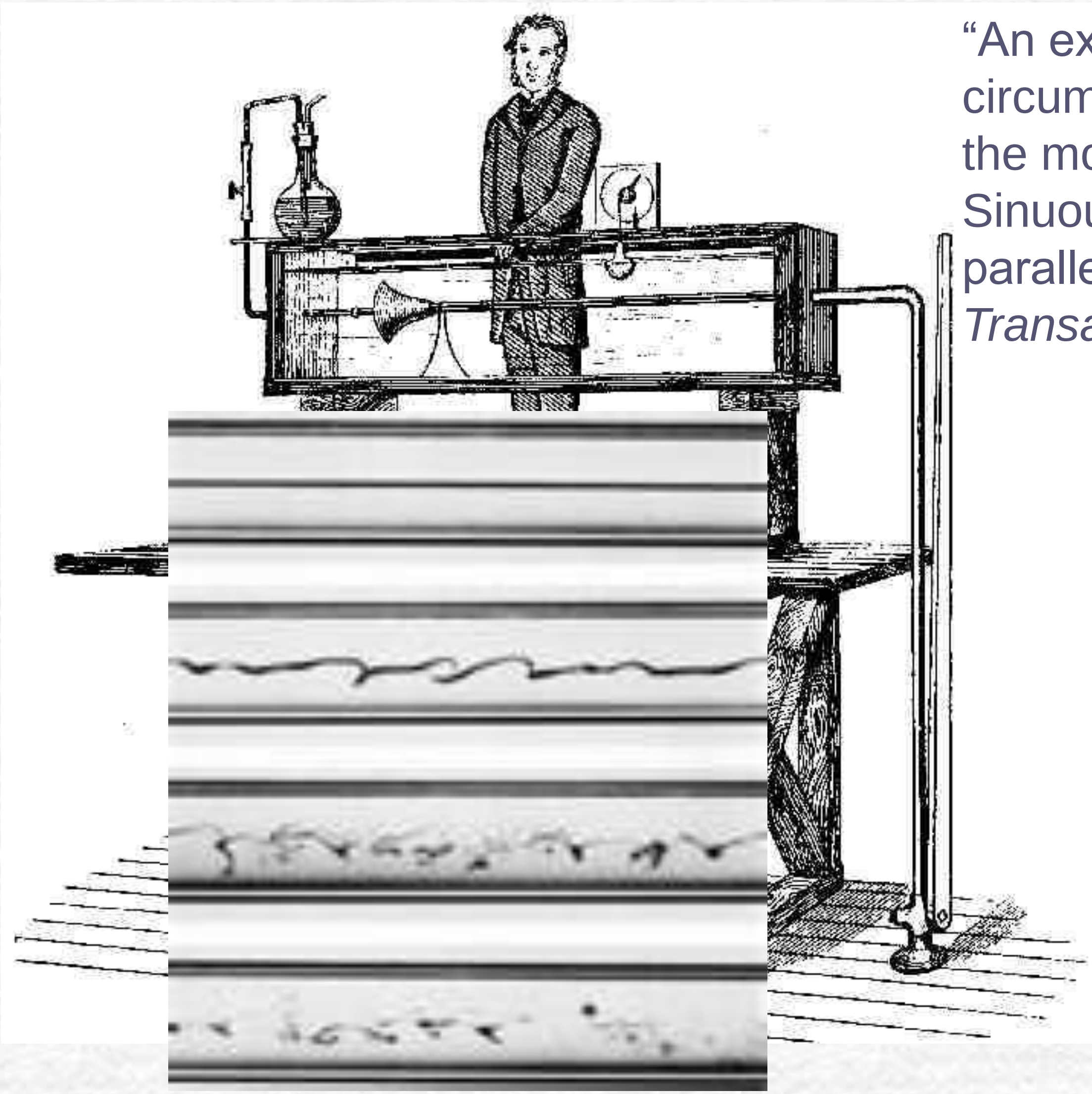
Turbulence in Geophysical Flows

Lecture 1

Osborne Reynolds (1842-1912)



“An experimental investigation of the circumstances which determine whether the motion of water shall be Direct or Sinuous, and of the law of resistance in parallel channels” *Philosophical Transactions, 1883*

Turbulence

- "Sinuous" motion (Reynolds 1883).
- Continuous state of instability (Monin & Yaglom 1961); proposed by Lev Landau, according to which flow undergoes an infinite sequence of bifurcations (instabilities) before it becomes unpredictable and chaotic.
- Irregular motion in space and time, which is not connected with the Brownian motion of the particles, so that average statistical values can be discerned (Hinze 1956; Corrsin 1961).
- Irregular motion of fluid that makes appearance during flow over boundaries or presence of velocity shear (Karman 1937; Taylor 1937)

Statistical Averaging

"On the Dynamical Theory of Incompressible Viscous Fluids and Determination of the Criterion," Phil Trans., Reynolds, 1894

Ensemble Averaging:

Conduct a large number (N) of identical experiments (realizations)

Ensemble Average

$$\overline{U}(\underset{\sim}{x},t) = \sum_{n=1}^{N} \frac{\tilde{u}_n(x,t)}{N}$$

Fluctuation

$$u'(\underset{\sim}{x},t) = \tilde{u}(x,t) - \overline{U}(x,t)$$

rms.

$$u(x,t) = \left(\overline{u'^2}\right)^{1/2} = \left\{ \frac{1}{N} \sum_{1}^{N} \left(\tilde{u}_n(x,t) - \overline{U}(x,t)\right)^2 \right\}^{1/2}$$

Stationary & Homogenous Turbulence

Stationary

$$\frac{\partial \overline{U(\underset{\sim}{x},t)}}{\partial t} = 0 \Rightarrow \overline{U}(\underset{\sim}{x},t) = \overline{U}(\underset{\sim}{x});$$

$$\overline{U}(\underset{\sim}{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} d\tau \, \tilde{u}(\underset{\sim}{x}, t+\tau)$$

homogeneous

$$\frac{\partial}{\partial x_j} \overline{U}(\underset{\sim}{x},t) = 0 \Rightarrow \overline{U}(\underset{\sim}{x},t) = \overline{U}(t);$$

$$\overline{U}(t) = \lim_{x \to 0} \frac{1}{2x} \int_{-x}^{x} dr \, \tilde{u}(\underset{\sim}{x}+\underset{\sim}{r},t)$$

Ergodicity – all converge to ensemble averaging at sufficiently large averaging periods (G.D. Birkhoff, circa 1930)

Reynold's (1894) Averaging Rules

$$\overline{f+g} = \overline{f} + \overline{g}$$

$$\overline{af} = a\overline{f}$$ If a is a constant

$$\overline{f\overline{g}} = \overline{f}\,\overline{g}$$

$$\frac{\overline{\partial f}}{\partial x_i} = \frac{\partial \overline{f}}{\partial x_i}$$

Examples:

$$\overline{\tilde{u}_i \tilde{u}_j} = \overline{(\overline{U}_i + u'_i)(\overline{U}_j + u'_j)} = \overline{U}_i \overline{U}_j + \overline{u'_i u'_j} \quad , \quad \overline{\overline{U}_i u'_j} = \overline{U}_i \overline{u'_j} = 0$$

Reynolds Averaged Equations

Continuity Equation:

$$\frac{\partial \tilde{u}_j}{\partial x_j} = 0$$

$$\frac{\overline{\partial(\overline{U}_j + u'_j)}}{\partial x_j} = \frac{\partial}{\partial x_j}(\overline{U}_j + \overline{u'}_j) = \frac{\partial \overline{U}_j}{\partial x_j} = 0$$

Momentum Equation

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \varepsilon_{ijk}\Omega_j \tilde{u}_k = \frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i} + \tilde{b}\,\delta_{i3} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}$$

$$\tilde{u}_i = \overline{U}_i + u'_i\,,\quad \tilde{b} = \overline{b} + b' \quad and \quad \tilde{p} = \overline{p} + p'$$

$$\frac{\partial\left(\overline{U}_i + u_i'\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{U}_i + u_i'\right)\left(\overline{U}_j + u_j'\right) + \varepsilon_{ijk}\Omega_j(\overline{U}_k + u_i') = \frac{1}{\rho_o}\frac{\partial\left(\overline{p} + p'\right)}{\partial x_i} + \left(\overline{b} + b'\right)\delta_{i3} + \frac{\nu\partial^2\left(\overline{U} + u_i'\right)}{\partial x_j \partial x_j}$$

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U}_j \frac{\partial \overline{U}}{\partial x_j} + \varepsilon_{ijk}\Omega_j \overline{U}_k = \frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x_i} + \overline{b}\delta_{iz} + \frac{\partial}{\partial x_j}\left(\left(\nu \frac{\partial \overline{U_i}}{\partial x_j}\right) - \overline{u_i' u_j'} \right)$$

$$-\overline{u_i' u_j'} = K_{ij\alpha\beta}\left(\frac{\partial \overline{U_\alpha}}{\partial x_\beta} + \frac{\partial \overline{U_\beta}}{\partial x_i}\right)$$

Turbulent Motions

- It remains to call attention to the chief outstanding difficulty of our subject… $u.\nabla u$

  – Sir Horace Lamb (1897, 1932)

- ……and this remains valid even today

Flow, Turbulence and Combustion 66: 241–

A Century of Turbulence*

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Abstract. A brief, superficial survey of some very
hundred years in turbulence. Some conclusions ca
have a pyramidal structure, like the best of physics.
experiments are exploratory experiments. What does this mean?
We believe it means that, even after 100 years, turbulence studies are still in their infancy. We are naturalists, observing butterflies in the wild. We are still discovering how turbulence behaves, in many respects. We do have a crude, practical, working understanding of many turbulence phenomena but certainly nothing approaching a comprehensive theory, and nothing that will provide predictions of an accuracy demanded by designers.

