Université de Montréal

NCAR

Solving the Monge-Ampère differential equation in the context of semi-Lagrangian advection schemes

Jean-Francois Cossette [a,*]

[a] *Université de Montréal, Montréal, Canada*

Piotr K. Smolarkiewicz [b]

[b] *National Center for Atmospheric Research, Boulder, CO80307, USA*

1 Context

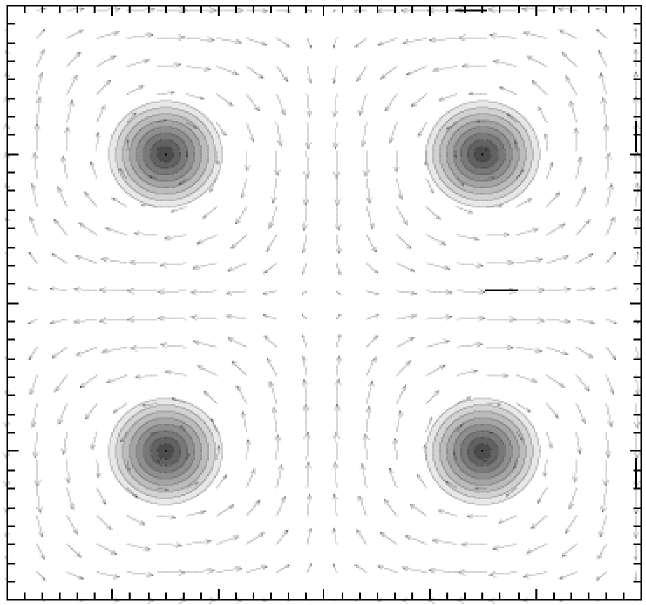
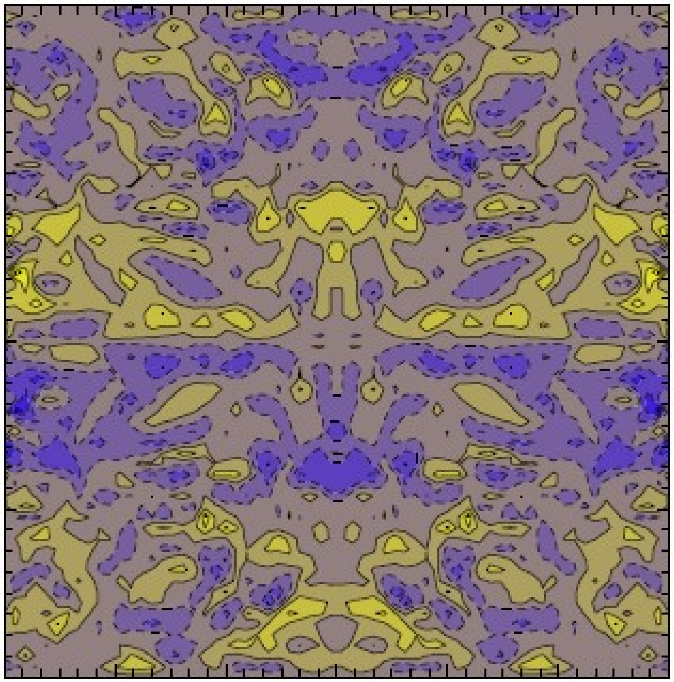
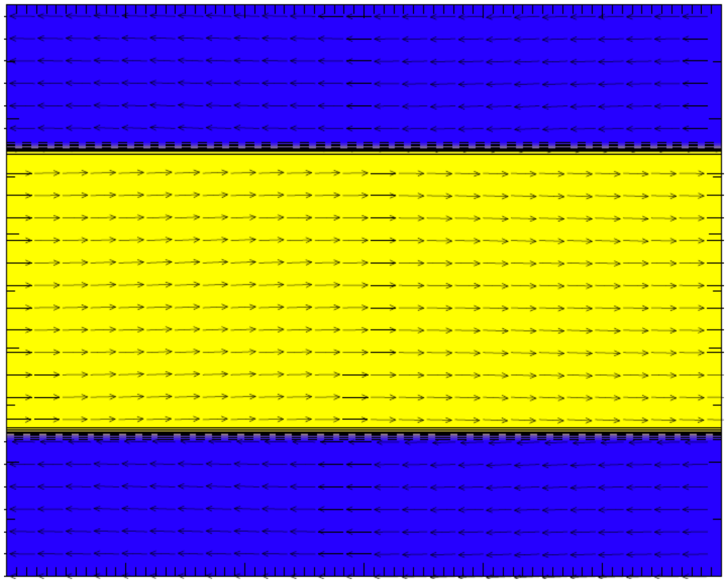
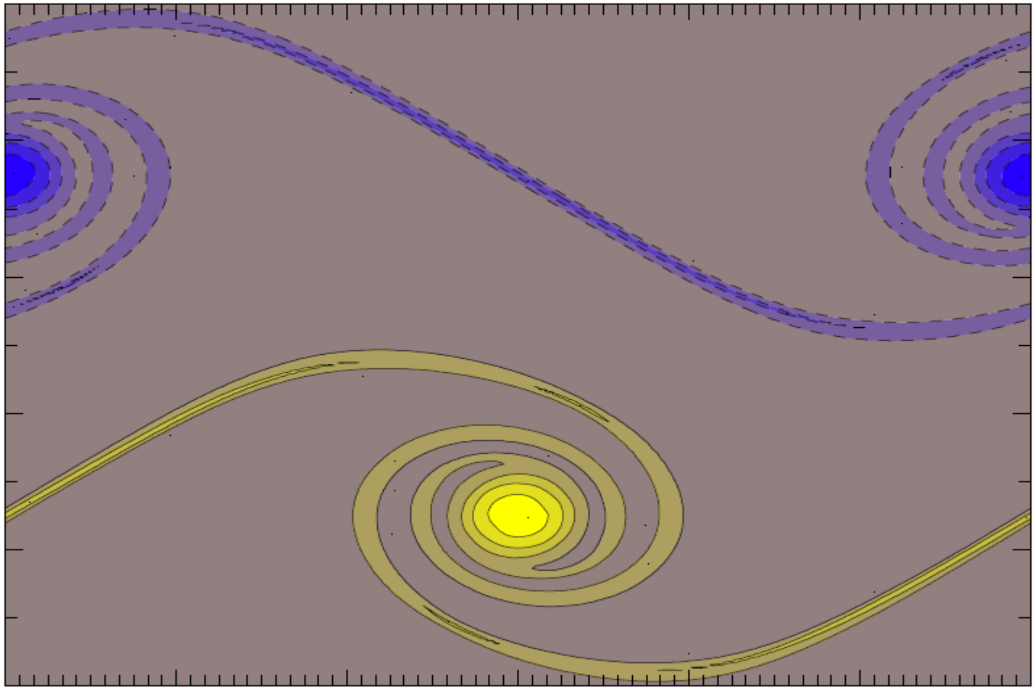
- Transporting a fluid variable in time and space:

