Lecture 3 Review
Edited

Majda - Class Notes - Fall Courant 2007

10/17/2007

1 Goal:

Starting from the model:

$$
\begin{aligned}
&\frac{D\mathbf{u}_h}{Dt} + \epsilon \sin\phi \mathbf{u}_h^{\perp} = -\nabla_h p, \\
&\frac{Dw}{Dt} = -p_z + \epsilon^{-1}(\theta - S_w), \\
&\frac{D\theta}{Dt} = \epsilon^{-1}(-w + S_\theta), \\
&div_h \mathbf{u}_h + w_z = 0,
\end{aligned} \tag{1.1}
$$

we are building a multiscale model on the following scales:

$$
\begin{aligned}
[t] &= 15min \\
[\mathbf{x}] &= 10km \\
[X] &= 100km
\end{aligned}
$$

It has two spatial scales and one time scale.

2 Notation

Denote the function by $f(\mathbf{x}, \epsilon\mathbf{x}, z, t)$ with $\epsilon\mathbf{x} = \mathbf{X}$. Then we have:

$$div_h(f) = div_{\mathbf{x}}(f) + \epsilon\, div_{\mathbf{X}}(f).$$

Spatial average of f is denoted by $\bar{f}$:

$$\bar{f}(\mathbf{X}, z, t) = \lim_{L\to\infty} \frac{1}{(2L)^2} \int_{-L}^{L}\int_{-L}^{L} f(\mathbf{x}, \mathbf{X}, z, t)d\mathbf{x} = \lim_{L\to\infty} \frac{1}{(2L)^2} \int_{-L}^{L}\int_{-L}^{L} f(\mathbf{x}, \mathbf{X}, z, t)dxdy. \tag{2.2}$$

Time average of f is denoted by $\langle f\rangle$:

$$\langle f\rangle(\mathbf{x}, \mathbf{X}, z) = \lim_{T^*\to\infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} f(\mathbf{x}, \mathbf{X}, z, t)dt. \tag{2.3}$$

Properties of average:

a) $f = \bar{f} + f'$ with $\bar{f'} = 0$;

b) g is sublinear $\Rightarrow \overline{\frac{\partial g}{\partial x}} = 0$ and $\overline{\frac{\partial g}{\partial y}} = 0$;

Proof:

$$\overline{\frac{\partial g}{\partial x}} = \lim_{L\to\infty} \frac{1}{2L} \int_{L}^{L} \frac{\partial g}{\partial x} dx = \lim_{L\to\infty} \frac{g(L) - g(-L)}{2L} = 0,$$

c) $\overline{fg} = \bar{f}\bar{g} + \overline{f'g'}$;

Proof:

$$\overline{fg} = \overline{(\bar{f} + f')(\bar{g} + g')} = \overline{fg} + \overline{f'g'} + \underbrace{\overline{f'\bar{g} + \bar{f}g'}}_{=0} = \overline{fg} + \overline{f'g'}.$$

As a general example consider the following Fourier representation in a small scale variable

$$u(x, X, t) = \overline{u}_0(X, t) + \left(\sum_{k_j>0} a_j(X, t)e^{ik_jx} + c.c.\right) \tag{2.4}$$

Here, we have

$$\overline{u(x, X, t)} = \overline{u}_0(X, t), \tag{2.5}$$

and

$$\overline{u^2(x, X, t)} = \overline{u}_0^2(X, t) + 2\sum_{k_j>0} |a_j(X, t)|^2. \tag{2.6}$$

In order to demonstrate this we use

$$\overline{e^{ik_j x}} = 0, \tag{2.7}$$

due to the property b) of space averaging.

3 Derivation of Model

Using the relation that:

$$\begin{aligned}
\frac{D}{Dt}f &= \frac{\partial}{\partial t}f + \mathbf{u}_h \cdot \nabla_h f + w\frac{\partial}{\partial z}f \\
&= \frac{\partial}{\partial t}f + div_h(\mathbf{u}_h f) - f\, div_h \mathbf{u}_h + (wf)_z - f w_z \\
&= \frac{\partial}{\partial t}f + div_h(\mathbf{u}_h f) + (wf)_z - f(div_h \mathbf{u}_h + w_z) \\
&= \frac{\partial}{\partial t}f + div_h(\mathbf{u}_h f) + (wf)_z
\end{aligned}$$

We can write the model (1.1) in the conservative form:

$$\begin{aligned}
&\frac{\partial \mathbf{u}_h}{\partial t} + div_h(\mathbf{u}_h \otimes \mathbf{u}_h) + (w\mathbf{u}_h)_z + \epsilon \sin\phi \mathbf{u}_h^{\perp} = -\nabla_h p, \\
&\frac{\partial w}{\partial t} + div_h(\mathbf{u}_h w) + (w^2)_z = -p_z + \epsilon^{-1}(\theta - S_w), \\
&\frac{\partial \theta}{\partial t} + div_h(\mathbf{u}_h \theta) + (w\theta)_z = \epsilon^{-1}(-w + S_\theta), \\
&div_h \mathbf{u}_h + w_z = 0. \qquad (3.8)
\end{aligned}$$

We begin with the following ansatz of scale separation:

$$
\begin{aligned}
w &= \epsilon\bar{w}(\mathbf{X}, z, t) + w'(\mathbf{x}, \mathbf{X}, z, t),\\
\mathbf{u}_h &= \bar{\mathbf{u}}_h(\mathbf{X}, z, t) + \mathbf{u}'_h(\mathbf{x}, \mathbf{X}, z, t),\\
\theta &= \bar{\theta}(\mathbf{X}, z, t) + \theta'(\mathbf{x}, \mathbf{X}, z, t),\\
p &= \epsilon^{-1}\bar{p}(\mathbf{X}, z, t) + p'(\mathbf{x}, \mathbf{X}, z, t),
\end{aligned}
\tag{3.9}
$$

and the forcing assumptions:

$$
\begin{aligned}
\epsilon^{-1}S_\theta &= \epsilon^{-1}S'_{\theta,-1} + \bar{S}_\theta,\\
\epsilon^{-1}S_w &= \epsilon^{-1}(S'_{w,-1} + \bar{S}_{w,-1}).
\end{aligned}
$$

