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Original Hydrostatic Balance

\[ \rho = \rho_o + \tilde{\rho}(x,t), \quad |\tilde{\rho}| \ll \rho_o \]
\[ T = T_o + \tilde{T}(x,t) \]
\[ p = p_o(z) + \tilde{p}(x,t), \quad \nabla p_o(z) = \rho_o g \]

Additional Hydrostatic Balance

\[ \tilde{\rho}(x,t) = \rho_1(z) + \rho^\prime(x,t) = -bz + \rho^\prime(x,t) \]
\[ \rho = \rho_o + \rho_1(z) + \rho^\prime(x,t) = \rho_o - bz + \rho^\prime(x,t), \quad |\rho^\prime| \ll \rho_o \]
\[ \tilde{p}(x,t) = p_1(z) + p^\prime(x,t), \quad -\frac{1}{\rho_o} \nabla p_1(z) = \frac{bz}{\rho_o} \]
\[ g = -g \hat{z} \]
\[ p = p_o(z) + p_1(z) + p^\prime \]

To add the additional layer of hydrostatic balance, similarly apply this new equation for \( \rho \) as done on page 13 of the lecture notes.

Thermal Energy

\[ \frac{\delta \rho}{\rho} \approx -\alpha \delta T \quad |\tilde{\rho}| \ll \rho_o \text{ and } |\rho^\prime| \ll \rho_o \]
\[ \frac{\rho - \rho_o}{\rho_o + \rho_1 + \rho^\prime} \approx \frac{\rho_1 + \rho^\prime}{\rho_o} \]
\[ T - T_o = -\frac{\rho_1 + \rho^\prime}{\alpha \rho_o} \]
\[ \frac{\rho c_p}{D} = k \nabla^2 T \]
\[ - (\rho_o + \rho_1 + \rho^\prime) c_p \frac{1}{\rho_o} \frac{D}{Dt} (\rho_0 + \rho_1 + \rho^\prime) = -k \frac{1}{\rho_o} \nabla^2 (\rho_1 + \rho^\prime) \]
\[ \frac{D\rho_1}{Dt} = -b(u \cdot \hat{z}) \]
\[ \frac{Dp^\prime}{Dt} - b(u \cdot \hat{z}) = \frac{k}{\rho_o c_p} \nabla^2 \rho^\prime \]
Mass

\[ \frac{\partial \rho}{\partial t} + \rho \nabla \cdot u + u \cdot \nabla \rho = 0 \]
\[ \frac{\partial}{\partial t} (\rho_o + \rho_1 + \rho') + (\rho_o + \rho_1 + \rho') \nabla \cdot u + u \cdot \nabla (\rho_o + \rho_1 + \rho') = 0 \]
Dominant Balance: \[ \nabla \cdot u = 0, \]

Momentum

\[ \rho \frac{D}{Dt} u = -\nabla p + \rho g + \nabla \left[ 2\mu S - \frac{2}{3} \mu \nabla \cdot u \right] \]
\[ (\rho_o + \rho_1 + \rho') \frac{D}{Dt} u = -\nabla (\rho_o + \rho_1 + \rho') + (\rho_o + \rho_1 + \rho') g + \mu \nabla^2 u \]
Subtract the two hydrostatic balances: \[ -\nabla p_o(z) + \rho_o g = 0 \text{ and } -\nabla p_1 - \rho_1 g = 0 \]
Divide by \( \rho_o + \rho_1 \): \[ \frac{D}{Dt} u = -\frac{1}{\rho_o} \nabla p' + \frac{\rho'}{\rho_o} g + \nu \nabla^2 u \]

Move the Momentum Equation to the Rotating Frame

\[ \Omega = \Omega \hat{\hat{z}} \]
\[ \left( \frac{\partial u_i}{\partial t} \right)_I = \left( \frac{\partial u_i}{\partial t} \right)_R + \Omega \times u_i + \left( \frac{\partial \Omega}{\partial t} \right)_R \times r + \Omega \times \left( \frac{\partial r}{\partial t} \right)_I \]
\[ \left( \frac{\partial r}{\partial t} \right)_I = \left( \frac{\partial r}{\partial t} \right)_R + \Omega \times r = u_R + \Omega \times r \]
\[ \left( \frac{\partial u_i}{\partial t} \right)_I = \left( \frac{\partial u_i}{\partial t} \right)_R + \Omega \times u_R + \Omega \times (u_R + \Omega \times r) \]
\[ \Omega \times (\Omega \times r) \approx 0 \]
\[ \frac{D}{Dt} u + 2\Omega \hat{\hat{z}} \times u = -\frac{1}{\rho_o} \nabla p' + \frac{\rho'}{\rho_o} g + \nu \nabla^2 u \]

