Group 7.

Tutorial 1 – Turbulence J. Fernando

Q12: Consider decay of isotropic turbulence, and the change of ε can be parameterized as $\frac{d\varepsilon}{dt} = f(\varepsilon, q^2)$.

Argue that the above expression is consistent with the Kolmogorov hypo., and that $\frac{d\varepsilon}{dt} = -C_1 \frac{\varepsilon^2}{q^2}$ [note: typo in the equation].

where C_1 is a constant. Also show that $q^2 = \overline{u_i' u_i'}$ varies according to

$$q^2 = q_0^2\, \theta^{-2/(C_1-2)}$$

$$\varepsilon = \varepsilon_0\, \theta^{-C_1/(C_1-2)}$$

where $\theta = 1 + (C_1 - 2)\left[\frac{t\varepsilon_0}{q_0^2}\right]$,

and that $C_1 > 2$. For $t \gg (q_0^2/\varepsilon_0)$, show that the decay occurs according to $q^2 \sim t^{-2/(C_1-2)} \sim t^{-1}$, and suggest a value for C_1.

Solution: ① Dimensions $q \sim \frac{1}{T}$ $\varepsilon \sim \frac{L^2}{T^3}$

$$\frac{d\varepsilon}{dt} \sim \frac{L^2}{T^4} \Leftrightarrow \left(\frac{L^2}{T^3}\right)^a \left(\frac{1}{T}\right)^b$$

$$\Rightarrow \begin{array}{l} 2a + b = 2 \\ 3a + b = 4 \end{array} \Rightarrow a = 2 \quad b = -2$$

$$\Rightarrow \frac{d\varepsilon}{dt} = -C_1 \frac{\varepsilon^2}{q^2}$$

② Let $E \equiv q^2$

$$\left.\begin{array}{l} \Rightarrow \frac{dE}{dt} = -2\varepsilon \\ \frac{d\varepsilon}{dt} = -C_1 \frac{\varepsilon^2}{E} \end{array}\right\} \Rightarrow \frac{dE}{d\varepsilon} = \frac{2}{C_1} \frac{E}{\varepsilon}$$

$$\Rightarrow \ln E = \frac{2}{C_1} \ln \varepsilon$$

$$\Rightarrow \frac{E}{E_0} = \frac{\varepsilon^{\frac{2}{C_1}}}{\varepsilon_0^{\frac{2}{C_1}}}$$

$$\Rightarrow \; E = E_0 \frac{\varepsilon^{\frac{2}{C_1}}}{\varepsilon_0^{\frac{2}{C_1}}}$$

$$\Rightarrow \; \cancel{E = t/C_1}$$

$$\frac{d\varepsilon}{dt} = -C_1 \frac{\varepsilon^2 \varepsilon_0^{2/C_1}}{E_0 \, \varepsilon^{2/C_1}}$$

$$\Rightarrow \; \frac{d\varepsilon}{\varepsilon^{(2-2/C_1)}} = -C_1 \frac{\varepsilon_0^{2/C_1}}{E_0} \, dt$$

$$\Rightarrow \; \frac{1}{\frac{2}{C_1} - 1} \varepsilon^{(\frac{2}{C_1} - 1)} = -C_1 \frac{\varepsilon_0^{2/C_1}}{E_0} t + \frac{1}{\frac{2}{C_1} - 1} \varepsilon_0^{(\frac{2}{C_1} - 1)}$$

$$\Rightarrow \; \varepsilon^{(\frac{2}{C_1} - 1)} = \varepsilon_0^{(\frac{2}{C_1} - 1)} - (\frac{2}{C_1} - 1) C_1 \frac{\varepsilon_0^{\frac{2}{C_1}}}{E_0} t$$

$$= \varepsilon_0^{(\frac{2}{C_1} - 1)} \left[1 + (C_1 - 2) \frac{\varepsilon_0 t}{E_0} \right]$$

$$\Rightarrow \; \varepsilon = \varepsilon_0 \left[1 + (C_1 - 2) \frac{\varepsilon_0 t}{E_0} \right]^{\frac{C_1}{2-C_1}}$$

$$E = E_0 \frac{\varepsilon_0^{\frac{2}{C_1}} \left[1 + (C_1 - 2) \frac{\varepsilon_0 t}{E_0} \right]^{\frac{C_1}{2-C_1} \frac{2}{C_1}}}{\varepsilon_0^{2/C_1}}$$

$$= E_0 \left[1 + (C_1 - 2) \frac{\varepsilon_0 t}{E_0} \right]^{\frac{2}{2-C_1}}$$

$$= q_0^2 \left[1 + (C_1 - 2) \frac{\varepsilon_0 t}{q_0^2} \right]^{\frac{2}{2-C_1}}$$

when $t > \frac{q_0^2}{\varepsilon_0}$

$$\Rightarrow \; E \sim t^{\frac{2}{2-C_1}} \sim t^{-n}$$