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Original Hydrostatic Balance

$$\rho = \rho_o + \tilde{\rho}(\underline{x}, t), \qquad |\tilde{\rho}| \ll \rho_o$$
$$T = T_o + \tilde{T}(\underline{x}, t)$$
$$p = p_o(z) + \tilde{p}(\underline{x}, t), \qquad \underline{\nabla} p_o(z) = \rho_o \underline{g}$$

Additional Hydrostatic Balance

$$\tilde{\rho}(\underline{x}, t) = \rho_1(z) + \rho'(x, t) = -bz + \rho'(\underline{x}, t)$$
$$\rho = \rho_o + \rho_1(z) + \rho'(x, t) = \rho_o - bz + \rho'(\underline{x}, t), \qquad |\rho'| \ll \rho_o$$
$$\tilde{p}(\underline{x}, t) = p_1(z) + p'(x, t), \qquad -\frac{1}{\rho_o}\underline{\nabla} p_1(z) = \frac{bz}{\rho_o}\underline{g}$$
$$\underline{g} = -g\hat{z}$$
$$p = p_o(z) + p_1(z) + p'$$

To add the additional layer of hydrostatic balance, similarly apply this new equation for ρ as done on page 13 of the lecture notes.

Thermal Energy

$$\frac{\delta\rho}{\rho} \approx -\alpha\delta T \qquad |\tilde{\rho}| \ll \rho_o \text{ and } |\rho'| \ll \rho_o$$
$$\frac{\rho - \rho_o}{\rho_o + \rho_1 + \rho'} \approx \frac{\rho_1 + \rho'}{\rho_o}$$
$$T - T_o = -\frac{\rho_1 + \rho'}{\alpha\rho_o}$$
$$\rho c_p \frac{DT}{Dt} = k\underline{\nabla}^2 T$$
$$-(\rho_o \cancel{+ \rho_1 + \rho'})c_p \frac{1}{\cancel{\alpha}\rho_o}\frac{D}{Dt}(\cancel{\rho_o} + \rho_1 + \rho') = -k\frac{1}{\cancel{\alpha}\rho_o}\underline{\nabla}^2(\cancel{\rho_1} + \rho')$$
$$\frac{D\rho_1}{Dt} = -b(\underline{u}\cdot\hat{z})$$
$$\frac{D\rho'}{Dt} - b(\underline{u}\cdot\hat{z}) = \frac{k}{\rho_o c_p}\underline{\nabla}^2\rho'$$

Mass

$$\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \rho = 0$$

$$\frac{\partial}{\partial t}(\cancel{\rho_o + \rho_1} + \rho') + (\rho_o + \rho_1 + \rho')\underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla}(\cancel{\rho_o} + \rho_1 + \rho') = 0$$

Dominant Balance: $\underline{\nabla} \cdot \underline{u} = 0$,

Momentum

$$\rho \frac{Du}{Dt} = -\underline{\nabla} p + \rho \underline{g} + \underline{\nabla} \cdot \left[2\mu \underline{\underline{S}} - \frac{2}{3}\mu \underline{\nabla} \cdot \underline{u} \underline{\underline{I}} \right]$$

$$(\rho_o + \rho_1 + \rho')\frac{Du}{Dt} = -\underline{\nabla}(p_o + p_1 + p') + (\rho_o + \rho_1 + \rho')\underline{g} + \mu \underline{\nabla}^2 \underline{u}$$

Subtract the two hydrostatic balances: $-\underline{\nabla} p_o(z) + \rho_o \underline{g} = 0$ and $-\underline{\nabla} p_1 - \rho_1 \underline{g} = 0$

Divide by $\rho_o + \rho_1$: $(1 + \cancel{\frac{\rho'}{\rho_o + \rho_1}})\frac{Du}{Dt} = \frac{1}{\rho_o + \cancel{\rho_1}}\left[-\underline{\nabla} p' + \rho' \underline{g} + \mu \underline{\nabla}^2 \underline{u} \right]$

$$\frac{Du}{Dt} = -\frac{1}{\rho_o}\underline{\nabla} p' + \frac{\rho'}{\rho_o}\underline{g} + \nu \underline{\nabla}^2 \underline{u}$$

