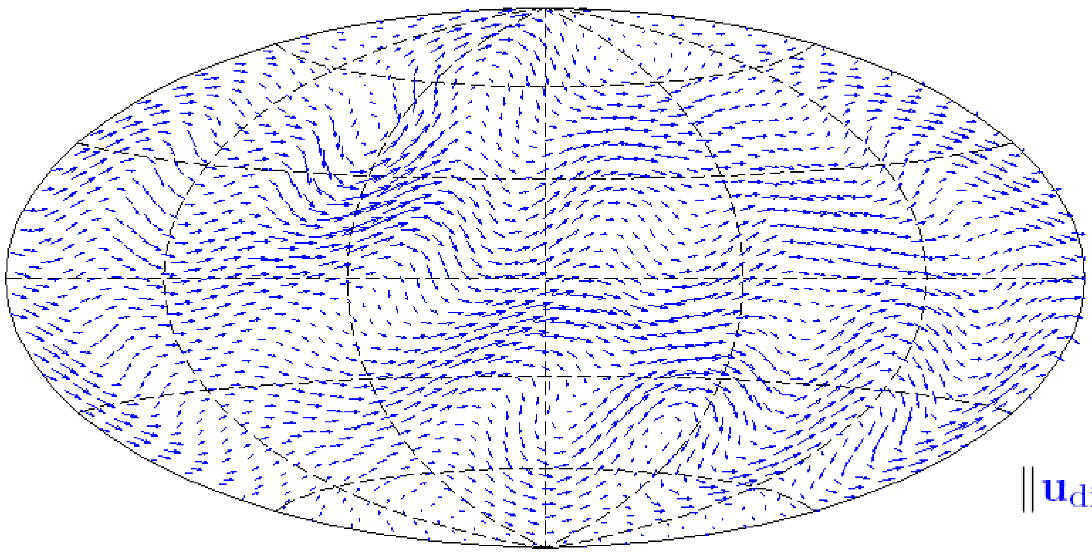
Sample details

- Sample velocity field of GME solution at N=1849 “scattered” nodes.
- Construct Helmholtz-Hodge RBF interpolant using Matérn $MA_{9/2}$ RBF

Helmholtz-Hodge Decomposition: Example 2

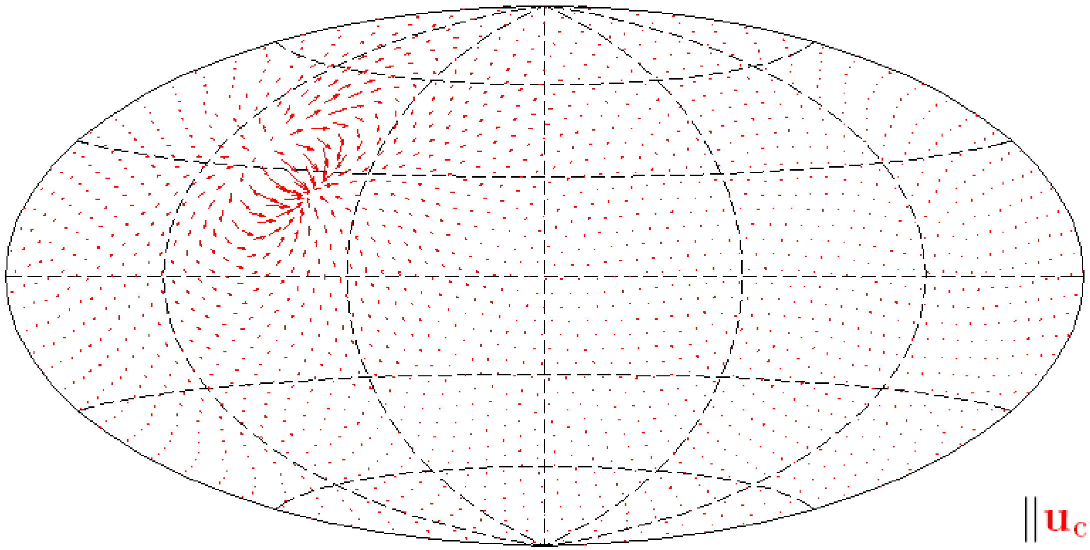
- Test case 5 (flow over an isolated mountain) from Williamson *et. al.* JCP (1992).

