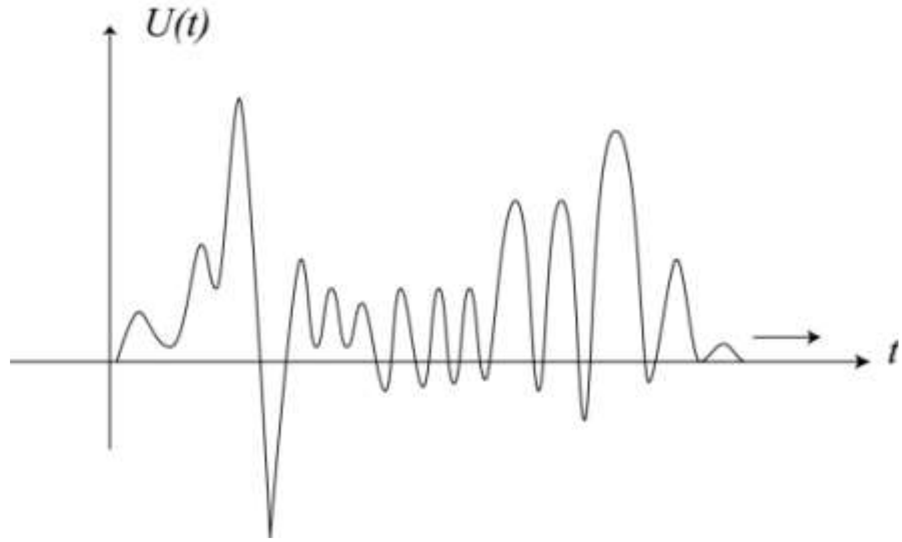
Tutorial 1 – Turbulence, J.H. Fernando

1. A velocity record taken from a probe placed in a certain flow is shown below.



What are the tests would you perform to ascertain that this flow is turbulent?

How do you experimentally show that a given turbulent flow is isotropic?

2. The Turbulent kinetic energy equation can be written in usual notation as:

$$\frac{\partial \overline{u_i^2}/2}{\partial t} + \overline{U}_j \frac{\partial \overline{u_i^2}/2}{\partial t} = -\frac{\partial}{\partial x_j}\left(\overline{pu_j} + \overline{u_i^2 u_j}/2 + \underbrace{v\frac{\partial \overline{u_i^2/2}}{\partial x_j}}_{I}\right) + \overline{bu_3} \underbrace{- v\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \overline{u_i u_j}\frac{d\overline{U}_i}{dx_j}}_{II}$$

A) Explain the physical meaning of each term.
B) Show that in the absence of body forces it will be impossible to obtain a homogeneous and stationary turbulent flow.
C) The above equation is independent of Coriolis forces, even if it is written in a frame of rotation. Why?
D) Estimate the order of magnitude if the ratio of terms II / I.

3. A) Define the turbulent kinetic energy dissipation rate ε in terms of the rate of strain tensor S_{ij}. Show that for a homogeneous turbulent flow

$$\varepsilon = v\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}},$$ where ν is the kinematic viscosity.

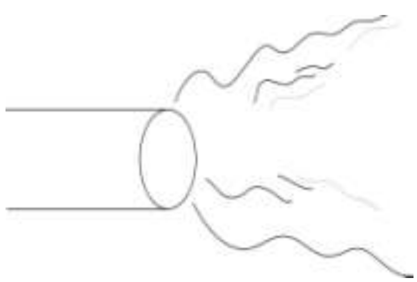
B) For homogeneous turbulent flows, how should vorticity fluctuations relate to ε ?

C) ε plays an important role in describing turbulent flows, and it links large and small scales. Discuss this statement.

D) Define the Taylor microscale. Using scaling arguments, find the scale relationship between Taylor microscale and the Integral lengthscale in the form of $\lambda/l \sim \mathrm{Re}^{-(power)}$.

