Lecture 2 Review.

Units analysis and self-similarity.

Majda - Class Notes - Fall Courant 2007

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1 Units

We start with the units of the hot tower scales:

$$
\begin{aligned}
[\vec{x}_h] &= [z] = 10km, \\
[\vec{u}_h] &= [w] = 10m/s, \\
[\theta] &= 3K, \\
[t] &= \frac{[\vec{x}_h]}{[\vec{u}]} \approx 15min, \\
[\theta_0] &= 300K, \\
[N] &= 10^{-2}s^{-1}, \text{ since } N = \sqrt{\frac{g}{\theta_0}\frac{\partial \theta_b}{\partial z}}
\end{aligned}
$$

2 Non-dimensional Boussinesq (Primitive) Equations

$$
\begin{aligned}
\frac{D\vec{u}_h}{Dt} + \varepsilon \sin\psi \vec{u}_h^{\perp} &= -\nabla_h p, \\
\frac{Dw}{Dt} &= -p_z + \varepsilon^{-1}(\theta - S_w), \\
\frac{D\theta}{Dt} &= \varepsilon^{-1}(-w + S_\theta), \\
\mathrm{div}_h \vec{u}_h + w_z &= 0.
\end{aligned}
$$

Note the horizontal momentum source from the clouds S_w. $S_\theta \neq 0$ is a physical source term which comes from Hot Towers.

$$[S_\theta] = 120K/hr.$$

What is ε? It is Froud number

$$\varepsilon = \frac{1/N}{H/[u_h]} = \frac{[u_h]}{NH} = \frac{\text{buoyancy time}}{\text{eddy turnover time}} = \frac{[10]}{[10^{-2}][10^4]} = 10^{-1},$$

So we have a low Froud number. Now, potential temperature

$$\theta_t = \varepsilon^{-2} + \varepsilon^{-1} z + \theta,$$

where $\varepsilon^{-2} = \theta/[\theta_0]$, $\varepsilon^{-1} = N^2\theta_0 [L]/g[\theta] = 10 = O(\varepsilon^{-1})$. We have weak horizontal temperature gradient (WTG).

3 Balanced dynamics (Hot Towers) $S_w \neq 0$.

In the zeroth order in ε we have

$$
\begin{aligned}
\frac{D\vec{u}_h}{Dt} &= -\nabla_h p, \\
\theta &= S_w \quad \text{never needed for dynamics}, \\
w &= S_\theta, \\
\mathrm{div}_h \vec{u}_h + w_z &= 0.
\end{aligned}
$$

4 Self-similarity

Consider a constant A and rescale horizontal coordinates and time

$$\vec{u}_h(A\vec{x}_h, z, At),$$
$$\theta(A\vec{x}_h, z, At),$$
$$p(A\vec{x}_h, z, At),$$
$$Aw_A = w(A\vec{x}_h, z, At),$$
$$AS_{\theta,A} = S_\theta.$$

Now, we rescale the equation from Section 2 using $\vec{X}_h = A\vec{x}_h$, $T = At$, and $D/DT = \partial/\partial T + \vec{u}_h \cdot \nabla_X + w_A \partial/\partial z$. We choose $A = \varepsilon$ because the rotation effect becomes of the order O(1).

$$\begin{aligned}\left[\vec{X}_h\right] &= 100km,\\ [T] &= 2.5hrs.\end{aligned}$$

We can drop the A subscript in the notations and by setting $f = \sin\psi$ we find

$$\begin{aligned}\frac{D\vec{u}_h}{DT} + f\vec{u}_h^{\perp} &= -\nabla_h p,\\ A^2\frac{Dw}{DT} &= -p_z + \varepsilon^{-1}(\theta - S_w),\\ \frac{D\theta}{DT} &= \varepsilon^{-1}(-w + S_\theta),\\ \mathrm{div}_X\vec{u}_h + w_z &= 0.\end{aligned}$$

Let us check that the balanced equation are also the same as for smaller scales except for the rotation, which now becomes important.

$$\begin{aligned}\frac{D\vec{u}_h}{DT} + f\vec{u}_h^{\perp} &= -\nabla_X p,\\ w &= S_\theta,\\ \mathrm{div}_X\vec{u}_h + w_z &= 0,\\ \theta &= S_w.\end{aligned}$$