RBF reconstructed div-free velocity field t=15 days



$\|\mathbf{u}_{\text{div}}\|_2 = 40.3$ m/s

RBF reconstructed curl-free velocity field t=15 days

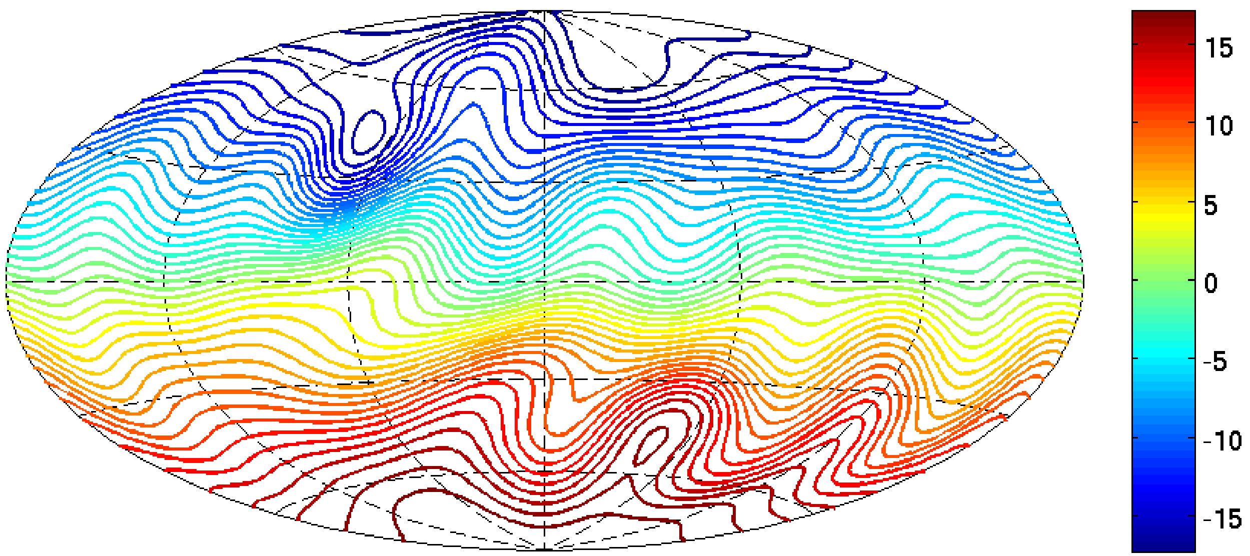


$\|\mathbf{u}_{\text{curl}}\|_2 = 4.3$ m/s

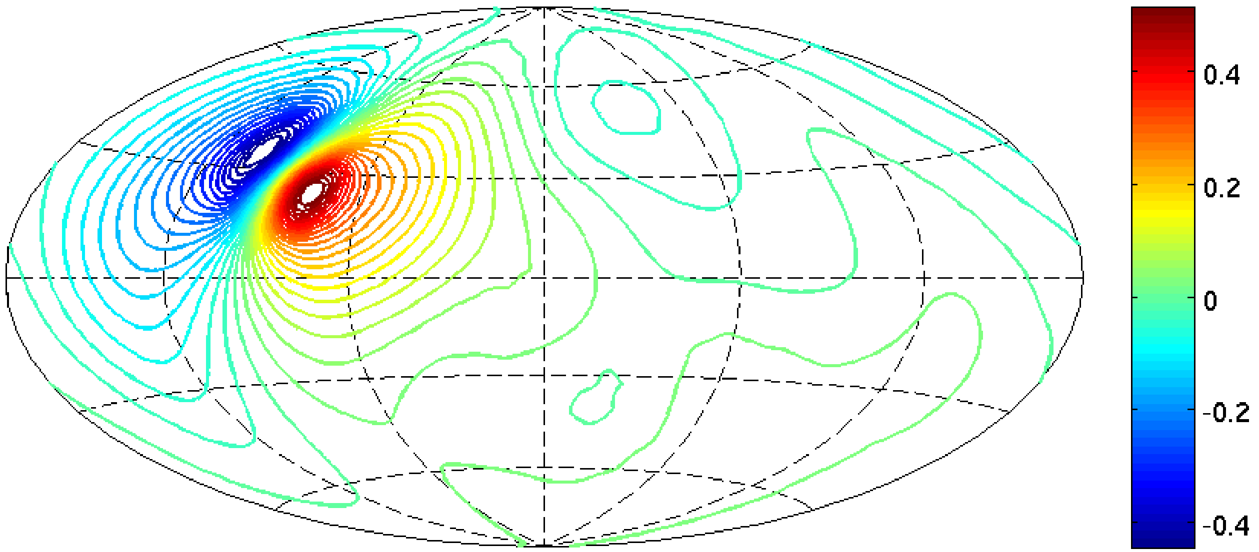
Helmholtz-Hodge Decomposition: Example 2

- Test case 5 (flow over an isolated mountain) from Williamson *et. al.* JCP (1992).