E) Use scaling arguments to show how the separation between large and small scales of a turbulent flow drastically increases with increasing Reynolds number. For a Reynolds number of 10^4, how many mesh points you think would be required to calculate a turbulent flow using direct numerical simulation? Assuming that the flow at a given grid point is dominated by small structures advected by large structures, can you estimate how many time steps per large structure passage time are required to calculate the flow.

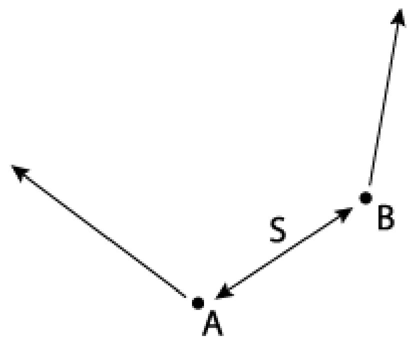
4.



Consider a turbulent jet ejecting into a quiescent ambience. The jet is colored with fluorescent dye (diffusivity $k_d = 10^{-9} m^2 s^{-1}$) and the average rate of dissipation in the jet is $\varepsilon \approx 10^{-6} m^2 s^{-3}$. The kinematic viscosity of water is $\nu = 10^{-6} m^2 s^{-1}$. It is planned to image the concentration field using a CCD camera and obtain the fractal dimension of the flow. What should be the appropriate resolution of the camera? Estimate the spatial and temporal resolution of a velocity probe that is need to measure the kinetic energy dissipation?

5. A) Using the two point correlation function $\overline{u_i(x)u_j(x+r)}$ for a homogeneous turbulent flow, explain the concept of integral length scale of turbulence. Why is L_{11} (longitudinal correlation function) typically larger than L_{22} (transverse velocity correlation function)?

B) Consider two particles that are placed at a distance *s* apart, and *s* is in the inertial subrange of background turbulence. How should $(\overline{s^2})^{1/2}$ vary with time?



4. Consider the vertical displacement $Y(t)$ of fluid particles released at $t=0$ from a line $Y=0$, where $\dfrac{dY}{dt}=v(t)$.

These fluid particles move with a velocity $v(t)$, which is a homogeneous, stationary velocity field. The Lagrangian auto-correlation of $v(t)$ is $\rho(\tau)=\overline{v(t)v(t+\tau)}$ and its frequency spectrum is $\Phi(\omega)$.

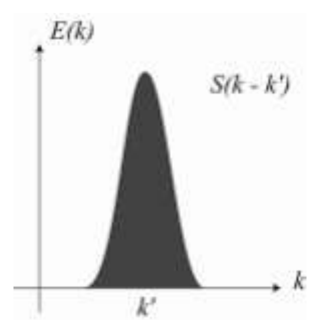
(i) Sketch a graph of $Y(t)$ for several realizations. Explain why $Y(t)$ is a non-stationary process.

(ii) Show that

$$\overline{Y}^2(t)=2\left[t\int_0^t \rho(\tau)\,d\tau-\int_0^t \tau\rho(\tau)\,d\tau\right]$$

(iii) Form the Fourier Transform pair relation between $\rho(\tau)$ and $\Phi(\omega)$ and show that as $\left(\dfrac{t}{T_L}\right)\to\infty,\ \overline{Y}^2\approx 2t\overline{v^2}T_L\approx 2\pi t\Phi(\omega=0)$, where $T_L=\dfrac{1}{\overline{v^2}}\int_0^\infty \rho(\tau)\,d\tau$

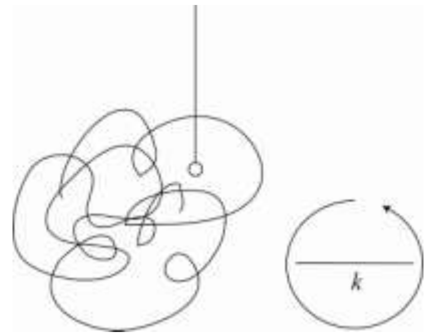
5. Consider a spiky spectra in wave number space. Draw approximate one dimensional spectral shapes $F_{11}(k_1)$ and $F_{22}(k_2)$ for this flow.



