Tutorial 3 – Waves in the Environment

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1. (a) What do you understand by the term “kinematic boundary condition?”

(b) Consider a moving surface in a fluid described by F(*x*,*y*,*z*,*t*) = 0. Show that the kinematic condition on the surface is given by

$$\underset{\sim}{u}\,.\,\underset{\sim}{n} = -\frac{\partial F / \partial t}{|\nabla F|}$$

(c) The flow situation in a two-dimensional U tube is shown in Figure 1 below. The undisturbed water surface is at *z* = 0 and the z-axis passes through the symmetry plane. The bottom surface is described by *z* = -*h*(x). The water oscillates so that the free surface displacement is $\eta = a\cos nt$.

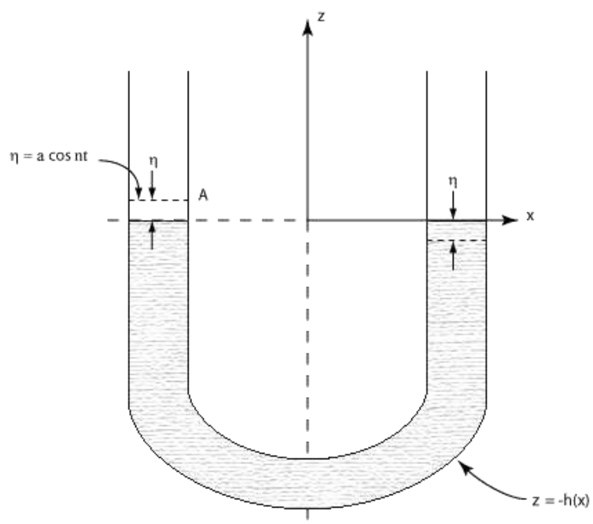

Figure 1.

Show that the kinematic condition at A is given by $w = \partial\eta/\partial t$. Similarly, derive the kinematic condition on the bottom surface at any location (*x*,*z*).

2. Consider an artificial pond with a flat bottom located at $z = -H$ (Figure 2). At equilibrium, the pond is at zero velocity and the pressure distribution is hydrostatic. A surface disturbance of $\eta(x, y, t)$ is given at time $t = 0$. The rotational speed of the coordinate system is Ω, around the *z* axis as shown.

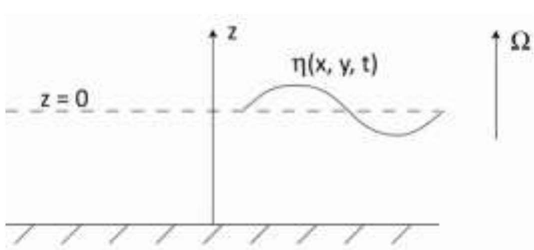

Figure 2.

(i) Write the perturbation (small amplitude) equations for continuity and all three components of momentum

(a)

(b) when $\Omega = 0$

(c) when $\Omega \neq 0$

(ii) Show that when $\Omega = 0$, the pressure is given by the Laplace equations, and the surface displacements can take the form of propagating waves.

(iii) Show that if the horizontal scale of motion L is much larger than the vertical scale H, then the hydrostatic approximation is valid for vertical motions. Show that, under these conditions, the continuity becomes (when $\Omega = 0$)

$$\frac{\partial \eta}{\partial t} = H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

(iv) Discuss briefly how the motion field with $\Omega \neq 0$ is different from the above cases. What type of pathlines would you expect in this case?

3. Consider a uniformly stratified fluid with Buoyancy frequency $N(z)$ and the vertical velocity of the perturbation of isopycnals in this fluid layer is described by the expression

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) + N^2(z)\nabla_h^2 w = 0 \qquad \text{(a)}$$

(i) Show that equation (a) describes wave-like motions in the layer, if the perturbation frequency ω is $\omega < N$. What happens to the perturbations that have frequencies $\omega > N$?

