13 Appendix II. Non-dimensionalization of the Equations

Here, we briefly present the derivation of the non-dimensionalized equations. We start from the Boussinesq equations

\[
\frac{Du}{Dt} + fu^\perp = - \nabla_h p + S_u, \quad (98)
\]

\[
\frac{Dw}{Dt} = - \frac{\partial p}{\partial z} - g \frac{\rho'}{\rho_0}, \quad (99)
\]

\[
\frac{D\theta_t}{Dt} = S_\theta, \quad (100)
\]

\[
\nabla_h \cdot u + w_z = 0, \quad (101)
\]

with

\[
\theta_t = \Theta_0 + \theta_{bg}(z) + \theta, \quad (102)
\]

\[
\Theta_0 = 300 \text{ K}, \quad (103)
\]

\[
\frac{d\theta_{bg}}{dz} = \frac{N^2 \Theta_0}{g}, \quad (104)
\]

\[
\frac{\rho'}{\rho_0} = \frac{\theta}{\Theta_0}, \quad (105)
\]

and

\[
N = -\frac{g}{\rho \frac{dp}{dz}} = 10^{-2} \text{s}^{-1}
\]

is the Brunt-Väisälä buoyancy frequency. Thus the system (98)-(101) becomes

\[
\frac{Du}{Dt} + fu^\perp = - \nabla_h p + S_u, \quad (106)
\]

\[
\frac{Dw}{Dt} = - \frac{\partial p}{\partial z} + g \frac{\theta}{\Theta_0}, \quad (107)
\]

\[
\frac{D\theta}{Dt} + w \frac{d\theta_{bg}}{dz} = S_\theta, \quad (108)
\]

\[
\nabla_h \cdot u + w_z = 0, \quad (109)
\]
Let

\[ u = U \hat{u} \tag{110} \]
\[ w = W \hat{w} \tag{111} \]
\[ p = P \hat{p} \tag{112} \]
\[ \theta = \Theta \hat{\theta} \tag{113} \]
\[ x = L \hat{x} \tag{114} \]
\[ z = H \hat{z} \tag{115} \]
\[ t = T \hat{t} \tag{116} \]
\[ S_\theta = \Sigma_\theta \hat{S}_\theta \tag{117} \]
\[ S_u = \Sigma_u \hat{S}_u \tag{118} \]

with

\[ L = H \tag{119} \]
\[ U = \frac{W}{T} = \frac{L}{T} \tag{120} \]
\[ \Theta = \frac{N \Theta_0 L}{g T} \tag{121} \]
\[ \Sigma_\theta = \frac{\Theta}{T} \tag{122} \]
\[ \Sigma_u = \frac{L}{T^2} \tag{123} \]
\[ P = \frac{WH}{T} = \frac{L^2}{T^2} \tag{124} \]

Note that in the above nondimensionalization, the horizontal and vertical scales and velocities are comparable and the unit of time is given by the advection time scale, \( T = \frac{L}{U} \).

Define

\[ \text{Fr} = \frac{U}{NL} = (NT)^{-1} \tag{125} \]
\[ \text{Ro} = \frac{U}{fL} = (fT)^{-1} \tag{126} \]
Substituting (110)-(118) in (106)-(109) we obtain

\[
\frac{L}{T^2} \frac{D\hat{u}}{Dt} + \frac{L}{T} f\hat{u}^1 = -\frac{1}{L} \hat{\nabla}_h \left( \frac{L^2}{T^2} \hat{p} \right) + \frac{L}{T^2} \hat{S}_u,
\]  
(127)

\[
\frac{L}{T^2} \frac{D\hat{w}}{Dt} = -\frac{1}{L} \frac{\partial}{\partial z} \left( \frac{L^2}{T^2} \hat{p} \right) + g \frac{N\Theta_0 L}{gT} \hat{\Theta},
\]  
(128)

\[
\Theta \frac{D\hat{\Theta}}{Dt} + \left( \frac{L}{T} \hat{w} \right) \frac{N^2 \Theta_0}{g} = \frac{\Theta}{T} \hat{S}_\Theta,
\]  
(129)

\[
\frac{L}{T^2} \left( \hat{\nabla} \cdot \hat{u} + \hat{w}_z \right) = 0,
\]  
(130)

or

\[
\frac{D\hat{u}}{Dt} + fT\hat{u}^1 = -\hat{\nabla}_h \hat{p} + \hat{S}_u,
\]  
(131)

\[
\frac{D\hat{w}}{Dt} = -\frac{\partial \hat{p}}{\partial z} + N\hat{T}\hat{\Theta},
\]  
(132)

\[
\frac{D\hat{\Theta}}{Dt} + g \frac{L N^2 \Theta_0}{\Theta} \hat{w} = \hat{S}_\Theta,
\]  
(133)

\[
\hat{\nabla} \cdot \hat{u} + \hat{w}_z = 0.
\]  
(134)