If a flow is injected with energy using the given spectra, qualitatively explain how the flow evolves at later times.

6. It has been found that the evolution of the integral length scale of decaying isotropic turbulence can be given by $l = c_1 t^{1/2}$. Using this empirical result, develop a simple one-equation turbulence model for the decay of isotropic turbulence.

7. Consider a shear-free turbulent flow situation with no mean flow. A probe is placed inside the fluid to detect the velocity fluctuations. Estimate the largest frequency recorded by the probe. *u* is the *r.m.s.* velocity.



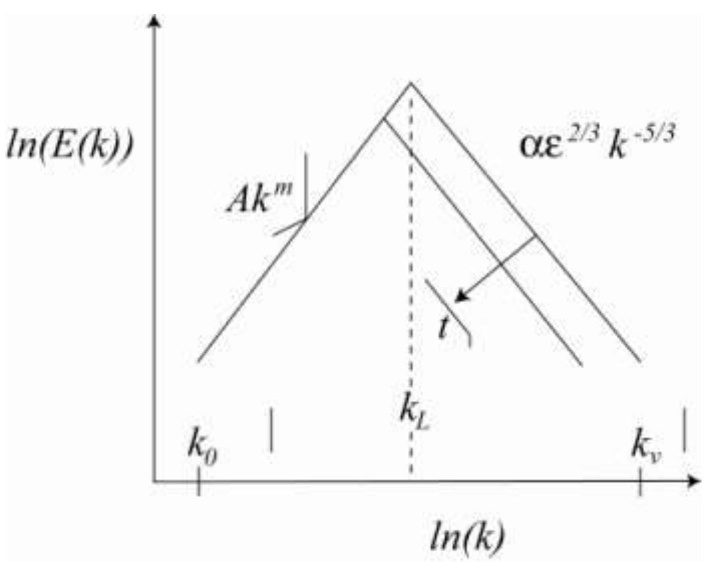
8. The dynamic equation for energy spectrum is written as (for isotropic turbulence)

$$\frac{\partial E(k)}{\partial t} = F(k) - 2\nu k^2 E(k)$$

(a) Identify each term and give its physical significance.
(b) Plot each of these terms on a wave number plot.
(c) Convert the above equation to the energy equation in physical space by using a simple manipulation.

10. What do you mean by 'return to isotropy'? Which terms in the energy equation perform this task? Write down a form of tensor (second order) that can be used to represent the return to isotropy.

(11) Consider the model spectrum shown in the figure:



It can be represented as:

$$E(k) = Ak^{m} \qquad \text{for } k_0 < k < k_L$$

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3} \text{ for } k_L < k < k_v$$

Assume that *A* is constant during the decay of turbulence, with spectrum changing only at wave numbers $k > k_L$ (based on the concept of permanence of large eddies). Show that turbulent kinetic energy $q^2/2$ should decay according to

$$\frac{q^2}{q_0{}^2} = \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\left(\frac{2m+2}{3m+5}\right)},$$

where '*0*' represents initial conditions.

(12) Consider decay of isotropic turbulence, and the change of ε can be parameterized as

$$\frac{d\varepsilon}{dt} = f(\varepsilon, q^2).$$

Argue that the above expression is consistent with the Kolmogorov hypothesis, and that

$$\frac{d\varepsilon}{dt} = c_1\frac{\varepsilon^2}{q^2}.$$

where c_1 is a constant. Also show that $q^2 = \overline{u_i' u_i'}$ varies according to

$$q^2 = q_0{}^2\theta^{-2/(c_1-2)}$$
$$\varepsilon = \varepsilon_0\theta^{-c_1/(c_1-2)}$$

where

$$\theta = 1 + (c_1 - 2)\left(\frac{t\varepsilon_0}{q_0{}^2}\right),$$

and that $c_1 > 2$. For $t >> q_0{}^2/\varepsilon_0$, show that the decay occurs according to

$$q^2 \sim t^{-2/(c_1-2)} \sim t^{-n},$$

and suggest a value for c_1.