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group #6

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#1 For 2-D non-rotating Flow,

$$\begin{cases} \vec{u}_t + \underline{u}\cdot\nabla\underline{u} = -\nabla p & ① \\ \nabla\cdot\underline{u} = 0 & ② \end{cases} \quad \text{use} \quad \begin{cases} u = -\frac{\partial\psi}{\partial y} \\ v = \frac{\partial\psi}{\partial x} \end{cases}, \text{ and } \omega_3 = \nabla^2\psi$$

For ①, use $\vec{u}_t + \vec{\omega}\times\vec{u} = -\nabla p^*$ ③ $p^* = p + \frac{uu}{2}$

$\underline{u}\cdot③$, $\Rightarrow$ $E = \frac{\underline{u}\cdot\underline{u}}{2}$

$$\frac{\partial E}{\partial t} = \underline{u}\cdot(-\nabla p^*) = -\nabla\cdot[p^*\underline{u}] + \cancel{p\,\nabla\cdot u}$$

$$\int_V \frac{\partial}{\partial t}E = -\int_V -\nabla\cdot[p^*u] = -\int_A p^*\underline{u}\cdot\hat{n}\,dA = 0$$

$\Rightarrow$ $E = \frac{\underline{u}\cdot\underline{u}}{2}$ is conserved

use stream function ψ, ① can be written as

$$\partial_t\nabla^2\psi + J(\psi, \nabla^2\psi) = 0 \quad ④$$

$\nabla^2\psi\cdot④$, $F = \frac{\omega_3^2}{2} = \frac{(\nabla^2\psi)^2}{2}$, $= -\int(\underline{u}\cdot\nabla\psi)\nabla^2\psi = \int\nabla\cdot(\underline{u}\,\psi\,\nabla^2\psi)\,dv$

$$\int_V \frac{\partial}{\partial t}F = -\int J(\psi, \nabla^2\psi)\cdot\nabla^2\psi$$

$= 0$

so enstrophy $F = \frac{\omega_3^2}{2}$ is conserved

in Fourier, means $\sum_k E(k)$ and $\sum_k k^2E(k)$ are conserved.

Using Kolmogorov's assumption, since energy is conserved, let ε be the energy transfer rate,

$$E(k) = C\varepsilon^\alpha k^\beta, \quad \text{since } [E] = \frac{l^3}{t^2}, \ [\varepsilon] = \frac{l^2}{t^3}$$

$$\Rightarrow E(k) = C\,\varepsilon^{2/3}\,k^{-5/3}$$

now let η be enerstrophy transfer rate, $[\eta] = [k^2\varepsilon] = \frac{1}{t^3}$

assume $k^2E(k) = C_0\,\eta^\alpha k^\beta$,

we have $k^2E(k) = C_0\,\eta^{2/3}k^{-1} \Rightarrow E(k) = C_0\,\eta^{2/3}k^{-3}$

now we have two scales, $E(k)\propto k^{-5/3}$, $E(k)\propto k^{-3}$

For QG: $\left(\frac{\partial}{\partial t} + \underline{u}_H\cdot\nabla_H\right)\left(\nabla_H^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi = 0$

where $u = -\frac{\partial\psi}{\partial y}$, $v = \frac{\partial\psi}{\partial x}$, $\theta = -\frac{f}{N}\frac{\partial^2}{\partial z^2}$ ⑤, $q = \left(\nabla_H^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z}\right)\psi$

$\psi\cdot⑤$, let $E = \frac{u_H\cdot u_H}{2} + \frac{\theta^2}{2}$, $= \frac{\nabla_H\psi\cdot\nabla_H\psi}{2} + \frac{f^2}{N^2}(\psi_z)^2$

$$\Rightarrow \frac{\partial}{\partial t}E = -\int_V \underline{u}_H\cdot\nabla_H\left(\nabla_H^2 + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi\cdot\psi$$

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$$\ldots = -\int_V (u_H \cdot \nabla_H q)\psi = -\int_V \nabla_H \cdot (u_H q\psi) = -\int_A q\psi\, u_H \cdot \hat{n}\, dA$$
$$= 0$$

$$\left(\nabla_H \cdot (u_H q\psi) = \cancel{(\nabla_H \cdot u_H)} q\psi + u_H \cdot \nabla (q\psi) = (u_H \cdot \nabla q)\psi + \cancel{(u_H \cdot \nabla_H \psi)} q \right.$$
$$= (u_H \cdot \nabla q)\psi$$

so Energy $E = \frac{u_H \cdot u_H + \theta^2}{2}$ is conserved.