Key words: history, turbulence.

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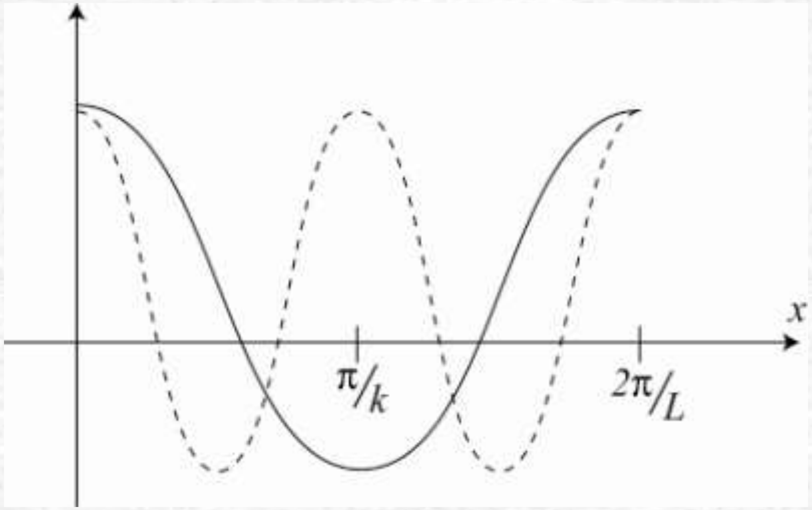
..nothing approaching a comprehensive theory…

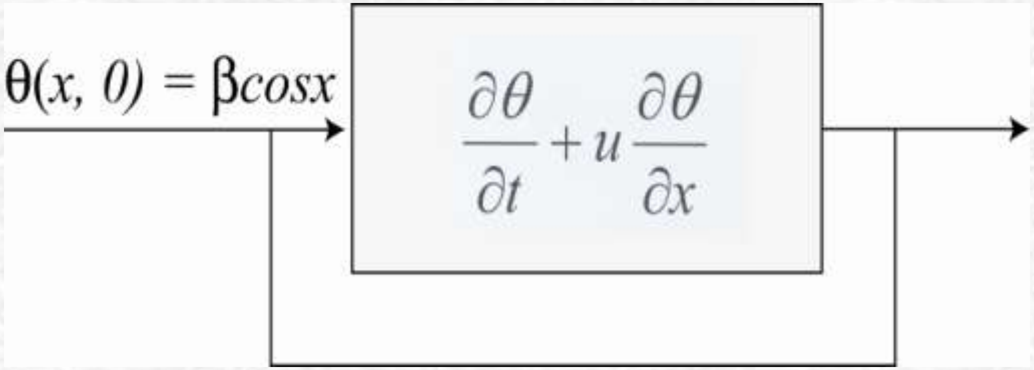
..and nothing that will provide predictions of an accuracy demanded by designers.

Non-Linearity

$$\frac{\partial\theta}{\partial t} + \theta\frac{\partial\theta}{\partial x} = 0$$

$$\theta(x,0) = \beta\cos kx,$$





$$\theta(x,t) \approx \beta\cos kx + \frac{\beta^2}{2}(kt)\sin(2kx)$$

Cascade and TKE dissipation

- Big whorls have little whorls, which feed on their velocity; and little whorls have lesser whorls, and so on to viscosity" –Richardson 1922

Turbulence

Viscous	Inertia	Inertia / Viscous
$\nu\nabla^2 u \sim \nu U/L^2$	$u \bullet \nabla u \sim U^2/L$	$\frac{UL}{\nu} = R_e$

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + w \frac{\partial u_\alpha}{\partial z} + \varepsilon_{\alpha jk} f \ell_j u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \left(\nu \frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\alpha} + \frac{\partial^2 u_\alpha}{\partial z^2} \right)$$

Energy Cascade

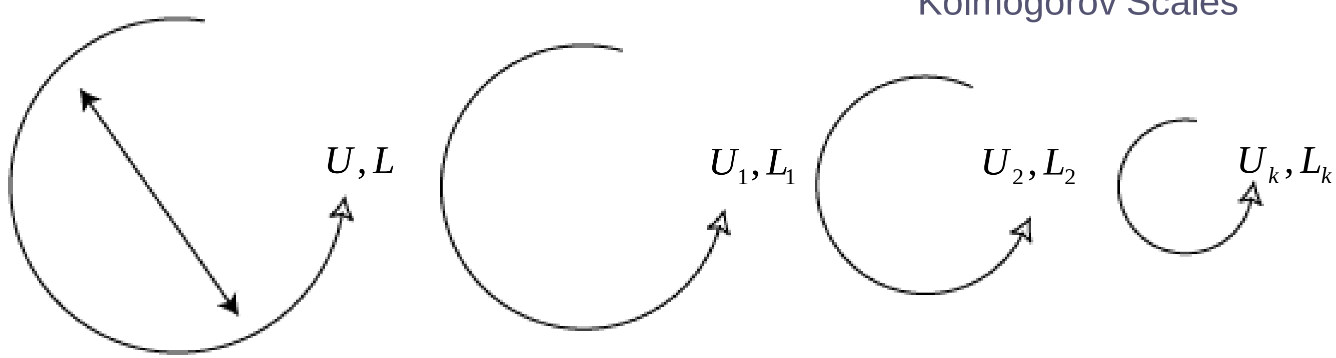
$$\varepsilon \propto \frac{U^3}{L}$$

$$\varepsilon \;\; units \;\; L^2 / T^3$$

Viscosity is unimportant
Energy flux is conserved

$$\varepsilon \;\; l_k, u_k = f[\varepsilon, \nu]$$

Kolmogorov Scales



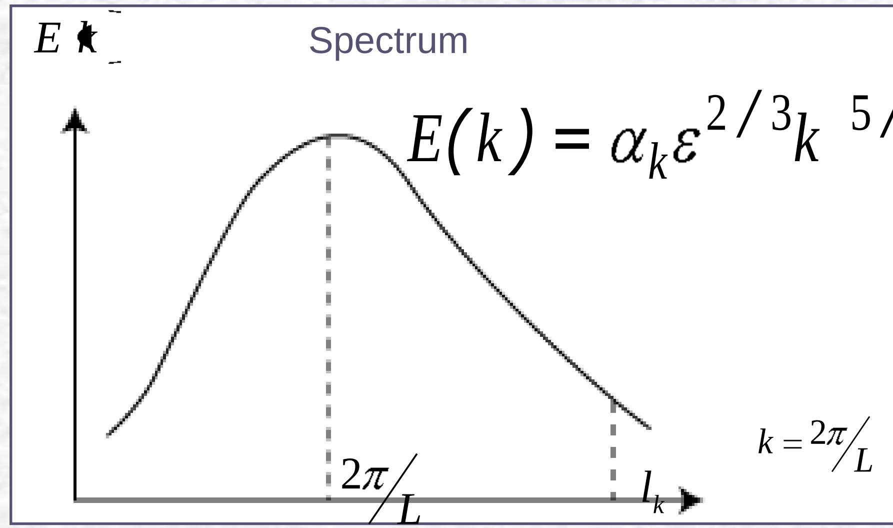
$$\frac{UL}{\nu} >>> 1 \qquad \frac{U_1 L_1}{\nu} >> 1 \qquad \frac{U_2 L_2}{\nu} > 1 \qquad \frac{U_k L_k}{\nu} \sim 1$$

Kolmogorov Hypothesis

$$\ell_k = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$ (length)

$$v_k = (\nu\varepsilon)^{1/4}$$ (velocity)

and $$T_k = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$ (time)



$$F_{11}(k_1) = \alpha_1 \varepsilon^{2/3} k_1^{\;\;5/3}$$

Spectral Analysis of Turbulent Flows (2/2)

$$u(x) = \sum_{-\infty}^{+\infty} a_n e^{\frac{i2\pi nx}{L}}, \; n = 1,2,3....$$

which is the Fourier representation of the signal. Further, $a_n = \frac{1}{L}\int_{-L/2}^{L/2} u(x) e^{\frac{-i2\pi nx}{L}} dx$