$$\psi(\mathbf{x}_i, t_0) \rightarrow \psi(\mathbf{x}_i, t_1) \qquad \forall i \in N$$

$$t_1 > t_0$$

ψ: A scalar (velocity component, temperature, concentration of a substance)

Special case:
- Incompressible fluid
- Cartesian geometry
- periodic boundaries

→ How?

- Eulerian (methods for PDEs)

$$\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = R$$

- Lagrangian (methods for ODEs)

$$\frac{D\psi}{Dt} = R$$

→ Our approach is *Semi-Lagrangian*

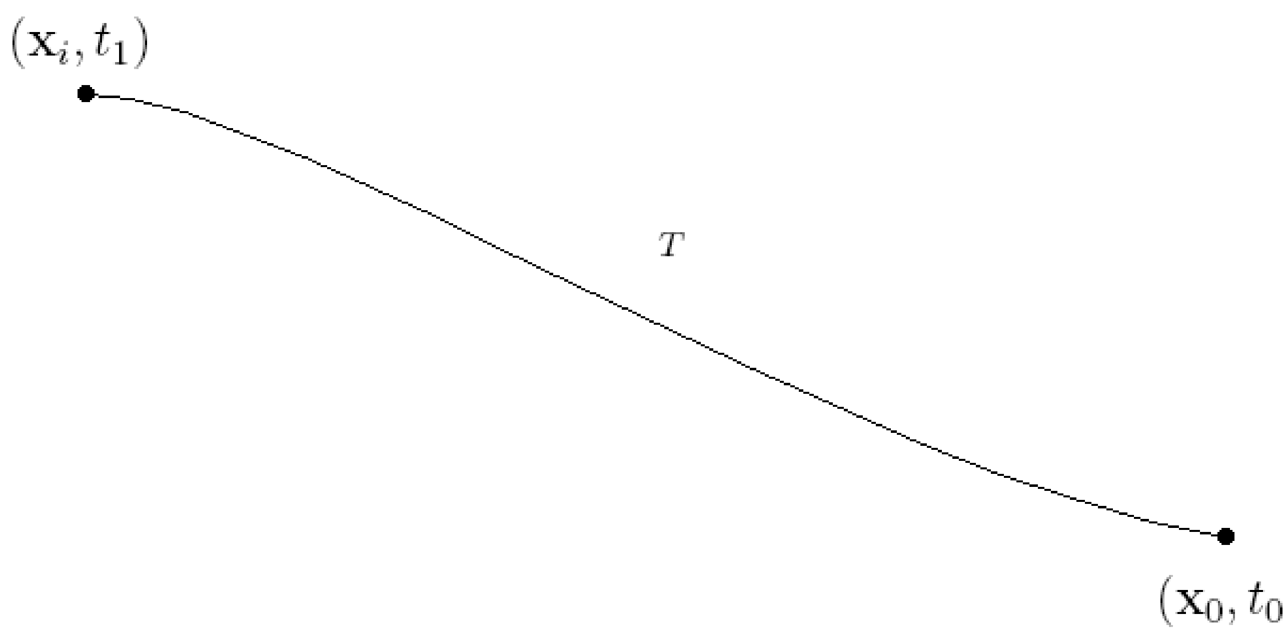
- Krisnamurti (1962), Robert(1981)
- Staniforth and Côté (1991), Smolarkiewicz and Pudykiewicz (1992)