RBF reconstructed stream function *t*=15 days



RBF reconstructed velocity potential *t*=15 days

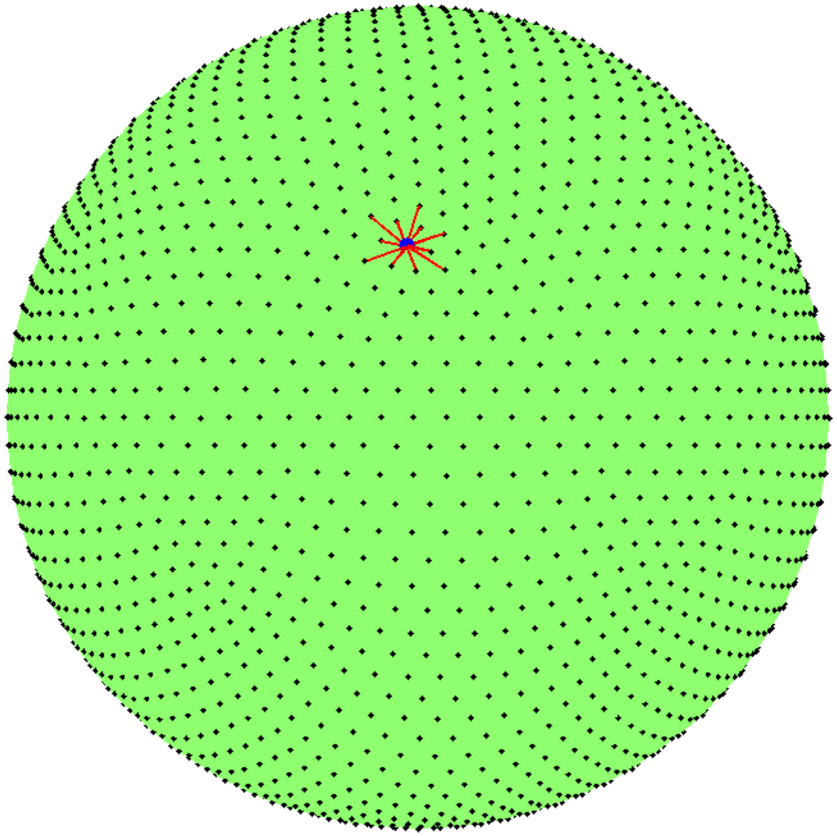


Scalability: moving from global to local

- This technique can also be used on patches of the sphere to locally approximate:

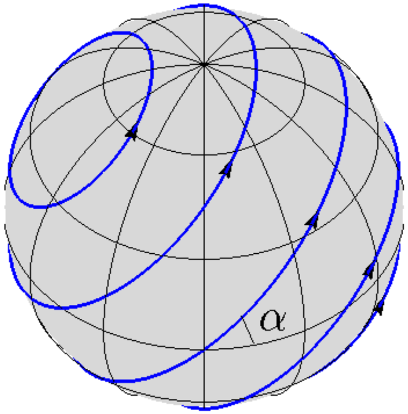
  1) Surface divergence-free or curl-free vector fields
  2) General vector fields tangent to the sphere
  3) Stream functions and velocity potentials
  4) Vorticity
  5) Divergence

- Illustration of a local approximation



Example: linear advection

- Governing PDE to simulate: $\dfrac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = 0$



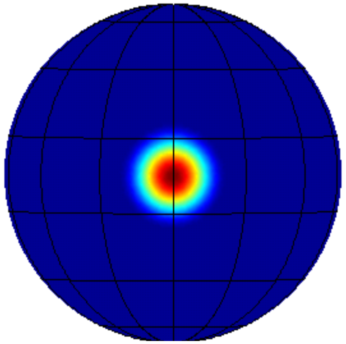
$$\begin{aligned} \mathbf{u} &= [u_s, v_s]^T = u_0[(\cos\alpha\cos\theta + \sin\alpha\cos\lambda\sin\theta), -\sin\alpha\sin\lambda]^T \\ \mathbf{u} &= [u_c, v_c, w_c]^T = u_0[-y\cos\alpha, x\cos\alpha + y\sin\alpha, -y\sin\alpha]^T \end{aligned}$$

- Simulation strategy (*method of lines*):
  - Rewrite advection equation as:

$$\frac{\partial h}{\partial t} = -\nabla \cdot \underbrace{(\mathbf{u}h)}_{\mathbf{f}}$$

  - Approximate $\mathbf{f}$ locally at each node point using Helmholtz-Hodge RBF interpolant.
  - Compute divergence of local interpolants.
  - Advance the system in time.

- Example: advection of a cosine bell (Test Case 1 of Williamson *et. al.* JCP (1992)).



- Icosahedral grid, *N=40962* nodes
- 13 point local approximation
- Standard RK4 with 30 min. time step
- MQ RBF

Simulation

Concluding remarks

- Can approximate vector fields that are either surface divergence or curl-free with RBFs.
  - For div-free fields, RBF reconstructed field will be div-free.
  - For curl-free fields, RBF reconstructed field will be curl-free.

- Can approximate general vector fields tangent to the sphere with RBFs. From the RBF reconstructed field, we can
  - Approximate the div-free and curl-free parts.
  - Generate a stream function and velocity potential for the field.
  - Approximate vorticity and divergence of the field.

- All this can also be done on local patches of the sphere.
  - Local technique offers nice scaling abilities (petascale?).
  - Can be used in many current geophysical models as a local interpolation method for vector fields.
  - Can also possibly be used as the basis functions for new computational models.

- Approximating “real” data.