Majda Problem #3 (Group 7)

The large scale slow time scales follow the nonlinear Boussinesq Hydrostatic equations

$$(\mathcal{O}(\varepsilon^{-1})): \quad \frac{\partial \underline{v}_h}{\partial t} + \underline{v}_h \cdot \nabla \underline{v}_h = -\nabla \bar{p}$$

$$\frac{\partial \bar{p}}{\partial z} = \Theta$$

$$\frac{\partial \Theta}{\partial t} + \underline{v}_h \cdot \nabla \Theta = -W + \bar{S}_\theta$$

$$\nabla_H \underline{v}_h + w_z = 0$$

$$\text{where} \quad \underset{\sim}{v}_h = (v, V, 0)$$

Small-scale fast time equations are

$$\frac{\partial u_h'}{\partial \tau} + \underline{v}_h \cdot \nabla_H u_h' + w' \frac{\partial v_h}{\partial z} = -\nabla p'$$

$$\frac{\partial \theta'}{\partial \tau} + \underline{v}_h \cdot \nabla_h u_h' + w' \frac{\partial \Theta}{\partial z} = -w' + S_\theta'$$

$$\nabla_h \underline{u}_h' + w_z' = 0$$

with a specified $\underset{\sim}{v}_h = (v_h(z), 0)$ these equations can be solved for u_h', w', θ'

The large-scale equations with $Fr = \mathcal{O}(1)$ are

$$\frac{D\underset{\sim}{u}_h}{Dt} + w\frac{\partial U_h}{\partial z} = -\nabla_h p - \frac{\partial}{\partial z}\langle w'\underset{\sim}{u}_h'\rangle$$

$$p_z = \theta - Sw$$

$$\frac{D\theta}{Dt} + \underset{\sim}{u}_h\cdot\nabla_h\Theta + w\frac{\partial\Theta}{\partial z} = -w + S_\theta - \frac{\partial}{\partial z}\overline{w'\theta'}$$

$\left(\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_h\cdot\nabla\right)$
$$\nabla_h\cdot\underset{\sim}{u}_h + w_z = 0$$

When making the low Fr approximation (as in Klein + Majda) the large scale equations are:

$$\frac{\partial\underset{\sim}{u}_h}{\partial t} = -\nabla_h p - \frac{\partial}{\partial z}\langle w'\underset{\sim}{u}_h'\rangle$$

$$p_z = \theta - Sw$$

$$\frac{\partial\theta}{\partial t} = -w + S_\theta - \frac{\partial}{\partial z}\overline{w'\theta'}$$

$$\nabla_h\cdot\underset{\sim}{u}_h + w_z = 0$$

When $U_h = 0$ and $\Theta = \text{const.}$ these equations reduce to the above set with $Fr = \mathcal{O}(1)$

Now, inserting a plane wave form into the fast-time equations, e.g.

$$u' = \hat{u}\, e^{i(kx + ly + mz - \omega t)} \quad \text{with } \underset{\sim}{U}_h = (U(z), 0)$$

$$(-i\omega)\,\hat{u} + ikU\hat{u} + \frac{dU}{dz}\,\hat{w} = -ik\hat{p}$$

$$(-i\omega)\,\hat{v} + ikU\hat{v} = -il\hat{p}$$

$$(-i\omega)\,\hat{w} + ikU\hat{w} = -im\hat{p} + \hat{\theta}$$

$$(-i\omega)\,\hat{\theta} + ikU\hat{\theta} + \hat{w}\,\frac{d\Theta}{dz} = -\hat{w}$$

$$ik\hat{u} + il\hat{v} + im\hat{w} = 0$$

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① $(kU - \omega)\,\hat{u} + \frac{dU}{dz}\,\hat{w} = -k\hat{p}$

② $(kU - \omega)\,\hat{v} = -l\hat{p}$

③ $(kU - \omega)\,\hat{w} = -m\hat{p} + \hat{\theta}$

④ $(kU - \omega)\,\hat{\theta} + \left(\frac{d\Theta}{dz} + 1\right)\hat{w} = 0$

⑤ $k\hat{u} + l\hat{v} + m\hat{w} = 0$

$k\,(1) + l\,(2)$:

$$(6) \quad -m(kU-\omega)\hat{w} + \frac{dU}{dz} k\hat{w} = (-k^2 - l^2)\hat{p}$$

$$m * (6): \quad -m^2(kU-\omega)\hat{w} + \frac{dU}{dz} km\hat{w} = (-k^2-l^2)m\hat{p}$$

From (3) : $\quad m\hat{p} = \hat{\theta} - (kU-\omega)\hat{w}$

$$(7) \Rightarrow -m^2(kU-\omega)\hat{w} + \frac{dU}{dz} km\hat{w} = (-k^2-l^2)(\hat{\theta} - (kU-\omega)\hat{w})$$

$(kU-\omega) * (7)$:

$$-m^2(kU-\omega)^2\hat{w} + \frac{dU}{dz} km(kU-\omega)\hat{w} = (-k^2-l^2)(kU-\omega)\hat{\theta} - (-k^2-l^2)(kU-\omega)^2\hat{w}$$

Using (4) :

$$-m^2(kU-\omega)^2\hat{w} + \frac{dU}{dz} km(kU-\omega)\hat{w} = (k^2+l^2)\left(\frac{d\Theta}{dz}+1\right)\hat{w} + (k^2+l^2)(kU-\omega)^2\hat{w}$$

$Fr = O(1)$: $\Rightarrow$

$$\boxed{\frac{dU}{dz} km(kU-\omega) - m^2(kU-\omega)^2 = (k^2+l^2)\left(\frac{d\Theta}{dz}+1+(kU-\omega)^2\right)}$$

with $U=0$, $\frac{d\Theta}{dz}=0 \quad (Fr \ll 1)$

$$-m^2\omega^2 = (k^2+l^2)(\omega^2+1) \Rightarrow \left(\frac{-m^2}{k^2+l^2} - \frac{k^2+l^2}{k^2+l^2}\right)\omega^2 = k^2+l^2$$

$$\Rightarrow \frac{|k|^2}{k_H^2}\omega^2 = k_H^2, \quad \boxed{\omega^2 = \frac{k_H^4}{|k|^2}}$$