→ Motivation: The improvement of mass continuity

2 Governing equations in Lagrangian form

$$\frac{D\psi}{Dt} = R$$

$$\psi(\mathbf{x}_i, t_1) = \psi(\mathbf{x}_0, t_0) + \int_T R dt$$



3 A semi-Lagrangian scheme

1 $\mathbf{x_0} = \mathbf{x_i} - APPROX(\int_{t0}^{t1} \mathbf{v} dt)$

2 $INTERP(\psi(\mathbf{x}_\star, t_0) \rightarrow \psi(\mathbf{x}_0, t_0))$

3 $\psi(\mathbf{x}_i, t_1) = \psi(\mathbf{x}_0, t_0) + APPROX(\int_T R dt)$

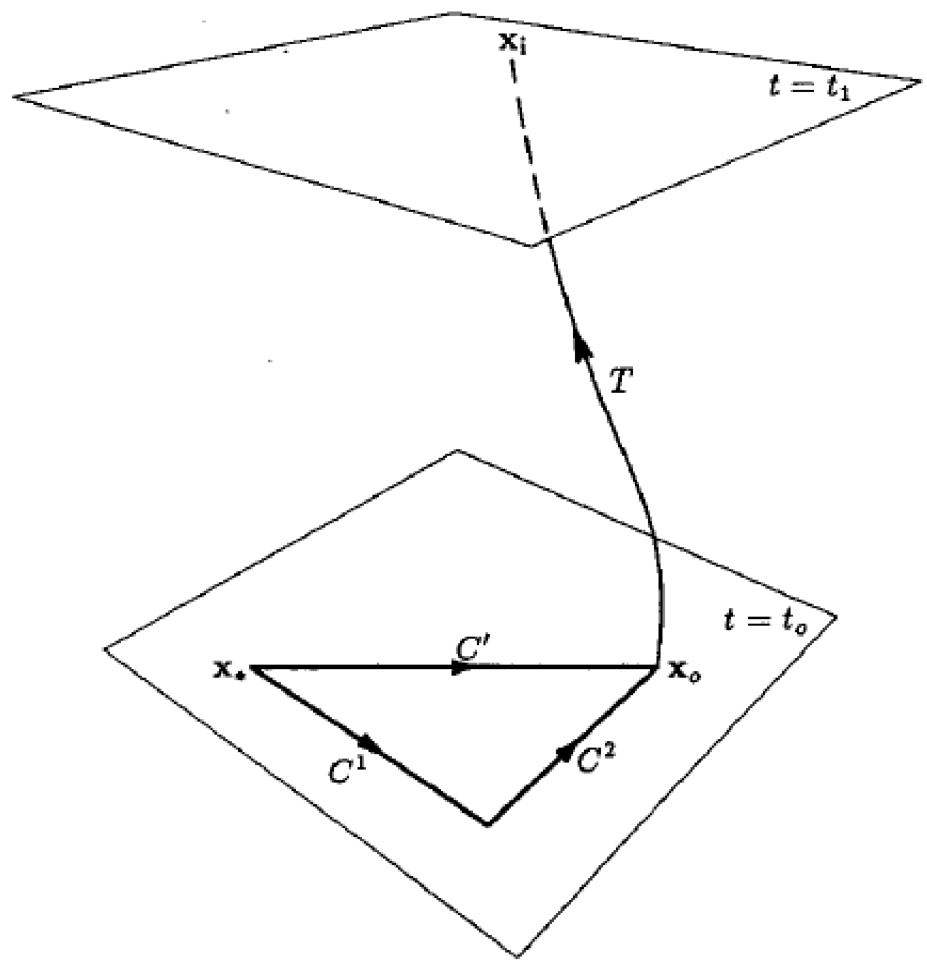


Fig. 1. A semi-Lagrangian contour of integration

4 Mass continuity in incompressible fluids

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \mathbf{v}$$

$$\Updownarrow$$

$$\rho(\mathbf{x_0}, t_0) = J^{-1}\rho(\mathbf{x_i}, t_1)$$

It follows that in incompressible fluids,

$$\nabla \cdot \mathbf{v} = 0 \Rightarrow J = 1$$

(c.f Chorin and Marsden)

5 Monge-Ampère equation correction

In general,

$$J = det\left\{\frac{\partial \mathbf{x_0}}{\partial \mathbf{x_i}}\right\} \neq 1$$

$\rightarrow$ so that J comes *closer* to unity, find the solution to

$$J = det\left\{\frac{\partial(\mathbf{x_0} + \nabla\phi)}{\partial \mathbf{x_i}}\right\} = 1$$

