

$$\begin{cases} E(k) = Ak^m & k_0 < k < k_L \\ E(k) = \alpha\varepsilon^{2/3}k^{-5/3} & k_L < k < k_v \end{cases}$$

Since at k , the energy $E(k)$ should be the same, we have

$$Ak^m = \alpha\varepsilon^{2/3}k^{-5/3} \Rightarrow k_L^{\frac{3m+5}{3}} = \alpha\varepsilon^{2/3} \Rightarrow k_L = \alpha\varepsilon^{\frac{2}{3m+5}} = C\varepsilon^{\frac{2}{3m+5}}$$

so

$$\begin{aligned} \frac{q^2}{2} &= \int_{k_0}^{k_v} E(k)dk = \int_{k_0}^{k_L} E(k)dk + \int_{k_L}^{k_v} E(k)dk \\ &= \int_{k_0}^{k_L} Ak^m dk + \int_{k_L}^{k_v} \alpha\varepsilon^{2/3}k^{-5/3}dk \\ &= \frac{Ak^{m+1}}{m+1} \Big|_{k_0}^{k_L} - \frac{3}{2}\varepsilon^{2/3}k^{-2/3} \Big|_{k_L}^{k_v} \\ &= \frac{Ak_L^{m+1}}{m+1} + \frac{3}{2}\varepsilon^{2/3}k_L \\ &= C_1 \frac{\varepsilon^{\frac{2m+2}{3m+5}}}{m+1} + C_2 \frac{3}{2}\varepsilon^{2/3}\varepsilon^{-\frac{4}{3(3m+5)}} \\ &= c_3\varepsilon^{\frac{2m+2}{3m+5}} \end{aligned}$$

here we assumed the energy in both very large scale k_0 and very small scale k_v are small. Also, from the derivation, we can see c_3 is a constant with respect to ε .

so

$$\frac{q^2}{q_0^2} = \frac{\varepsilon^{\frac{2m+2}{3m+5}}}{\varepsilon_0^{\frac{2m+2}{3m+5}}} = \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{2m+2}{3m+5}}$$