Write in Terms of \( \Theta \) and \( N \)

\[ \rho' = \left( \frac{b \rho_o}{g} \right)^{\frac{1}{2}} \Theta \]
\[ N = \left( \frac{bg}{\rho_o} \right)^{\frac{1}{2}} \]
Thermal Energy

\[
\left( \frac{b \rho_0}{g} \right)^{\frac{1}{2}} \frac{D \Theta}{Dt} - b(u \cdot \hat{z}) = \left( \frac{b \rho_0}{g} \right)^{\frac{1}{2}} \frac{k}{\rho_0 c_p} \nabla^2 \Theta
\]

Multiply by \( \left( \frac{b \rho_0}{g} \right)^{-\frac{1}{2}} \)

\[
\frac{D \Theta}{Dt} - \left( \frac{bg}{\rho_o} \right)^{\frac{1}{2}} (u \cdot \hat{z}) = \frac{k}{\rho_0 c_p} \nabla^2 \Theta
\]

\[
\frac{D \Theta}{Dt} - N(u \cdot \hat{z}) = \frac{k}{\rho_0 c_p} \nabla^2 \Theta
\]

Momentum

\[
\frac{Du}{Dt} + 2 \Omega \hat{z} \times u = -\frac{1}{\rho_o} \nabla p' - \left( \frac{bg}{\rho_o} \right)^{\frac{1}{2}} \Theta + \nu \nabla^2 u
\]

\[
\frac{Du}{Dt} + 2 \Omega \hat{z} \times u + N \Theta = -\frac{1}{\rho_o} \nabla p' + \nu \nabla^2 u
\]
Show that $\frac{\rho}{\rho^\gamma}$ is constant in an adiabatic ideal gas: This is done using the first law of thermodynamics $dQ = d\hat{u} + dW$ and the equation of state for an ideal gas $p = \rho RT = \frac{1}{V}RT$.

First, note that in an adiabatic gas, $dQ = 0$. Also, by definition $d\hat{u} = c_v dT$. The work done by the gas is $dW = pdV$. Since $V = \rho^{-1}$, we have $dV = -\frac{dp}{\rho^2}$, so for an adiabatic ideal gas, the first law becomes $c_v dT = p \frac{dp}{\rho^2} = RT \frac{dp}{\rho}$. To show that the value $\frac{\rho}{\rho^\gamma}$ is constant, we examine its differential.

\[
d(\frac{p}{\rho^\gamma}) = d(\rho^{1-\gamma}RT) = R\rho^{1-\gamma}[(1-\gamma)\frac{dp}{\rho} + dT]
\]

\[
= R\rho^{1-\gamma}[(1-\gamma)T \frac{dp}{\rho} + \frac{p}{c_v} \rho^2] = R\rho^{1-\gamma}[(1-\gamma)T \frac{dp}{\rho} + \frac{RT}{c_v} \frac{dp}{\rho}]
\]

\[
= R\rho^{1-\gamma}T \frac{dp}{\rho} [(1-\gamma) + \frac{R}{c_v}] = 0
\]

The proof that the potential temperature $\theta = T(\frac{\rho}{p})^{\frac{\gamma}{c_p}} = CTp^{\gamma-1}$ is constant in an adiabatic ideal gas is similar: This time, we use the first law of thermodynamics in a different form. The work done by the gas is $dW = pdV = R(dT - T \frac{dp}{p})$, so we write the first law as $c_p dT - RT \frac{dp}{p} = 0$. From this we find $dT = (1-\gamma)T \frac{dp}{p}$. Again, to show that $\theta$ is constant, we examine its differential.

\[
d\theta = d(CTp^{\gamma-1}) = Cp^{\gamma-1}dT + (\gamma - 1)T \frac{dp}{p}
\]

\[
= Cp^{\gamma-1}[(1-\gamma)T \frac{dp}{p} + (\gamma - 1)T \frac{dp}{p}] = 0
\]