- e.g. in 2D:

$$a\frac{\partial^2\phi}{\partial x^2} + 2b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + e\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right) + d = 0$$

is the *Monge-Ampère equation*

6 A simple test: passive advection (R=0)

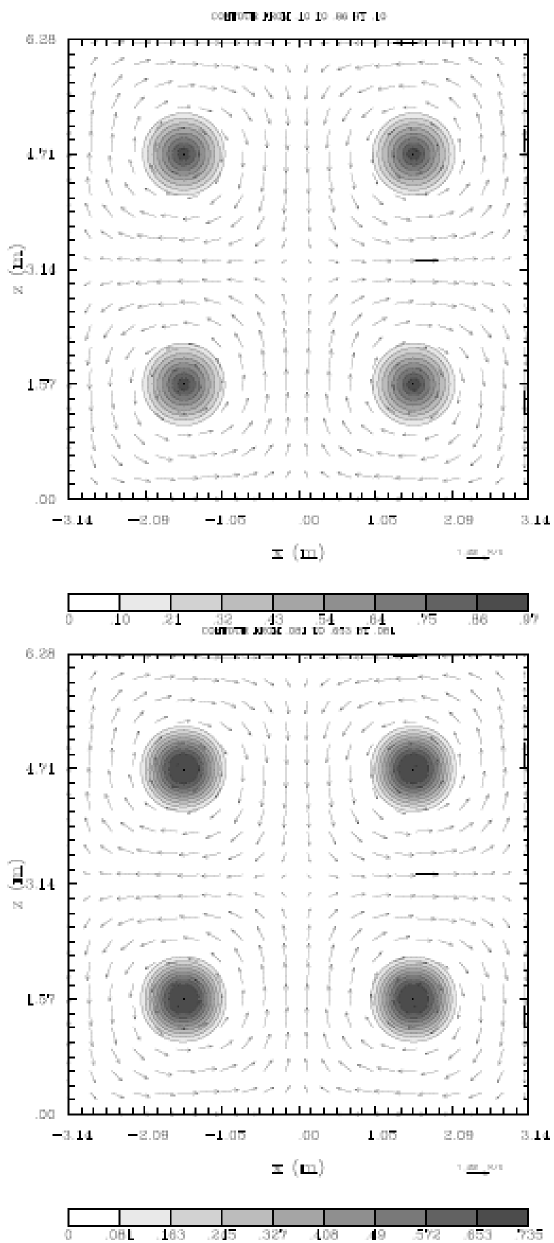
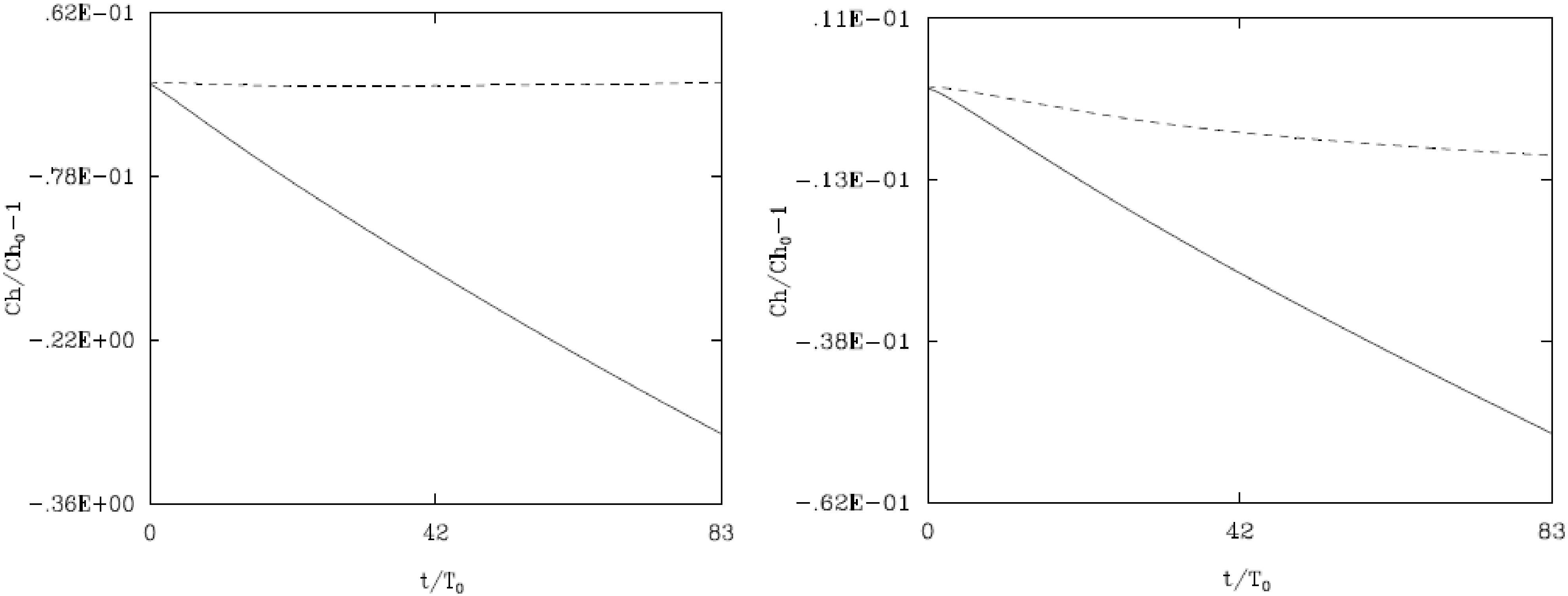