Hence the model (3.8) takes the form:

$$
\begin{aligned}
&\frac{\partial \mathbf{u}_h}{\partial t} + div_h(\mathbf{u}_h \otimes \mathbf{u}_h) + (w\mathbf{u}_h)_z + \epsilon \sin\phi \mathbf{u}_h^{\perp} = -\nabla_h p,\\
&\frac{\partial w}{\partial t} + div_h(\mathbf{u}_h w) + (w^2)_z = -p_z + \epsilon^{-1}(\theta - S'_{w,-1} - \bar{S}_{w,-1}),\\
&\frac{\partial \theta}{\partial t} + div_h(\mathbf{u}_h\theta) + (w\theta)_z = \epsilon^{-1}(-w + S'_{\theta,-1}) + \bar{S}_\theta,\\
&div_h\mathbf{u}_h + w_z = 0.
\end{aligned}
\tag{3.10}
$$

The reason we separate w and p in a different way (as in (3.9) is explained here. We have

$$
\begin{aligned}
&div_h\mathbf{u}_h + w_z = 0\\
\Rightarrow\quad & div_{\mathbf{x}}\mathbf{u}_h + \epsilon div_{\mathbf{X}}\mathbf{u}_h + w_z = 0\\
\Rightarrow\quad & \overline{div_{\mathbf{x}}\mathbf{u}_h + \epsilon\, div_{\mathbf{X}}\mathbf{u}_h + w_z} = 0\\
\Rightarrow\quad & \epsilon\,\overline{div_{\mathbf{X}}\mathbf{u}_h} + \bar{w}_z = 0.
\end{aligned}
$$

$\bar{w}_z$ is of order ϵ, so does $\bar{w}$. Hence we assume $w = \epsilon\bar{w}(\mathbf{X}, z, t) + w'(\mathbf{x}, \mathbf{X}, z, t)$. For the similar reason, we separate p as $\epsilon^{-1}\bar{p}(\mathbf{X}, z, t) + p'(\mathbf{x}, \mathbf{X}, z, t)$ to make $\overline{\nabla_h p}$ of order 1.

The balanced law for the model (3.10) is:

$$\theta = S'_{w,-1} + \bar{S}_{w,-1},$$
$$w = S'_{\theta,-1},$$
$$\frac{D\mathbf{u}_h}{Dt} = -\nabla_h p,$$
$$div_h \mathbf{u}_h + w_z = 0, \tag{3.11}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\bar{\mathbf{u}}_h + \mathbf{u}'_h) \cdot \nabla_h + w' \frac{\partial}{\partial z}$. We first take the spatial average of (3.11) and then

subtract it from (3.11) to obtain the following balanced law for the **Microscale Model**:

$$\theta' = S'_{w,-1},$$
$$w' = S'_{\theta,-1},$$
$$\frac{D\mathbf{u}'_h}{Dt} = -\nabla_h p' + (\overline{w'\mathbf{u}'_h})_z,$$
$$div_{\mathbf{x}} \mathbf{u}'_h + w'_z = 0. \tag{3.12}$$

Put the ansatz (3.9) into model (3.10) and take the spatial average to obtain the mesoscale model. Taking into account the following relations:

$$
\begin{aligned}
\overline{div_h(\mathbf{u}_h f)} &= \overline{div_{\mathbf{x}}(\mathbf{u}_h f) + \epsilon\, div_{\mathbf{X}}(\mathbf{u}_h f)} \\
&= \epsilon\, \overline{div_{\mathbf{X}}(\mathbf{u}_h f)} = O(\epsilon) \\
\overline{(wf)_z} &= (\bar{w}\bar{f})_z + \overline{(w'f')_z} = \overline{(w'f')_z} + O(\epsilon)
\end{aligned}
$$

we have the following **Macroscale Model**:

$$
\begin{aligned}
&\frac{\partial \bar{\mathbf{u}}_h}{\partial t} = -\nabla_{\mathbf{X}}\bar{p} - (\overline{w'\mathbf{u}'_h})_z, \\
&\bar{p}_z = \bar{\theta} - \bar{S}_{w,-1}, \\
&\frac{\partial \bar{\theta}}{\partial t} = -\bar{w} + \bar{S}_\theta - (\overline{w'\theta'})_z, \\
&div_{\mathbf{X}}\bar{\mathbf{u}}_h + \bar{w}_z = 0.
\end{aligned} \tag{3.13}
$$

We have derived two coupled systems of equations for microscales (3.12) and macroscales (3.13). In system (3.12), the first equation is often disregarded in the sense that θ' is assumed to vanish. This is called Weak Temperature Gradient approximation. The source term in the second equation comes from condensation (rain) and/or sun radiation, which influence vertical dynamics. The horizontal dynamics is described by the third equation. Note that, in the third equation, we have the term $(\overline{w'\mathbf{u}'_h})_z$ which is in accordance with a fact that by definition $\overline{\mathbf{u}'}_h = 0$. Note that the averaged horyzontal velocity $\bar{\mathbf{u}}_h$ is taken into account here through the material derivative. Thus we have the effect of the macroscales on the microscales. On the other hand, the system (3.13) is a traditional linearized system of Boussinesq equations for large scales. Note that here, the coupling of the small and large scales shows up in the first and third equations for the horizontal and vertical system dynamics.