Group 1: ①

Majda's Problem #1

A) Set
$$\begin{pmatrix} \vec{u}_h \\ w \\ p \\ \theta \end{pmatrix} = \begin{pmatrix} \hat{u}_h(z) \\ \hat{w}(z) \\ \hat{p}(z) \\ \hat{\theta}(z) \end{pmatrix} e^{i(\vec{k}_h\cdot\vec{x}_h - \sigma t)}$$

plug into the equations, then we can get (without force terms)

$$\begin{cases} -i\sigma\hat{u}_h = -i\vec{k}_h\hat{p} & ① \\ \dfrac{\partial\hat{p}}{\partial z} = \hat{\theta} & ② \\ -i\sigma\hat{\theta} + \hat{w} = 0 & ③ \\ i\vec{k}_h\cdot\hat{u}_h + \dfrac{\partial\hat{w}}{\partial z} = 0 & ④ \end{cases}$$

$$③ \Rightarrow \hat{\theta} = \frac{\hat{w}}{i\sigma}$$

$$② \Rightarrow \frac{\partial\hat{p}}{\partial z} = \frac{\hat{w}}{i\sigma}$$

$$① \Rightarrow \vec{k}_h\cdot\hat{u}_h = \frac{|\vec{k}_h|^2}{\sigma}\hat{p}$$

$$④ \Rightarrow i\frac{|\vec{k}_h|^2}{\sigma}\hat{p} + \frac{\partial\hat{w}}{\partial z} = 0 \qquad ⑤$$

$$\frac{\partial}{\partial z}⑤ \Rightarrow \frac{\partial^2\hat{w}}{\partial z^2} + \frac{|\vec{k}_h|^2}{\sigma^2}\hat{w} = 0 \quad \Rightarrow \quad \hat{w} = A\cos\left(\frac{|\vec{k}_h|}{\sigma}z\right) + B\sin\left(\frac{|\vec{k}_h|}{\sigma}z\right)$$

with Boundary conditions $\hat{w}(0) = \hat{w}(H) = 0$

$\Rightarrow A = 0, \quad \frac{|\vec{k}_h|}{\sigma}H = j\pi.$ j can be any integer.

$$\Rightarrow \sigma_j = \frac{|\vec{k}_h|H}{j\pi}, \quad j\in\mathbb{Z}.$$

$$\hat{w}\sim\sin\left(\tfrac{j\pi}{H}z\right),\ \hat{u}_h\sim\cos\left(\tfrac{j\pi}{H}z\right),\ \hat{p}\sim\cos\left(\tfrac{j\pi}{H}z\right),\ \hat{\theta}\sim\sin\left(\tfrac{j\pi}{H}z\right)$$

So, we can set
$$\begin{pmatrix} \vec{u}_h \\ w \\ p \\ \theta \end{pmatrix} = \sum_{\vec{k}}\sum_j \begin{pmatrix} \tilde{u}_{hj}(k,t)\cos\left(\frac{j\pi}{H}z\right) \\ \tilde{w}_j(k,t)\sin\left(\frac{j\pi}{H}z\right) \\ \tilde{p}_j(k,t)\cos\left(\frac{j\pi}{H}z\right) \\ \tilde{\theta}(k,t)\sin\left(\frac{j\pi}{H}z\right) \end{pmatrix} e^{i\vec{k}_h\cdot\vec{x}_h} \qquad (*)$$

Plug into the original equs,

$$\begin{cases} \dfrac{\partial \tilde{\vec{u}}_{h,j}}{\partial t} = -i\vec{k}_h \tilde{p} + \tilde{F}_{u,j} \\ -\dfrac{j\pi}{H}\tilde{p}_j = \tilde{\theta}_j \\ \dfrac{\partial \tilde{\theta}_j}{\partial t} + \tilde{w}_j = \tilde{F}_{\theta j} \\ i\vec{k}_h \cdot \tilde{u}_{h,j} + \dfrac{j\pi}{H}\tilde{w}_j = 0 \end{cases}$$

$$\left(\begin{array}{l} \text{Assume} \\ F_{\vec{u}} = \sum_{\vec{k}}\sum_{j} \tilde{F}_{u,j} \cos\left(\frac{j\pi}{H}z\right) e^{i\vec{k}_h\cdot\vec{x}_h} \\ F_\theta = \sum_{\vec{k}}\sum_{j} \tilde{F}_{\theta j} \sin\left(\frac{j\pi}{H}z\right) e^{i\vec{k}_h\cdot\vec{x}_h} \end{array} \right)$$

combine them together to get

$$\frac{\partial \tilde{\theta}_j}{\partial t^2} + \frac{H^2|\vec{k}_h|^2}{(j\pi)^2}\tilde{\theta}_j - \frac{H}{j\pi}\left(\frac{j\pi}{H}\frac{\partial \tilde{F}_{\theta j}}{\partial t} + i\vec{k}_h\cdot\tilde{F}_{u,j}\right) = 0$$

$e^{\pm i\frac{H|\vec{k}_h|}{j\pi}t}$ are the homogenous solutions, so we can use the following formula to construct a particular solution $\tilde{\theta}_j^p$

(Assume $y_1(t)$, $y_2(t)$ are two linearly independent homogenous solutions of $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$,

then we have one particular solution,

$$y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{\Omega(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{\Omega(t)}dt,$$

$$\Omega(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)\ )$$

$$\Rightarrow \tilde{\theta}_j = A e^{i\frac{H|\vec{k}_h|}{j\pi}t} + B e^{-i\frac{H|\vec{k}_h|}{j\pi}t} + \tilde{\theta}_j^p$$

then we can get $\tilde{p}_j$, $\tilde{w}_j$ and $\tilde{u}_{h,j}$.

Plug into $(*)$, we can get $\vec{u}_h$, w, p and θ. #