Now define $\frac{2\pi n}{L} = k$ and $a_n L = \hat{U}(k)$

Then $u(x) = \sum_{-\infty}^{+\infty} a_n e^{ikx}$ and $\hat{U}(k) = \int_{L/2}^{L/2} u(x) e^{ikx} dx$

Now that *k* changes as $\frac{2\pi}{L}, \frac{4\pi}{L}, \frac{6\pi}{L} \ldots$ as n takes $n = 1,2,3.......$

So that $dk = \frac{2\pi}{L}$

$$u(x) = \frac{1}{2\pi}\int_{L/2}^{L/2} \hat{U}(k) e^{-ikx} dx$$

G.I. Taylor Simplification

$$u_i(t) = u_i(t) \quad for \quad -T < t \le T$$

$$= 0 \quad for \ |t| > T$$

Then, $$u_i(t) = \int_{-\infty}^{+\infty} \hat{u}_i(\omega) e^{i\omega t} d\omega$$

and $$\hat{u}_i(\omega) = \frac{1}{2\pi} \int_{-T}^{T} u_i(t) e^{-i\omega t} dt$$

Hence, $$\overline{u_i(t) u_j^*(t+\tau)} = \int_{-\infty}^{\infty} \psi_{ij}(\omega) e^{+i\omega\tau} d\omega$$

$$\overline{u_i(x) u_j^*(x+r)} = \int_{\infty}^{\infty} F_{ij}(k) e^{+ikr} d\omega$$

$$|u_i|^2 = \int \psi_{ii}(\omega) d\omega$$

A Fourier mode

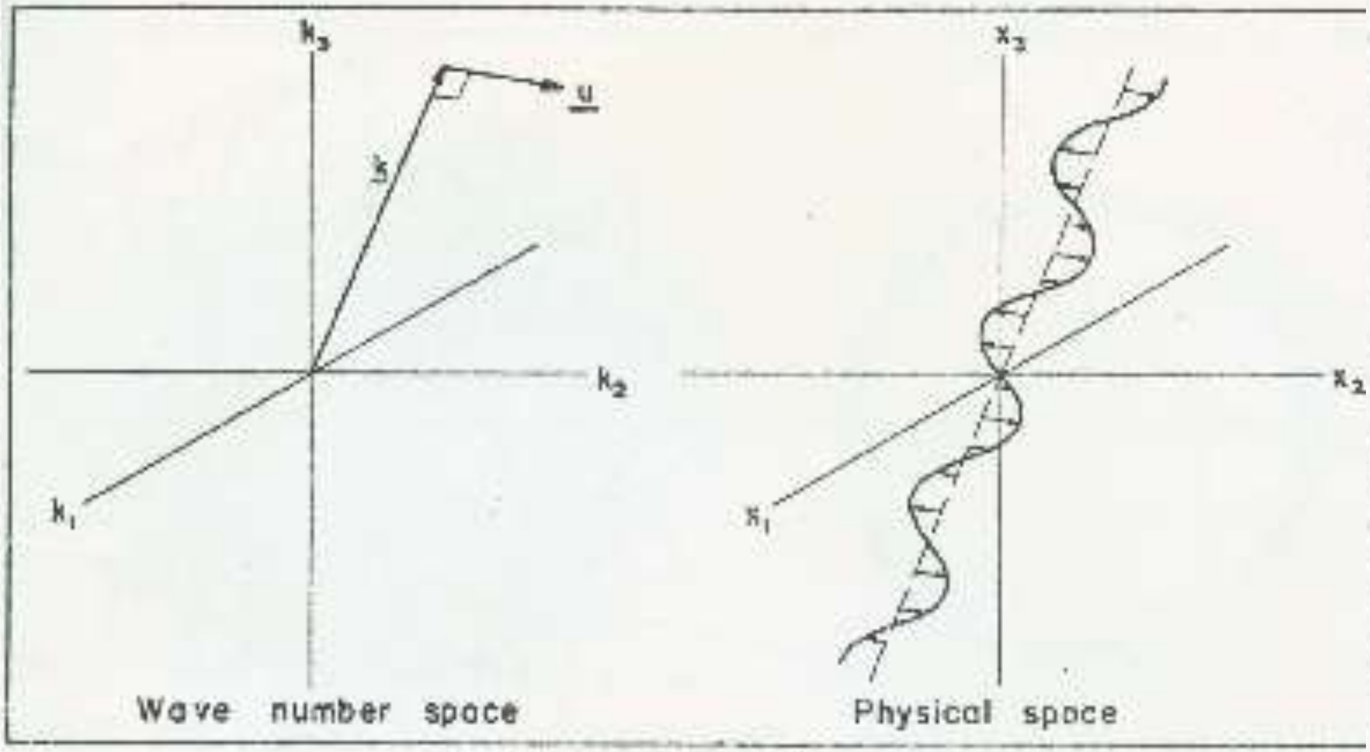


Fig. 1—Representation of shear waves.

$$\underset{\sim}{u}(\underset{\sim}{x}) = \sum_{\infty}^{+\infty} \underset{\sim}{a_n} e^{i\underset{\sim}{k}.\underset{\sim}{x}}$$

$$\underset{\sim}{u} . \underset{\sim}{k} = 0$$

One and Three Dimensional Spectra

$$\overline{u'_i(\underset{\sim}{x})u'_j(\underset{\sim}{x}+\underset{\sim}{r})} = R_{ij}(\underset{\sim}{r}) = \int_{\infty}^{+\infty}\int\int \phi_{ij}(\underset{\sim}{k})e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\,d\underset{\sim}{k}$$

where $\underset{\sim}{r} = (r_1, r_2, r_3)$ and $\underset{\sim}{k} = (k_1, k_2, k_3)$

$$R_{ij}(r,0,0) = \int_{\infty}^{+\infty}\int\int \phi_{ij}(\underset{\sim}{k})e^{ik_1 r}\,dk_1 dk_2 dk_3$$

$$F_{ij}(k_1) = \int_{-\infty}^{+\infty} \varnothing_{ij}(\underset{\sim}{k})\,dk_2 dk_3$$ one-dimensional spectra

$$\overline{u_i(x)u_j^*(x+r)} = \int_{\infty}^{\infty} F_{ij}(k)\,e^{+ikr}\,d\omega$$

The wave number k

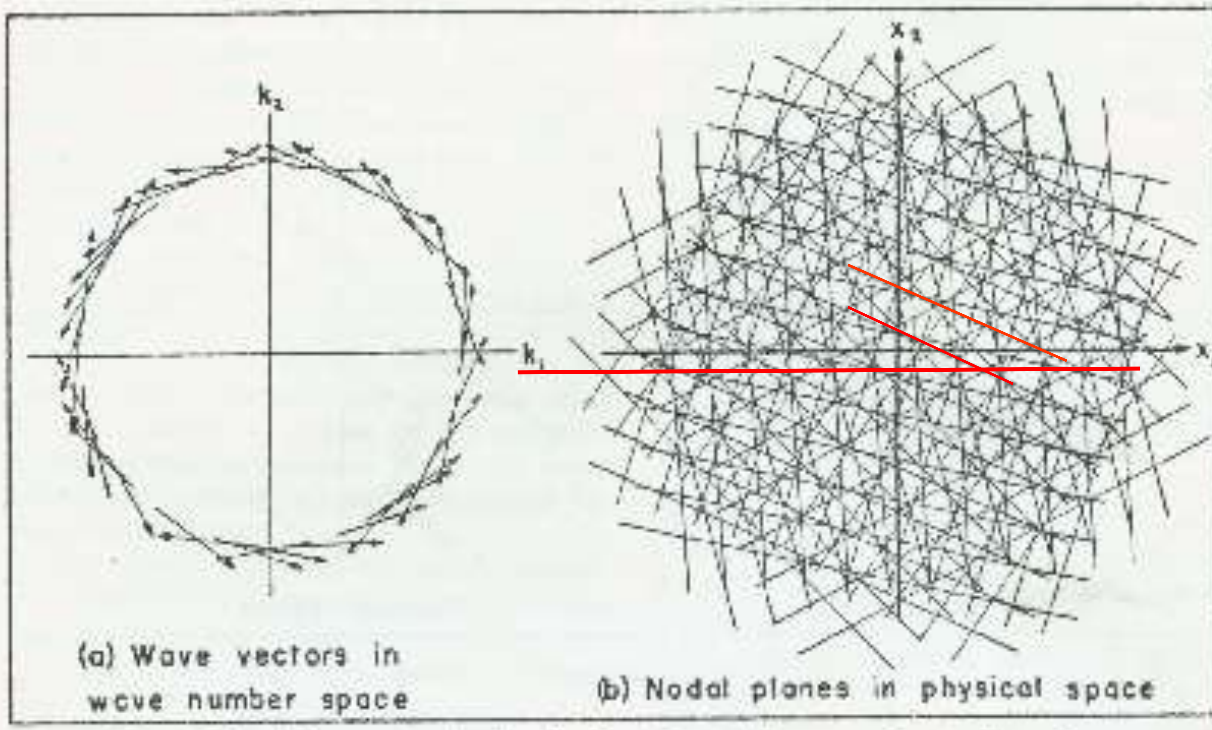


FIG. 2—Sample from an isotropic collection of equal shear waves in two dimensions