Move the Momentum Equation to the Rotating Frame

$$\underline{\Omega} = \Omega \hat{z}$$

$$\left(\frac{\partial \underline{u}_I}{\partial t}\right)_I = \left(\frac{\partial \underline{u}_R}{\partial t}\right)_R + \underline{\Omega} \times \underline{u}_R + \cancel{\frac{\partial \underline{\Omega}}{\partial t}} \times r + \underline{\Omega} \times \left(\frac{\partial r}{\partial t}\right)_I$$

$$\left(\frac{\partial r}{\partial t}\right)_I = \left(\frac{\partial r}{\partial t}\right)_R + \underline{\Omega} \times r = \underline{u}_R + \underline{\Omega} \times r$$

$$\left(\frac{\partial \underline{u}_I}{\partial t}\right)_I = \left(\frac{\partial \underline{u}_R}{\partial t}\right)_R + \underline{\Omega} \times \underline{u}_R + \underline{\Omega} \times (\underline{u}_R + \underline{\Omega} \times r)$$

$$\underline{\Omega} \times (\underline{\Omega} \times r) \approx 0$$

$$\frac{Du}{Dt} + 2\Omega \hat{z} \times \underline{u} = -\frac{1}{\rho_o}\underline{\nabla} p' + \frac{\rho'}{\rho_o}\underline{g} + \nu \underline{\nabla}^2 \underline{u}$$

Write in Terms of Θ and N

$$\rho' = \left(\frac{b\rho_o}{g}\right)^{\frac{1}{2}} \Theta$$

$$N = \left(\frac{bg}{\rho_o}\right)^{\frac{1}{2}}$$

Thermal Energy

$$\left(\frac{b\rho_o}{g}\right)^{\frac{1}{2}}\frac{D\Theta}{Dt} - b(\underline{u}\cdot\hat{z}) = \left(\frac{b\rho_o}{g}\right)^{\frac{1}{2}}\frac{k}{\rho_o c_p}\underline{\nabla}^2\Theta$$

Multiply by $\left(\frac{b\rho_o}{g}\right)^{-\frac{1}{2}}$

$$\frac{D\Theta}{Dt} - \left(\frac{bg}{\rho_o}\right)^{\frac{1}{2}}(\underline{u}\cdot\hat{z}) = \frac{k}{\rho_o c_p}\underline{\nabla}^2\Theta$$

$$\frac{D\Theta}{Dt} - N(\underline{u}\cdot\hat{z}) = \frac{k}{\rho_o c_p}\underline{\nabla}^2\Theta$$

Momentum

$$\frac{Du}{Dt} + 2\Omega\hat{z}\times\underline{u} = -\frac{1}{\rho_o}\underline{\nabla}p' - \left(\frac{bg}{\rho_o}\right)^{\frac{1}{2}}\Theta + \nu\underline{\nabla}^2\underline{u}$$

$$\frac{Du}{Dt} + 2\Omega\hat{z}\times\underline{u} + N\Theta = -\frac{1}{\rho_o}\underline{\nabla}p' + \nu\underline{\nabla}^2\underline{u}$$

2

Show that $\frac{p}{\rho^\gamma}$ is constant in an adiabatic ideal gas: This is done using the first law of thermodynamics $dQ = d\hat{u} + dW$ and the equation of state for an ideal gas $p = \rho RT = \frac{1}{V}RT$.

First, note that in an adiabatic gas, $dQ = 0$. Also, by definition $d\hat{u} = c_v dT$. The work done by the gas is $dW = pdV$. Since $V = \rho^{-1}$, we have $dV = -\frac{d\rho}{\rho^2}$, so for an adiabatic ideal gas, the first law becomes $c_v dT = p\frac{d\rho}{\rho^2} = RT\frac{d\rho}{\rho}$. To show that the value $\frac{p}{\rho^\gamma}$ is constant, we examine its differential.

$$d(\frac{p}{\rho^\gamma}) = d(\rho^{1-\gamma}RT) = R\rho^{1-\gamma}[(1-\gamma)T\frac{d\rho}{\rho} + dT]$$
$$= R\rho^{1-\gamma}[(1-\gamma)T\frac{d\rho}{\rho} + \frac{p}{c_v}\frac{d\rho}{\rho^2}] = R\rho^{1-\gamma}[(1-\gamma)T\frac{d\rho}{\rho} + \frac{RT}{c_v}\frac{d\rho}{\rho}]$$
$$= R\rho^{1-\gamma}\frac{Td\rho}{\rho}[(1-\gamma) + \frac{R}{c_v}] = 0$$

The proof that the potential temperature $\theta = T(\frac{p_s}{p})^{\frac{R}{c_p}} = CTp^{\gamma-1}$ is constant in an adiabatic ideal gas is similar: This time, we use the first law of thermodynamics in a different form. The work done by the gas is $dW = pdV = R(dT - T\frac{dp}{p})$, so we write the first law as $c_p dT - RT\frac{dp}{p} = 0$. From this we find $dT = (1-\gamma)T\frac{dp}{p}$. Again, to show that θ is constant, we examine its differential.

$$d\theta = d(CTp^{\gamma-1}) = Cp^{\gamma-1}[dT + (\gamma-1)T\frac{dp}{p}]$$
$$= Cp^{\gamma-1}[(1-\gamma)T\frac{dp}{p} + (\gamma-1)T\frac{dp}{p}] = 0$$