For ⑤, $\left(\frac{\partial}{\partial t} + u_H \cdot \nabla_H\right) q = 0$, $\quad q \cdot$ ⑤.

let $F = \frac{q^2}{2} = \left[\left(\nabla_H^2 + \frac{F^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi\right]^2$,

we have $\frac{\partial}{\partial t} F = -\int_V (u_H \cdot \nabla_H q) q = -\int_V \nabla_H \cdot (u_H q^2) = 0$

so potential enstrophy $F = \left(\left(\nabla_H^2 + \frac{F^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\psi\right)^2$ is conserved.

now if rescale $\zeta = \frac{N}{F} z \Rightarrow d\zeta = \frac{N}{F} dz$

we will have the similar result as 2-D flow

$E(K) = \frac{1}{2}(u^2 + v^2 + \theta^2)$ and $F(K) = K^2 E(K)$ are conserved

so with similar argument. $E(K) \propto K^{-5/3}$ and $E(K) \propto K^{-3}$

#2 Since $C_{kpq} = \frac{iN(p \times q)\cdot \hat{z}}{\sigma_k k\, \sigma_p p\, \sigma_q q}\left(\sigma_q^2 q^2 - \sigma_p^2 p^2\right)$,

with $\sigma_k k = (N^2 k_h^2 + F^2 k_z^2)^{1/2}$

Now if $k + p + q = 0$.

$(q \times k)\cdot \hat{z} = (q \times (-p - q))\cdot \hat{z} = -(q \times p)\cdot \hat{z} = (p \times q)\cdot \hat{z}$

so $p \times q = q \times k = k \times p$

so $C_{kpq} + C_{pqk} + C_{qkp}$

$$= \frac{iN(p \times q)\cdot \hat{z}}{\sigma_k k\, \sigma_p p\, \sigma_q q}\left(\sigma_q^2 q^2 - \sigma_p^2 p^2\right) + \frac{iN(q \times k)\cdot \hat{z}}{\sigma_p p\, \sigma_q q\, \sigma_k k}\left(\sigma_k^2 k^2 - \sigma_q^2 q^2\right)$$

$$+ \frac{iN(k \times p)\cdot \hat{z}}{\sigma_q q\, \sigma_k k\, \sigma_p p}\left(\sigma_p^2 p^2 - \sigma_k^2 k^2\right)$$

$$= \frac{iN(p \times q)\cdot \hat{z}}{\sigma_k k\, \sigma_p p\, \sigma_q q}\left(\sigma_q^2 q^2 - \sigma_p^2 p^2 + \sigma_k^2 k^2 - \sigma_q^2 q^2 + \sigma_p^2 p^2 - \sigma_k^2 k^2\right) = 0$$

and $\sigma_k^2 k^2 C_{kpq} + \sigma_p^2 p^2 C_{pqk} + \sigma_q^2 q^2 C_{qkp}$

$$= \frac{iN\,(\mathbf{p}\times\mathbf{q})\cdot\hat{z}}{\sigma_k k\,\sigma_p p\,\sigma_q q}\left[\sigma_k^2 k^2(\sigma_q^2 q^2 - \sigma_p^2 p^2) + \sigma_p^2 p^2(\sigma_q^2 k^2 - \sigma_q^2 q^2) + \sigma_q^2 q^2(\sigma_p^2 p^2 - \sigma_k^2 k^2)\right]$$

$= 0$

so $\quad C_{kpq} + C_{pqk} + C_{qkp} = 0$

and $\sigma_k^2 k^2 C_{kpq} + \sigma_p^2 p\, C_{pqk} + \sigma_q^2 q^2 C_{qkp} = 0$

in one triad, $\quad \underline{k} + \underline{p} + \underline{q} = 0.$

$$\begin{aligned} a_k^* \cdot &\left\{ \frac{\partial}{\partial t} a_k = C_{kpq}\, a_p^* a_q^* \right. \\ a_p^* \cdot &\left\{ \frac{\partial}{\partial t} a_p = C_{pqk}\, a_q^* a_k^* \right. \\ a_q^* \cdot &\left\{ \frac{\partial}{\partial t} a_q = C_{qkp}\, a_k^* a_p^* \right. \end{aligned} \Rightarrow \left\{ \begin{aligned} \frac{\partial}{\partial t}\left(\frac{a_k^2}{2}\right) &= C_{kpq}\, a_k^* a_p^* a_q^* \\ \frac{\partial}{\partial t}\left(\frac{a_p^2}{2}\right) &= C_{pqk}\, a_k^* a_p^* a_q^* \\ \frac{\partial}{\partial t}\left(\frac{a_q^2}{2}\right) &= C_{qkp}\, a_k^* a_p^* a_p^* \end{aligned} \right.$$

take sum, $\Rightarrow \frac{\partial}{\partial t}\left(\frac{a_k^2 + a_p^2 + a_q^2}{2}\right) = 0$

$\Rightarrow |a_k^0|^2 + |a_p^0|^2 + |a_q^0|^2$ is conserved

similarly, $\sigma_k^2 k^2 |a_k^0|^2 + \sigma_p^2 p^2 |a_p^0|^2 + \sigma_q q^2 |a_q^0|^2$ is conserved