Spectra for a single wave number

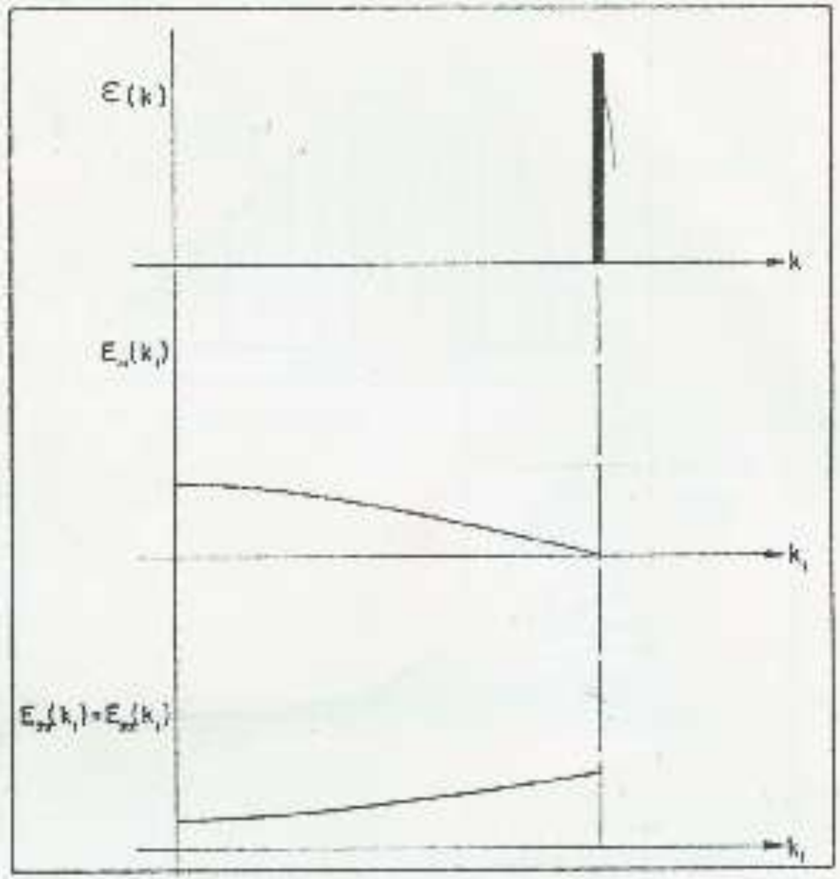
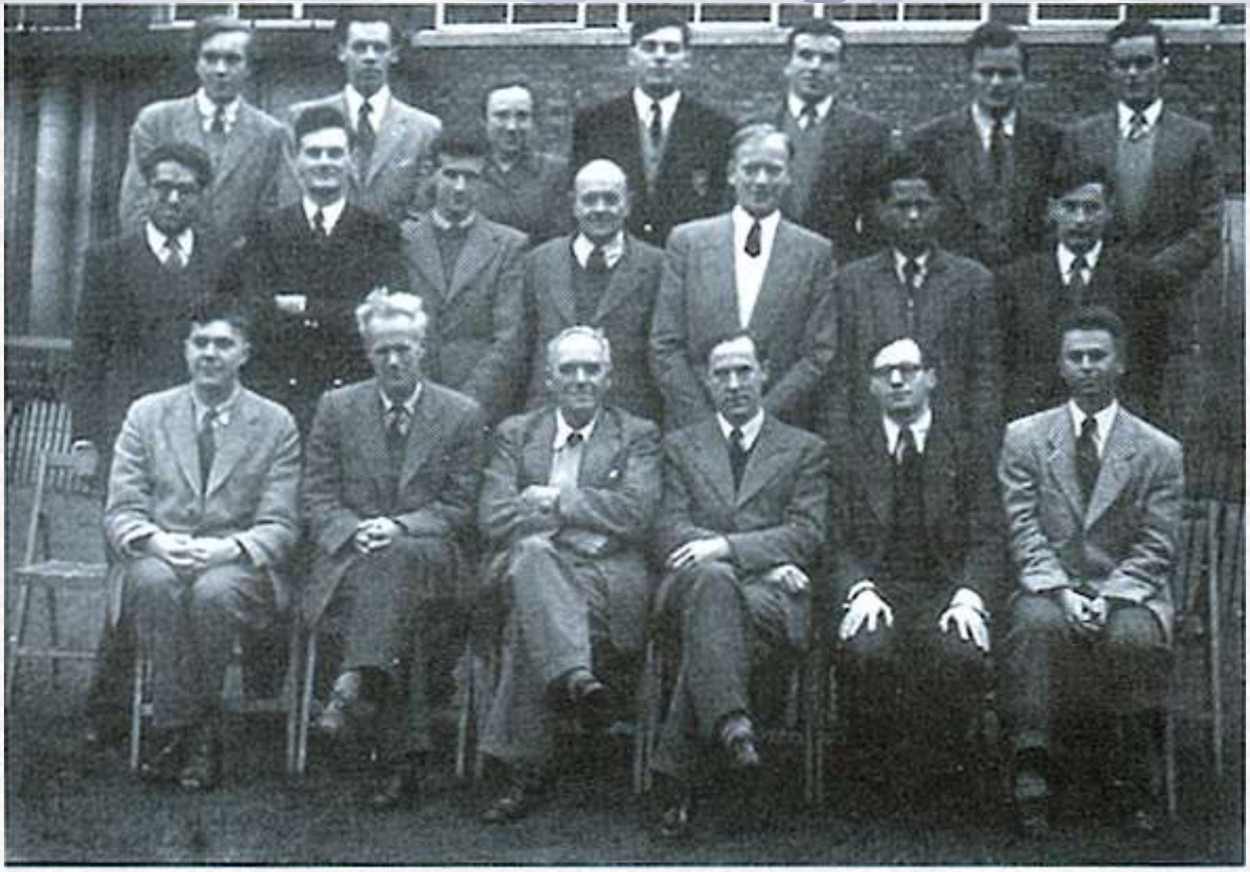


FIG. 4—A 'Dirac type' three-dimensional isotropic spectrum and the two corresponding kinds of one-dimensional spectra.



The Fluid Dynamics group in the Cavendish Laboratory, April 1955. Front row: Ellison, Townsend, Taylor, Batchelor, Ursell, van Dyke. Middle row: Barua, Thomas, Morton, Thompson, Philips, Bartholomeusz, Thorne. Back row: Nisbet, Grant, Hawk, Saffman, Wood, Hutson, Turner. *From Huppert (2000).*

Turbulent Kinetic Energy Equation

$$\frac{\partial \frac{\overline{u'^2_i}}{2}}{\partial t} + \overline{U}_j \frac{\partial \frac{\overline{u'^2_i}}{2}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{u'_j u'^2_i / 2} + \frac{\overline{p' u'_j}}{\rho_0} \right) + \overline{b' u'_3} + \nu \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j \partial x_j}}$$

(A) (B) (C) (D) (E) (F)

- (A) rate of increase of TKE a given point
- (B) gain of TKE by the advection caused by mean flow
- (C) production of turbulent kinetic energy by the working of mean flow on the Reynolds stresses
- (D) transport of turbulent kinetic energy by the advection by the turbulent fluctuations $\overline{u'_j u'^2_i / 2}$ and turbulent pressure
- $\overline{b'w'}$ > 0 represents convection and <0 stably stratified turbulence

TKE Dissipation

$$\varepsilon = 2\nu \overline{s'_{ij} s'_{ij}} \approx \nu \left(\overline{\omega'_i \omega'_i} \right)$$