Note that \(NT = Fr^{-1}\) in (132). Moreover, the coefficient of \(\hat{w}\) in (133) becomes

\[
\frac{L N^2 \Theta}{\Theta g} = \frac{NT N\Theta_0 L}{\Theta gT} = \frac{NT \Theta}{\Theta} = NT = Fr^{-1}.
\]  
(135)

Dropping the “hat” in (131)-(134), one obtains the non-dimensional system

\[
\frac{Du}{Dt} + Ro^{-1}u^1 = -\nabla_h p + S_u
\]  
(136)

\[
\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + Fr^{-1}\theta
\]  
(137)

\[
\frac{D\Theta}{Dt} + Fr^{-1}w = S_\Theta
\]  
(138)

\[
\nabla_h \cdot u + w_z = 0
\]  
(139)
We now fix the reference magnitudes as following:

\[ L = H = 10 \text{km} = 10^4 \text{m} \quad (140) \]
\[ T = 15 \text{min} = 900 \text{s} \quad (141) \]
\[ \Theta = \frac{N \Theta_0 L}{gT} = \frac{0.01 \times 300 \times 10^4}{10 \times 900} \approx 3 \text{ K} \quad (142) \]
\[ \Sigma_\theta = \frac{\Theta}{T} = \frac{3 \text{ K}}{15 \text{ min}} \quad (143) \]
\[ \Sigma_u = \frac{L}{T_T^2} \quad (144) \]
\[ P = \frac{WH}{T} = \frac{L^2}{T_T^2} \quad (145) \]

Note that the above reference magnitudes correspond to small Froude number

\[ Fr = \frac{U}{NL} = (NT)^{-1} = \frac{1}{0.01 \times 900} \approx 0.1 = \epsilon, \quad (146) \]

with \( \epsilon = 0.1 \). Further,

\[ \text{Ro}^{-1} = fT = 2\Omega \sin \phi_0 T = 2 \left( \frac{2\pi}{24 \times 3600 \text{s}} \right) (\sin \phi_0) (15 \times 60 \text{s}) \approx \epsilon \sin \phi_0, \quad (147) \]

where \( \Omega = \frac{2\pi}{24 \times 3600 \text{s}} \) is the Earth rotation frequency. The approximate magnitude of the heating rate observed in the “hot towers” is \( \frac{60^\circ \text{K}}{\text{hour}} \) or \( \frac{30^\circ \text{K}}{15 \text{ minutes}} \) which occur on scales of order 10 kilometers through moist deep convection in the hurricane embryo (Hendricks et al. 2004; Montgomery et al. 2006). Such a heating rate \( \frac{30^\circ \text{K}}{15 \text{ minutes}} \) is strong compared with the reference magnitude \( \Sigma_\theta = \frac{3 \text{K}}{15 \text{min}} \). Hence, in this paper, \( S_\theta \) is given the magnitude \( \epsilon^{-1} \), i.e. \( S_\theta = \epsilon^{-1} S_\theta^* \), and the star is dropped for simplicity. Thus, we obtain

\[ \frac{Du}{Dt} + \epsilon \sin \phi_0 u^1 \mathbf{u} = -\nabla_h p + S_u \quad (148) \]
\[ \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \epsilon^{-1} \theta \quad (149) \]
\[ \frac{D\theta}{Dt} = -\epsilon^{-1}(w + S_\theta) \quad (150) \]
\[ \nabla_h \cdot \mathbf{u} + w_z = 0 \quad (151) \]
According the nondimensionalization procedure and the reference magnitude employed here, the above system corresponds to the following situation:

- Weak temperature gradient (WTG); $\Theta = \Theta(z) + \epsilon\theta(x_h, z, t)$
- Small Froude number ($Fr = \frac{U}{N L} = \epsilon$),
- Isotropic scale $L = H$, where $L$ and $H$ are horizontal and vertical spatial scale, respectively,
- Comparable horizontal and vertical velocity magnitudes $U = W$,
- Strong heat sources.