Fig. 3. Initial and final spatial distribution of ψ for a semi-Lagragian run at a 128^2 grid points resolution



C	max(J-1) (before)	max(J-1) (after)	min(J-1) (before)	min(J-1) (after)	I	N
4	$4.3e-2$	$2e-7$	$-4e-2$	$-2e-7$	4	128^2
4	$2.9e-3$	$5.2e-7$	$-3.6e-3$	$-5.6e-7$	4	256^2
4	$3.5e-4$	$1e-5$	$-5.5e-4$	$-7.6e-6$	4	512^2

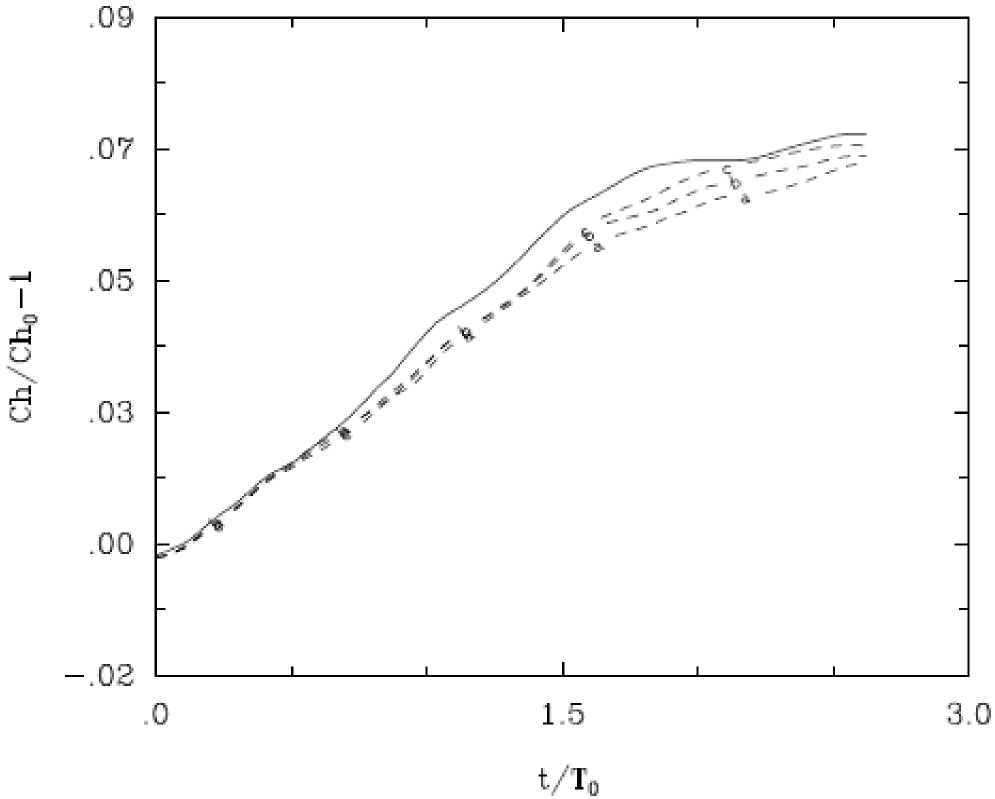
7 Comments on the Monge-Ampère equation

- Non-linear Elliptic PDE
- Pionneering work: Monge (1784), André-Marie Ampère (1820)
- System of nonlinear coupled ODEs: Iyanaga and Kawada (1980)
- differential geometry, optimal mesh adaptivity
