Leslie Smith, Problem 2

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1 Problem A

We require two pieces of information: the definition of σ and a piece of the definition of resonance,

$$\sigma_i^{\pm} = \pm \frac{1}{i} \left(k_z^2 f^2 + k_h^2 N^2 \right)^{\frac{1}{2}} \text{ for } i = k, p, q \tag{1}$$

$$\sigma_k^{\pm} + \sigma_p^{\pm} + \sigma_q^{\pm} = 0, \tag{2}$$

where the latter equation holds for some particular combination of signed terms. Squaring Equation (1),

$$\sigma_i^2 = \left(\frac{k_z^2}{k^2} \right) f^2 + \left(\frac{k_h^2}{k^2} \right) N^2 \text{ for } i = k, p, q. \tag{3}$$

By definition, $k^2 = k_z^2 + k_h^2$, so $\left(\frac{k_z^2}{k^2} \right) + \left(\frac{k_h^2}{k^2} \right) = 1$, meaning that Equation (3) is a weighted average of f and N. Hence,

$$\min(f, N)^2 \leq \sigma_i^2 \leq \max(f, N)^2 \text{ for } i = k, p, q.$$

Since $f, N \geq 0$, positive σ values are bounded analogously,

$$\min(f, N) \leq \sigma_i^+ \leq \max(f, N) \text{ for } i = k, p, q. \tag{4}$$

Equation (2) cannot hold if all terms have the same sign; one must have a different sign from the other two. Without loss of generality, suppose σ_k has the unique sign. Then,

$$\sigma_k^+ = \sigma_p^+ + \sigma_q^+.$$

Applying Equation (4) to bound σ_k from above and $\sigma_p + \sigma_q$ from below yields,

$$\max(f, N) \geq 2\min(f, N),$$

which implies either $\frac{N}{f} \geq 2$ or $\frac{N}{f} \leq \frac{1}{2}$. This result has followed from the assumption (2), which occurs when $\left(\vec{k}, \vec{p}, \vec{q}\right)$ is a resonant mode. Conversely, there are no three-wave resonances with $\frac{1}{2} < \frac{N}{f} < 2$. ■

2 Problem B

Assume $\vec{u}\,(x,t) = \sum_k \sum_{s_k} a_{s_k}\,(t)\,\Phi_{s_k} e^{i\left(\vec{k}\cdot\vec{x}-\sigma_{s_k}t\right)}$ and plug it into

$$\vec{u}_t + 2\Omega \times \vec{u} + \vec{\omega} \times \vec{u} = \nabla \tilde{P}$$

where $\tilde{P} = P - \frac{1}{2}\vec{u}\cdot\vec{u}$ and use $\vec{\omega} = \nabla \times \vec{u} = -s_k k\vec{u}$.

$$\sum_k \sum_{s_k} \left(\frac{\partial a_{s_k}}{\partial t} - \imath\sigma_{s_k} a_{s_k}\right) \Phi_{s_k} e^{i\left(\vec{k}\cdot\vec{x}-\sigma_{s_k}t\right)} + 2\Omega \sum_k \sum_{s_k} \hat{z} \times a_{s_k} \Phi_{s_k} e^{i\left(\vec{k}\cdot\vec{x}-\sigma_{s_k}t\right)} +$$
$$\left[\sum_p \sum_{s_p} \left\{-s_p p a_{s_p} \Phi_{s_p} e^{i\left(\vec{k}\cdot\vec{x}-\sigma_{s_k}t\right)}\right\}\right] \times \left[\sum_q \sum_{s_q} \left\{a_{s_q} \Phi_{s_q} e^{i\left(\vec{q}\cdot\vec{x}-\sigma_{s_q}t\right)}\right\}\right] =$$
$$- \imath \sum_k \sum_{s_k} \vec{k}\hat{p} e^{i\left(\vec{q}\cdot\vec{x}-\sigma_{s_q}t\right)} \quad (5)$$

Multiply (5) by $e^{-\imath\vec{l}\cdot\vec{x}}$ and integrate over x, then only the terms with $e^{-\imath\vec{l}\cdot\vec{x}}$ will remain. If $\vec{l}$ is changed back to $\vec{k}$, we get

$$\left(\frac{\partial a_{s_k}}{\partial t} - \imath\sigma_{s_k} a_{s_k}\right) \Phi_{s_k} + 2\Omega\hat{z} \times a_{s_k}\Phi_{s_k} + \imath\vec{k}\hat{P}$$
$$= \sum_{k=p+q} \sum_{s_p,s_q} s_p p a_{s_p} a_{s_q} \Phi_{s_p} \times \Phi_{s_q} e^{i\left(\sigma_{s_k}-\sigma_{s_p}-\sigma_{s_q}\right)t}$$
$$= \frac{1}{2} \sum_{k=p+q} \sum_{s_p,s_q} \left(s_p p - s_q q\right) a_{s_p} a_{s_q} \Phi_{s_p} \times \Phi_{s_q} e^{i\left(\sigma_{s_k}-\sigma_{s_p}-\sigma_{s_q}\right)t} \quad (6)$$

Take the inner product of (6) by $\Phi^*_{s_k}$ and use $\Phi_{s_k}\cdot\Phi^*_{s_k} = 2$ and $\vec{k}\cdot\Phi^*_{s_k} = 0$. Then the fourth term on RHS vanishes, and the second and the third terms cancel out because

$$-\imath\sigma_{s_k} a_{s_k} \Phi_{s_k} \cdot \Phi^*_{s_k} = -2\imath\sigma_{s_k} a_{s_k} = -4\imath\Omega s a_{s_k} \frac{k_z}{k} \text{ , and}$$

$$\begin{aligned}
(\hat{z} \times \Phi_{s_k}) \cdot \Phi^*_{s_k} &= \hat{z} \cdot \left(\Phi_{s_k} \times \Phi^*_{s_k}\right) \\
&= \hat{z} \cdot \left(\hat{k} \times \hat{\psi} + \imath s \hat{\psi}\right) \times \left(\hat{k} \times \hat{\psi} - \imath s \hat{\psi}\right) \\
&= \hat{z} \cdot \left\{2\imath s \hat{\psi} \times \left(\hat{k} \times \hat{\psi}\right)\right\} \\
&= 2\imath s \hat{z} \cdot \left\{\hat{k}\left(\hat{\psi} \cdot \hat{\psi}\right) - \hat{\psi}\left(\hat{\psi} \cdot \hat{k}\right)\right\} \\
&= 2\imath s \frac{k_z}{k}
\end{aligned}$$

since $\Phi_{s_k} = \hat{k} \times \hat{\psi} + \imath s \psi$ where $\hat{k} = \frac{\vec{k}}{k}$ and $\hat{\psi} = \frac{\vec{k} \times \hat{z}}{|\vec{k} \times \hat{z}|}$. Finally (6) becomes

$$C^{s_k s_p s_q}_{\vec{k}\vec{p}\vec{q}} \equiv \frac{\partial a_{s_k}}{\partial t} = \frac{1}{4} \sum_{k=p+q} \sum_{s_p, s_q} \left(s_p p - s_q q\right) a_{s_p} a_{s_q} \left(\Phi_{s_p} \times \Phi_{s_q}\right) \cdot \Phi^*_{s_k} e^{i\left(\sigma_{s_k} - \sigma_{s_p} - \sigma_{s_q}\right)t}$$

$\blacksquare$