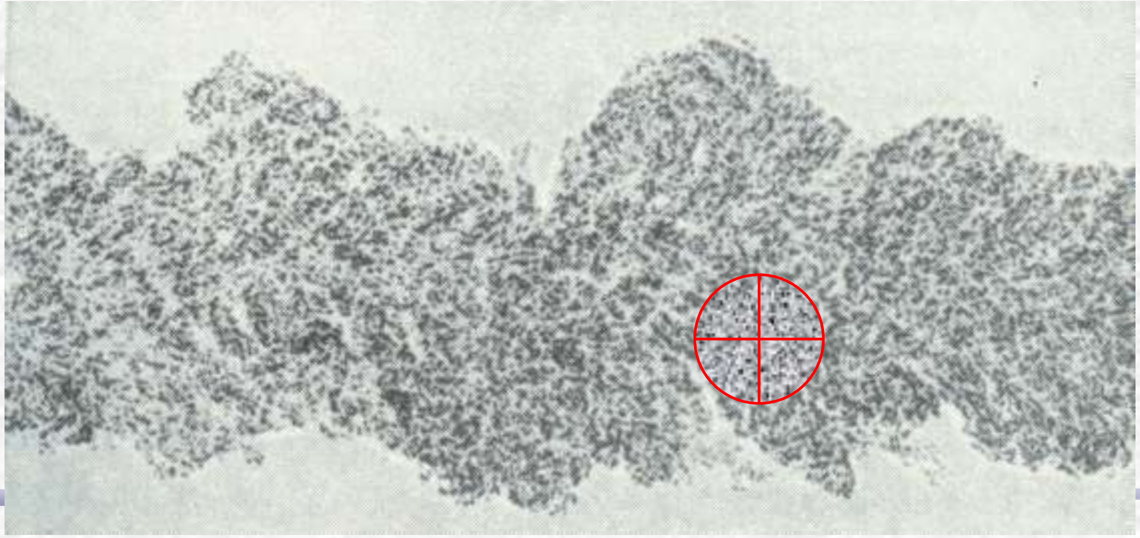
$$s'_{ij} = \frac{1}{2} \left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)$$

$$\varepsilon \propto \frac{U^3}{L}$$

$$\underbrace{\nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}}}_{F} = \underbrace{\nu \frac{\partial^2 \frac{\overline{u'^2_i}}{2}}{\partial x_j \partial x_j}}_{F_1} + \underbrace{\nu \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j}}_{F_2} - \underbrace{\varepsilon}_{F_3}$$

Isotropic Turbulence

- A flow in which any relationship between turbulent quantities (in averaged sense) is invariant under rotation of the coordinate system, translation of the coordinate system and under the reflection with respect to the coordinate system.



$$\varepsilon = 2\nu \overline{s'_{ij}\, s'_{ij}} = \nu \overline{\omega_i \omega_i}$$

$$\varepsilon = 15\nu \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}$$

For isotropic turbulence

$$\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \frac{\overline{u_1^2}}{\lambda^2} \qquad \lambda = l\left(uL/\nu\right)^{1/2}$$

Microscale Reynolds Number

$$R_\lambda = \frac{u\lambda}{\nu}$$

WHERE IN THE SPECTRUM IS DISSIPATION?

Kolmogorov scales?

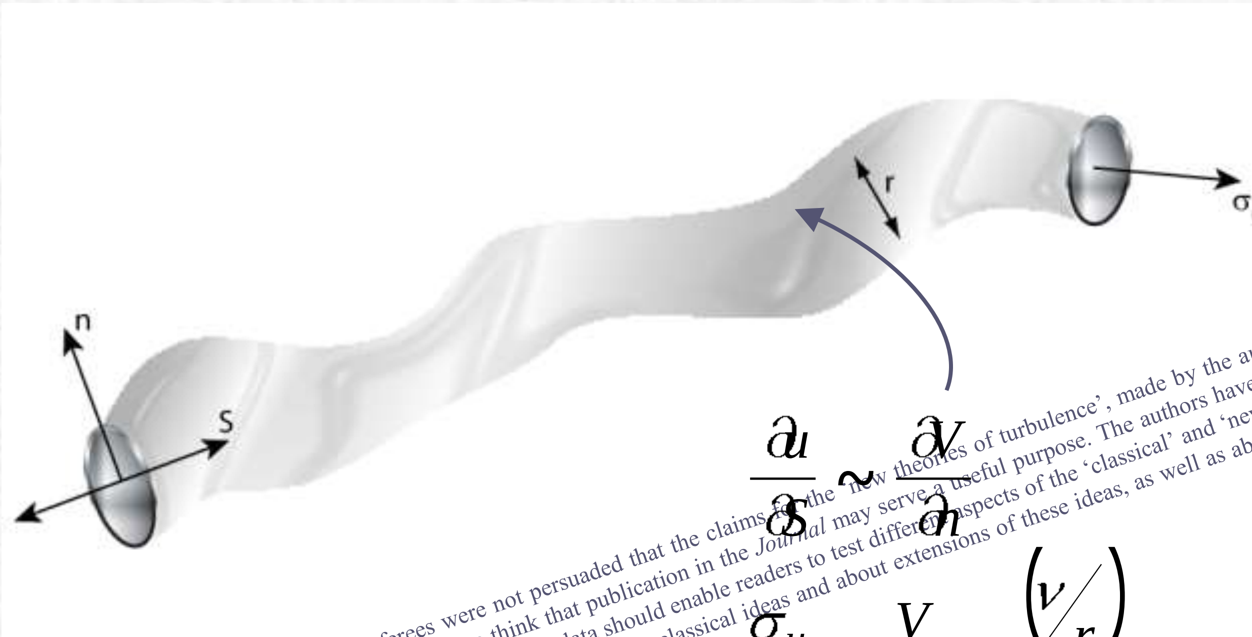
$$\eta \equiv (\upsilon^3/\varepsilon)^{1/4}$$

$$u_\eta \equiv (\varepsilon\upsilon)^{1/4}$$

$$\tau_0 \equiv (\upsilon/\varepsilon)^{1/2}$$

$$\frac{u_\eta \eta}{\nu} = 1$$

R.R. Long. 1982. A new theory of energy spectrum, *B.L.M.*, 24, 136-160.



† *Editorial footnote*. Although the referees were not persuaded that the claims for the 'new theories of turbulence', made by the authors in the abstract and elsewhere in this paper, are justified, we think that publication in the *Journal* may serve a useful purpose. The authors have assembled a large body of data for various turbulent flow systems. These data should enable readers to test different aspects of the 'classical' and 'new' theories for themselves and should stimulate thought about the foundations of the classical ideas and about extensions of these ideas, as well as about the validity of the new theories.

$$\frac{\partial u}{\partial s} \sim \frac{\partial V}{\partial n}$$

$$\frac{\sigma_u}{l} \sim \frac{V}{r} \sim \frac{\left(\nu/r\right)}{r}$$

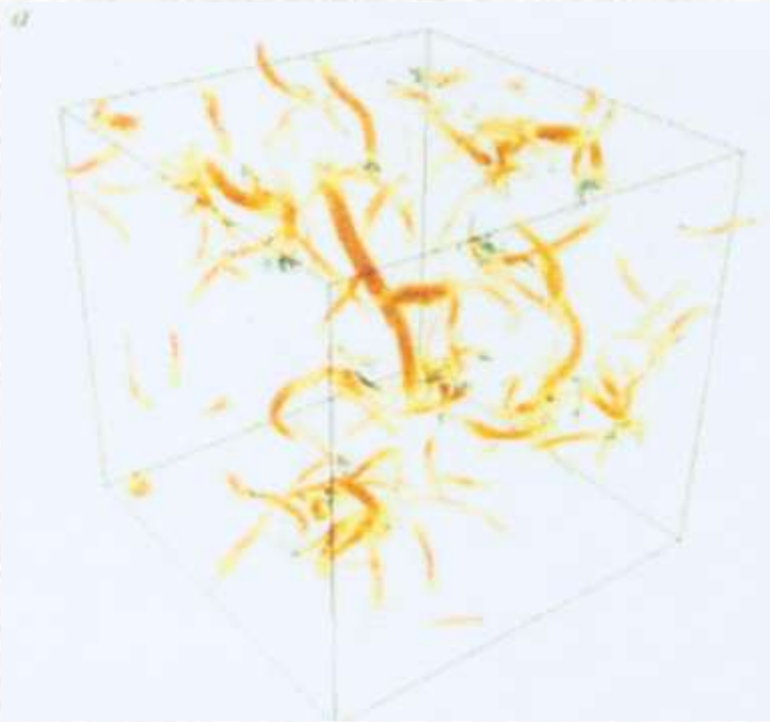
$$r \sim l \quad (\sigma_d l/\nu)^{\frac{1}{2}}$$