- Algorithms for Full-nonlinear MA, Dean and Glowinski (2006)

- Rellich's theorem on the Monge-Ampère equation (1933)

There exist at most two solutions of the Monge-Ampère equation which satisfy the same boundary conditions

(c.f. Courant and Hilbert)



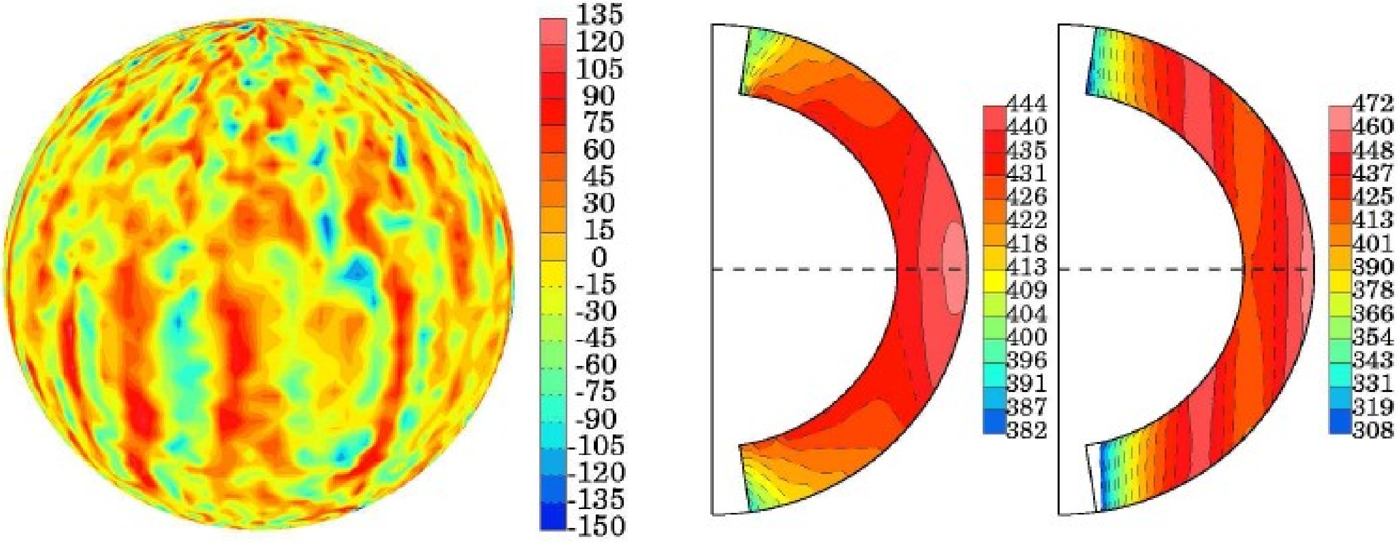
$$max|J_c| < max|J_b| < max|J_a| < 1$$

Objectives

• To gain insight on Monge-Ampère equation solutions' behavior in a varieties of incompressible and fully-compressible fluids

• curvilinear coordinates and open boudaries

• Ultimately: Solar convection



Elliott and Smolarkiewicz (2002)