(ii) Consider a channel filled with a stratified fluid as shown in Figure 3. The vertical distribution of the Brunt-Vaisala frequency is also shown. If the system is excited with frequency $N_1 < N(z)$, explain what kind of motions you expect to see in the channel.

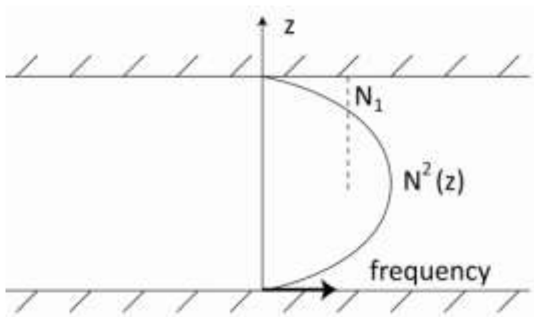

Figure 3.

(Hint: look solutions for (a) in the form of $W(z)e^{i(kx-\omega t)}$, and consider the behavior of disturbance for $N > \omega$ and $N < \omega$.)

(iii) If N^2 is constant, show that the internal waves in an unbounded fluid propagate at an angle θ to the horizontal, where

$$cos\theta = \frac{\omega}{N}$$

and ω is the frequency of the wave.

(4) Consider the propagation of a periodic two-dimensional wave in a tank, as shown in Figure 4. The usual equations for irrotational flow as well as free surface kinematic ($w = \partial\eta/\partial t$) and dynamic boundary conditions, *viz*.,

$$\frac{\partial \phi_1}{\partial t} + g\eta_1 = 0,$$

is applied at *z* = 0. The bottom is solid. Because of the side walls, periodic lateral boundary conditions apply, that is

$\phi_1(x,t) = \phi_1(x_1 + h,T)$ and $\phi(x,t) = \phi_1(x_1, t+T)$.

Assuming solutions of the form

$\phi_1(x,z,t) = Z(z)e^{ikx}\sin(\sigma t)$

show that

$\phi_1 = C\cosh k(h+z)\cos kx \sin\sigma t$

and evaluate *C* in terms of the wave length *H.* Also obtain the dispersion relation for those waves.

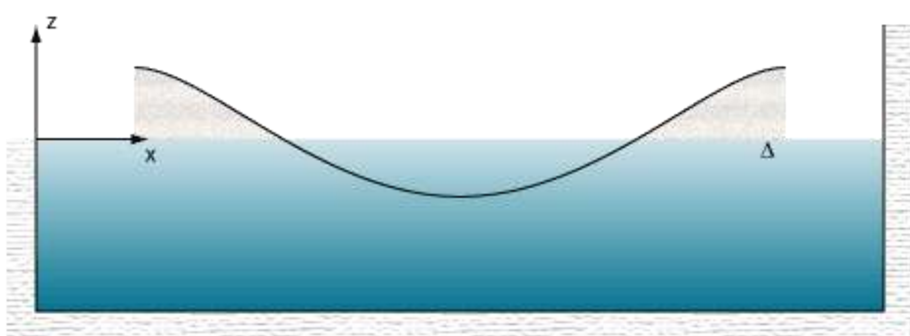

Figure 4.

5. (a) Write down the governing equations of motion for all velocity (*u,v,w*) components and buoyancy (or density) fluctuations for linear internal waves.

   (b) If the vertical velocity of an internal wave propagating with a frequency *n* and wave number $\underset{\sim}{k}$ is given by

   $$w = \alpha \cos\theta \cos(\underset{\sim}{k} \cdot \underset{\sim}{x} - nt),$$

   where α is the speed of oscillations of fluid particles, obtain expressions for other velocity components, pressure and density (buoyancy) oscillations.

   (c) Calculate the averaged buoyancy flux (or the density flux $\overline{\rho' w'}$), momentum fluxes ($\overline{u'w'}$, $\overline{u'v'}$, and $\overline{v'w'}$) and explain why internal waves are poor mass transport agents but good momentum transport agents in stratified atmosphere and oceans.

   (d) Evaluate the energy transport due to pressure flux in the vertical direction.