Solving the Monge-Ampère differential equation in the context of semi-Lagrangian advection schemes

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1 Context

- Transporting a fluid variable in time and space:

\[ \psi(x_i, t_0) \rightarrow \psi(x_i, t_1) \quad \forall i \in N \]

\[ t_1 > t_0 \]

\( \psi \): A scalar (velocity component, temperature, concentration of a substance)

Special case:
- Incompressible fluid
- Cartesian geometry
- periodic boundaries
→ How?

- Eulerian (methods for PDEs)
  \[
  \frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = R
  \]

- Lagrangian (methods for ODEs)
  \[
  \frac{D\psi}{Dt} = R
  \]

→ Our approach is Semi-Lagrangian

- Staniforth and Côté (1991), Smolarkiewicz and Pudykiewicz (1992)

→ Motivation: The improvement of mass continuity
2  Governing equations in Lagrangian form

\[ \frac{D\psi}{Dt} = R \]

\[ \psi(x_i, t_1) = \psi(x_0, t_0) + \int_T Rdt \]
3 A semi-Lagrangian scheme

1 $x_0 = x_i - \text{APPROX}(\int_{t_0}^{t_1} v dt)$

2 $\text{INTERP}(\psi(x_*, t_0) \rightarrow \psi(x_0, t_0))$

3 $\psi(x_i, t_1) = \psi(x_0, t_0) + \text{APPROX}(\int_T R dt)$
Fig. 1. A semi-Lagrangian contour of integration
4 Mass continuity in incompressible fluids

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}
\]

\[\uparrow\]

\[
\rho(x_0, t_0) = J^{-1} \rho(x_i, t_1)
\]

It follows that in incompressible fluids,

\[
\nabla \cdot \mathbf{v} = 0 \Rightarrow J = 1
\]

(c.f Chorin and Marsden)
5 Monge-Ampère equation correction

In general,

\[ J = \text{det} \left\{ \frac{\partial x_0}{\partial x_i} \right\} \neq 1 \]

→ so that \( J \) comes closer to unity, find the solution to

\[ J = \text{det} \left\{ \frac{\partial (x_0 + \nabla \phi)}{\partial x_i} \right\} = 1 \]

• e.g. in 2D:

\[ a \frac{\partial^2 \phi}{\partial x^2} + 2b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + e \left( \frac{\partial^2 \phi \partial^2 \phi}{\partial x^2 \partial y^2} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right) + d = 0 \]

is the Monge-Ampère equation
6. A simple test: passive advection (R=0)

Fig. 3. Initial and final spatial distribution of $\psi$ for a semi-Lagrangian run at a $128^2$ grid points resolution
\begin{tabular}{cccccc}
C & max(J-1) & max(J-1) & min(J-1) & min(J-1) & I & N \\
(before) & (after) & (before) & (after) & & & \\
4 & 4.3e-2 & 2e-7 & -4e-2 & -2e-7 & 4 & 128^2 \\
4 & 2.9e-3 & 5.2e-7 & -3.6e-3 & -5.6e-7 & 4 & 256^2 \\
4 & 3.5e-4 & 1e-5 & -5.5e-4 & -7.6e-6 & 4 & 512^2 \\
\end{tabular}
7 Comments on the Monge-Ampère equation

- Non-linear Elliptic PDE
- Pioneering work: Monge (1784), André-Marie Ampère (1820)
- Differential geometry, optimal mesh adaptivity

- Rellich’s theorem on the Monge-Ampère equation (1933)

There exist at most two solutions of the Monge-Ampère equation which satisfy the same boundary conditions

(c.f. Courant and Hilbert)
\[ \max |J_c| < \max |J_b| < \max |J_a| < 1 \]
Objectives

- To gain insight on Monge-Ampère equation solutions’ behavior in a varieties of incompressible and fully-compressible fluids

- curvilinear coordinates and open boudaries

- Ultimately: Solar convection

Elliott and Smolarkiewicz (2002)