$$\sim l\,Re^{-1/2} \equiv \lambda$$

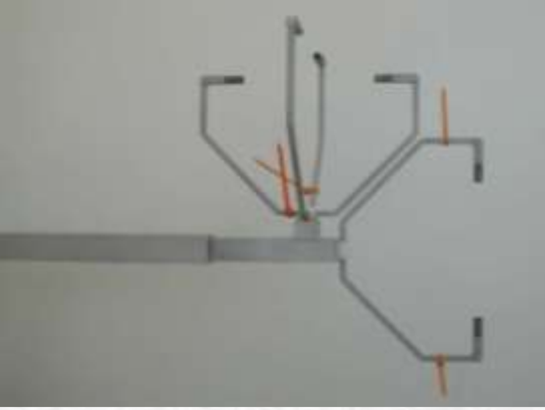
$$l_k < \lambda < l$$

$$\left(l\,Re^{-3/4}\right) < \left(l\,Re^{-1/2}\right) < l$$

Structure of turbulence and dissipation



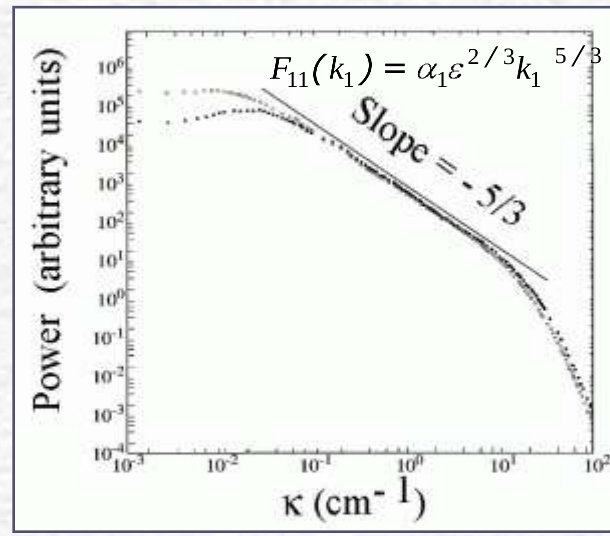
She et al., Nature, Vol. 334, 226-228, 1990



Dissipation Estimation

Taylor's hypothesis: $\overline{\left(\frac{\partial u'}{\partial t}\right)^2} = \overline{U}^2 \overline{\left(\frac{\partial u'}{\partial x}\right)^2}$ if $\frac{u}{U} << 1$ then $t = \frac{x}{U}$

spatial autocorrelation: $R_{xx}(r) = \langle u(x)u(x+r) \rangle$

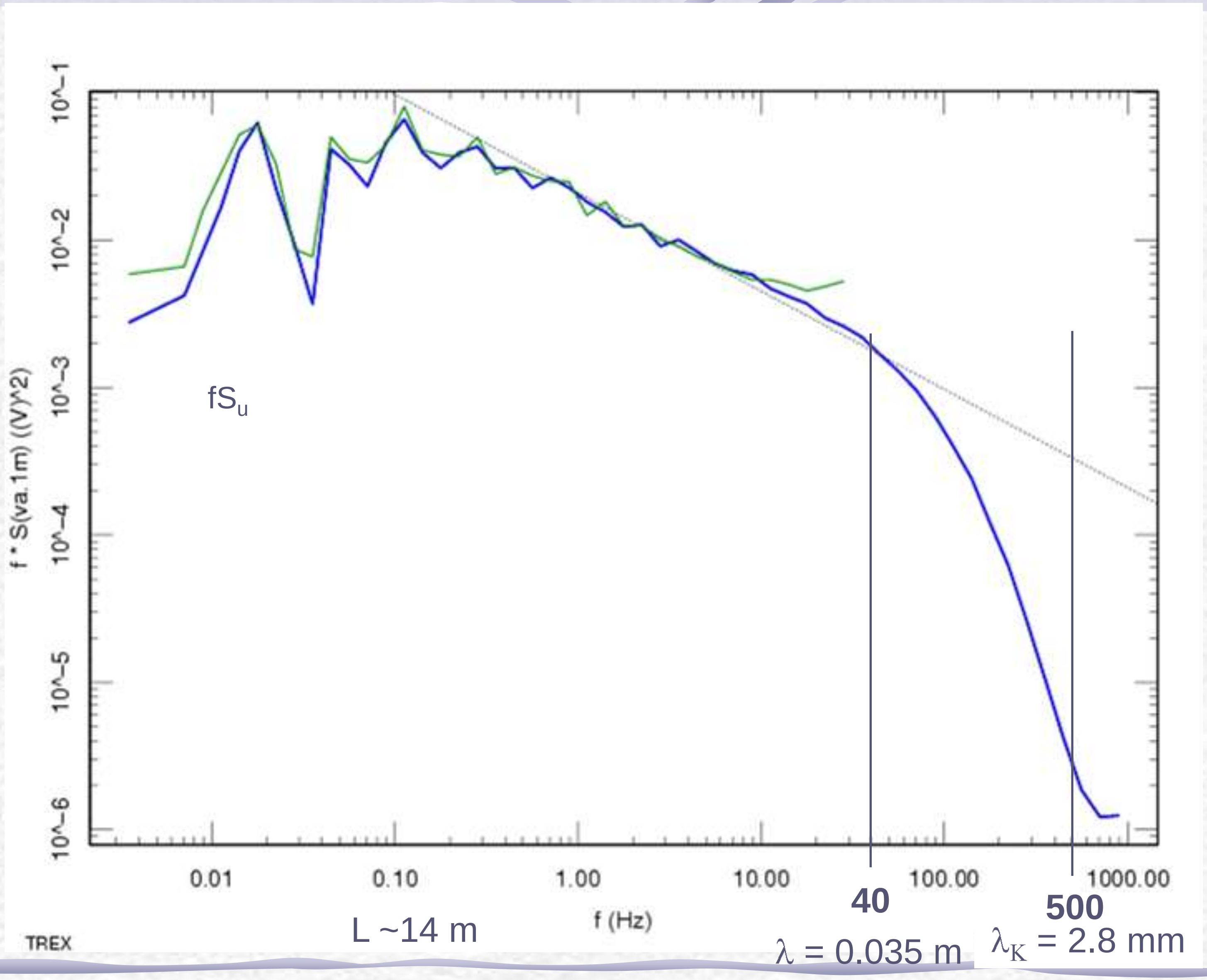


1-D velocity spectrum: $F_{11}(k_1) = \int_0^\infty R_{xx} e^{-iwx} dx$

Kolmogorov spectrum: $F_{11}(k_1) = \alpha_1 \varepsilon^{2/3} k_1^{\;5/3}$ for inertial subrange

Outdoor Three-dimensional In-situ calibrated Hot-film anemometry System (OTIHS) – 3D probe

Poulos, Semmer, Militzer, Maclean, Horst, Oncley…..



Courtesy: Greg Poulos

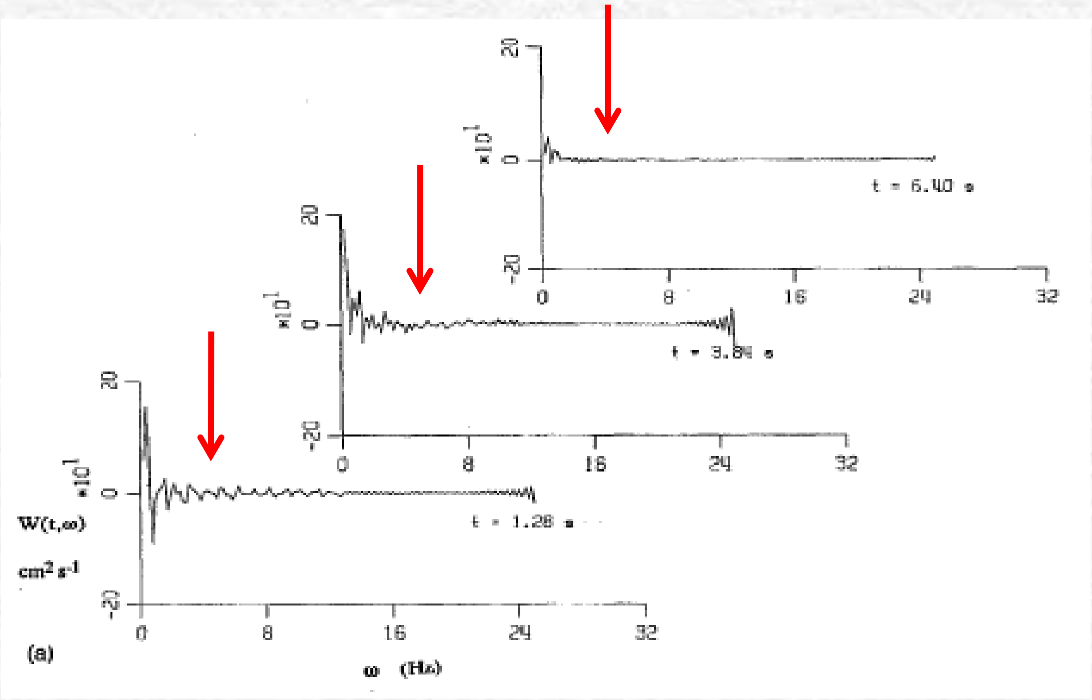
Turbulent Kinetic Energy Equation

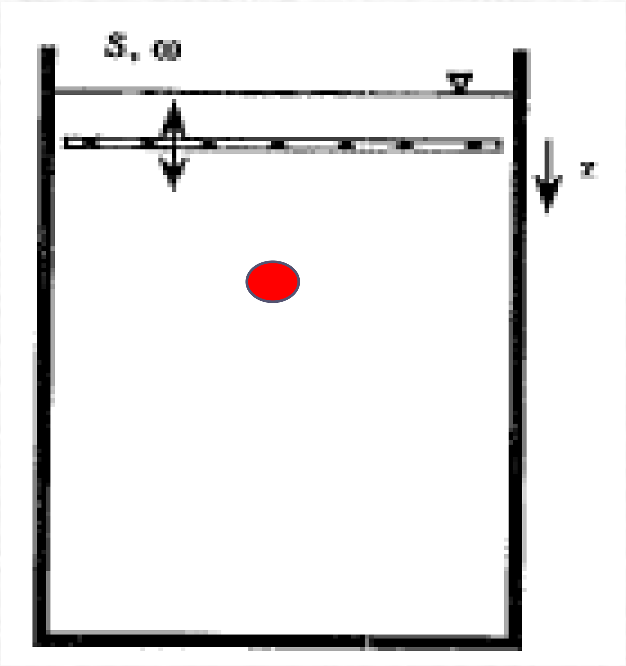
$$\frac{\partial \frac{\overline{u_i'^2}}{2}}{\partial t} + \overline{U}_j \frac{\partial \frac{\overline{u_i'^2}}{2}}{\partial x_j} = -\overline{u_i' u_j'} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\overline{u_j' u_i'^2 / 2} + \frac{\overline{p' u_j'}}{\rho_0} \right) + \overline{b' u'_3} + \nu \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j \partial x_j}} = -\varepsilon$$

(A) (B) (C) (D) (E) (F)

- Typically -- homogeneous turbulence-- **production balances dissipation** (really large scales feeding the smaller scales)

$$0 \approx -\overline{u_i' u_j'} \frac{\partial \overline{U}_i}{\partial x_j} + \overline{b' u'_3} - \varepsilon$$





$$W(t,\omega)=\int_{-\infty}^{\infty} w(\tau)\tilde{u}(t+\frac{\tau}{2})\tilde{u}^{*}(t-\frac{\tau}{2})e^{-2\pi j\omega\tau}d\tau$$

Wigner-Ville Transforms

De Silva, I.P.D. and H.J.S. Fernando. 1994. Oscillating grids as a source of nearly isotropic turbulence, *Phys. Fluids*, **6**(7), 2455-2464.

Turbulent Kinetic Energy Equation

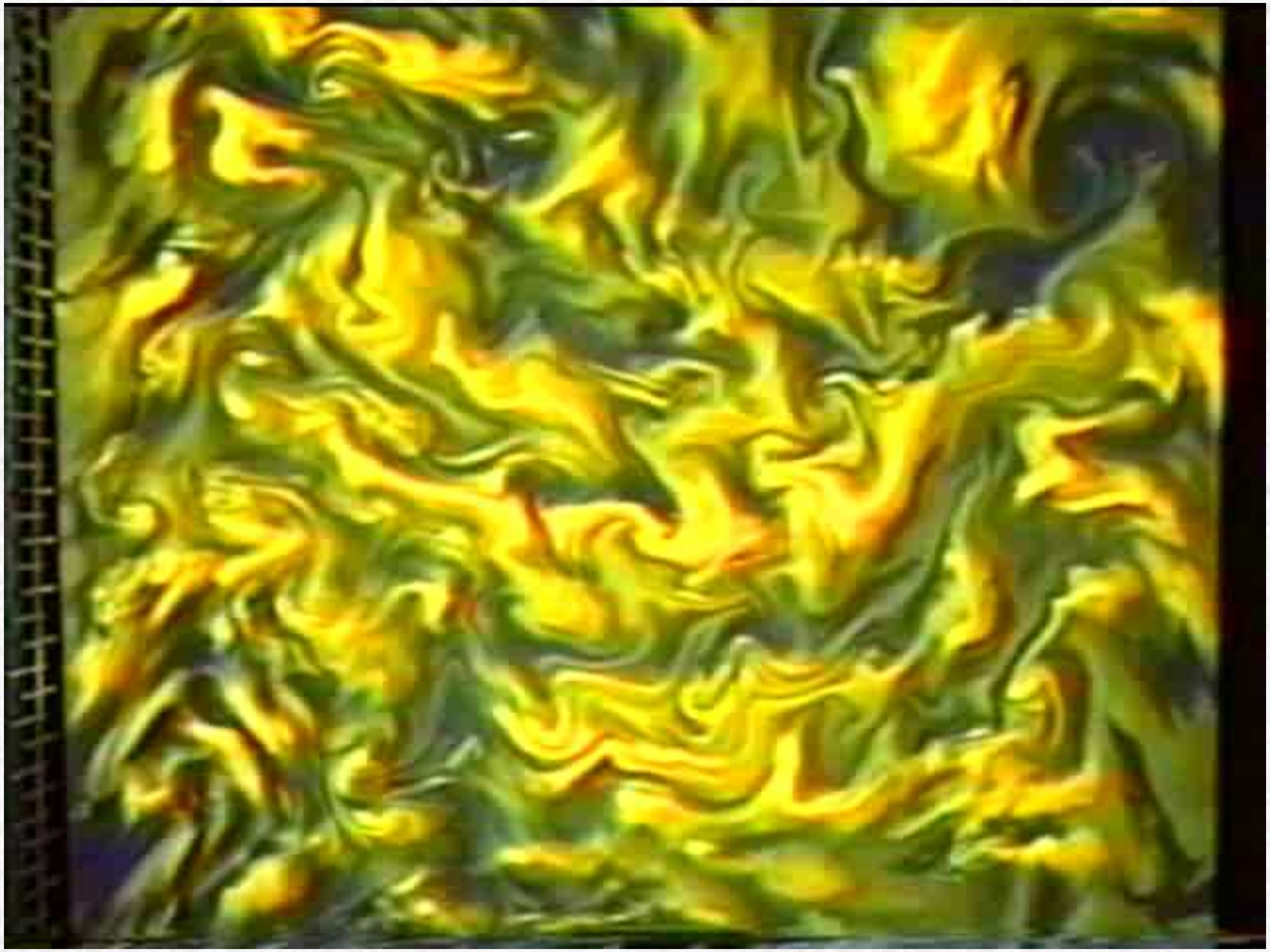
$$\frac{\partial \frac{\overline{u_i'^2}}{2}}{\partial t} + \overline{U}_j \frac{\partial \frac{\overline{u_i'^2}}{2}}{\partial x_j} = -\overline{u_i' u_j'} \frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\overline{u_j' u_i'^2 / 2} + \frac{\overline{p' u_j'}}{\rho_0} \right) + \overline{b' u'_3} + \nu \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j \partial x_j}} = -\varepsilon$$

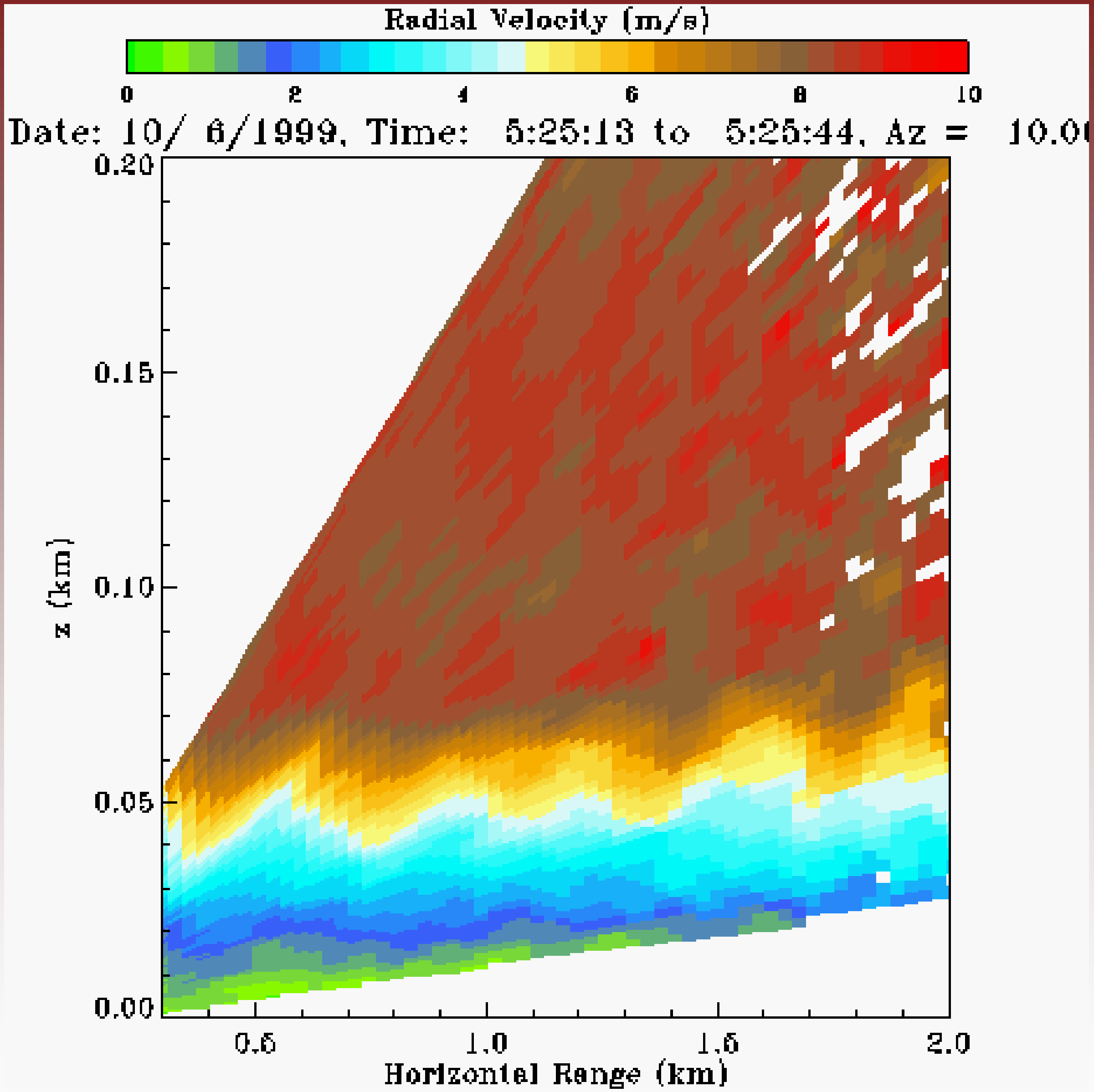
(A) (B) (C) (D) (E) (F)

- **No production** in homogeneous turbulence -- **turbulence is decaying** (small scales decay first and gradually seeps into larger scales)

$$\frac{\partial \frac{\overline{u_i'^2}}{2}}{\partial t} \approx -\varepsilon$$

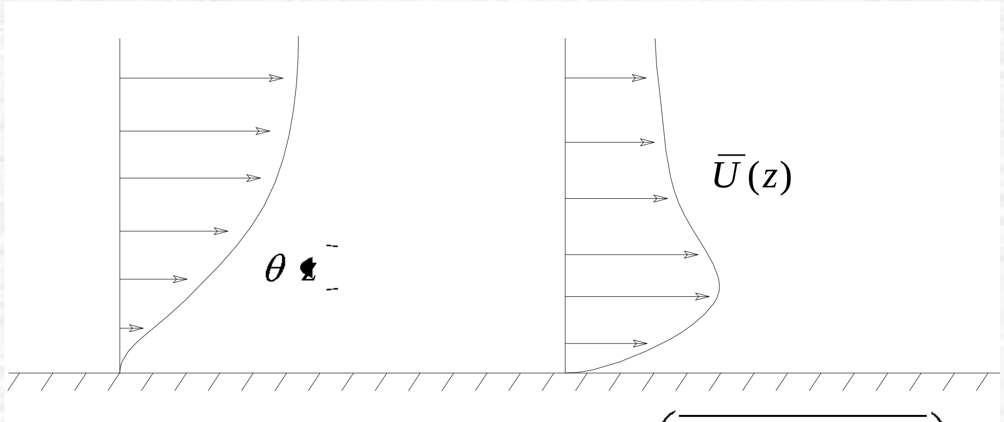
Smaller scales decay fast





Radial Velocity (m/s)
0
2
4
6
8
10
Date: 10/ 6/1999, Time: 5:25:13 to 5:25:44, Az = 10.0
0.20
0.15
0.10
0.05
0.00
z (km)
0.5
1.0
1.5
2.0
Horizontal Range (km)

Energy Budgets



$$\frac{\partial \overline{q^2/2}}{\partial t} + U_j \frac{\partial \overline{q^2/2}}{\partial x_j} = \underbrace{-\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{P} + \underbrace{\overline{bw}}_{B} - \frac{\partial\left(\overline{u_j q^2/2} + \overline{p u_j}\right)}{\partial x_j} - \underbrace{\varepsilon}_{\varepsilon}$$

Flux Richardson Number

$$R_f = \frac{-\overline{bw}}{-\overline{u_i u_j}\dfrac{\partial \overline{U_i}}{\partial x_j}}$$

Gradient Richardson Number

$$R_{ig} = \frac{N^2}{\left(\dfrac{\partial \overline{U}}{\partial z}\right)^2}$$

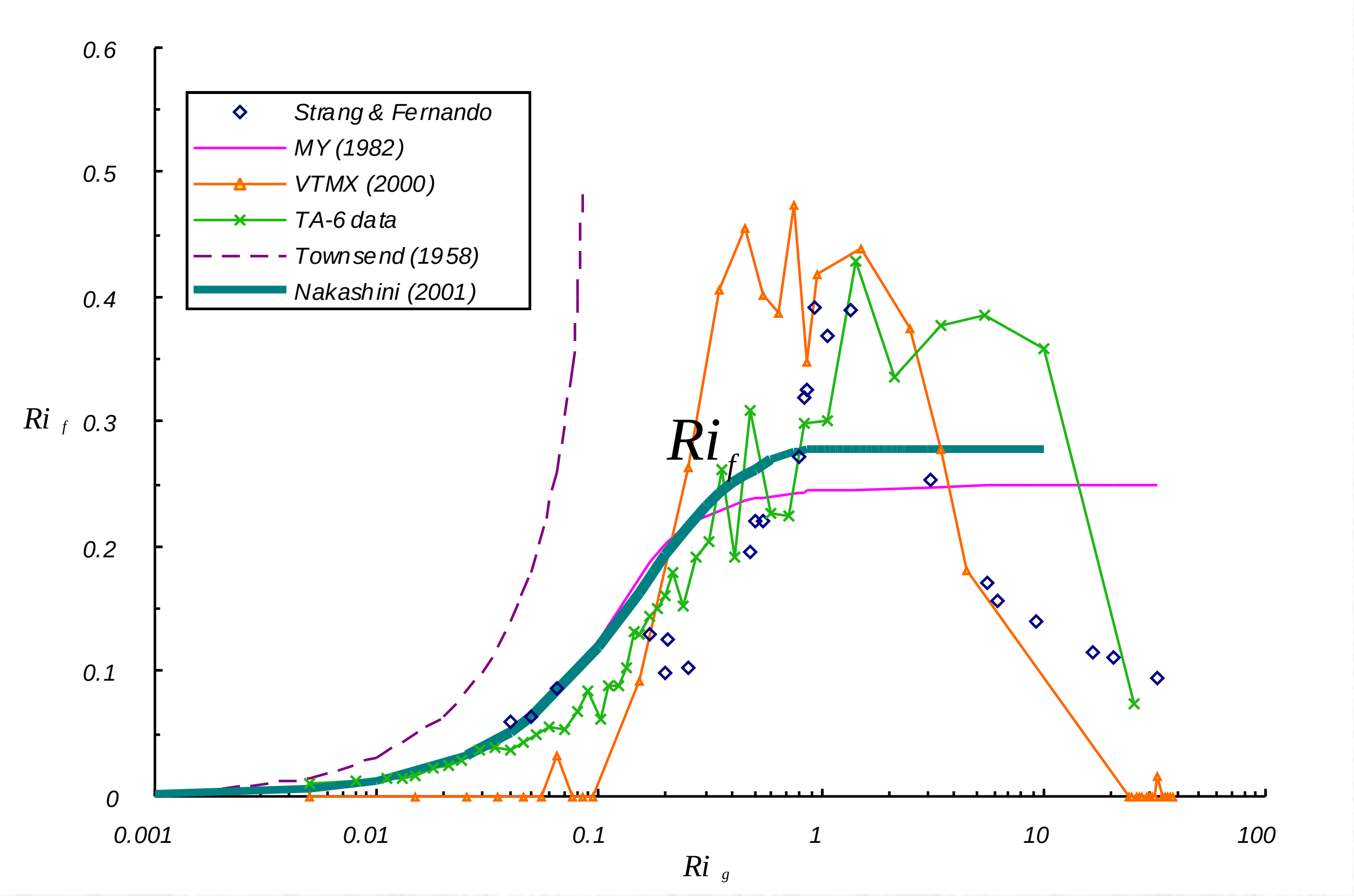
Diffusion Coefficients

$$-\overline{u_i u_j} = K_m\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

eg. $-\overline{uw} = K_m \dfrac{\partial \overline{U}}{\partial z}$

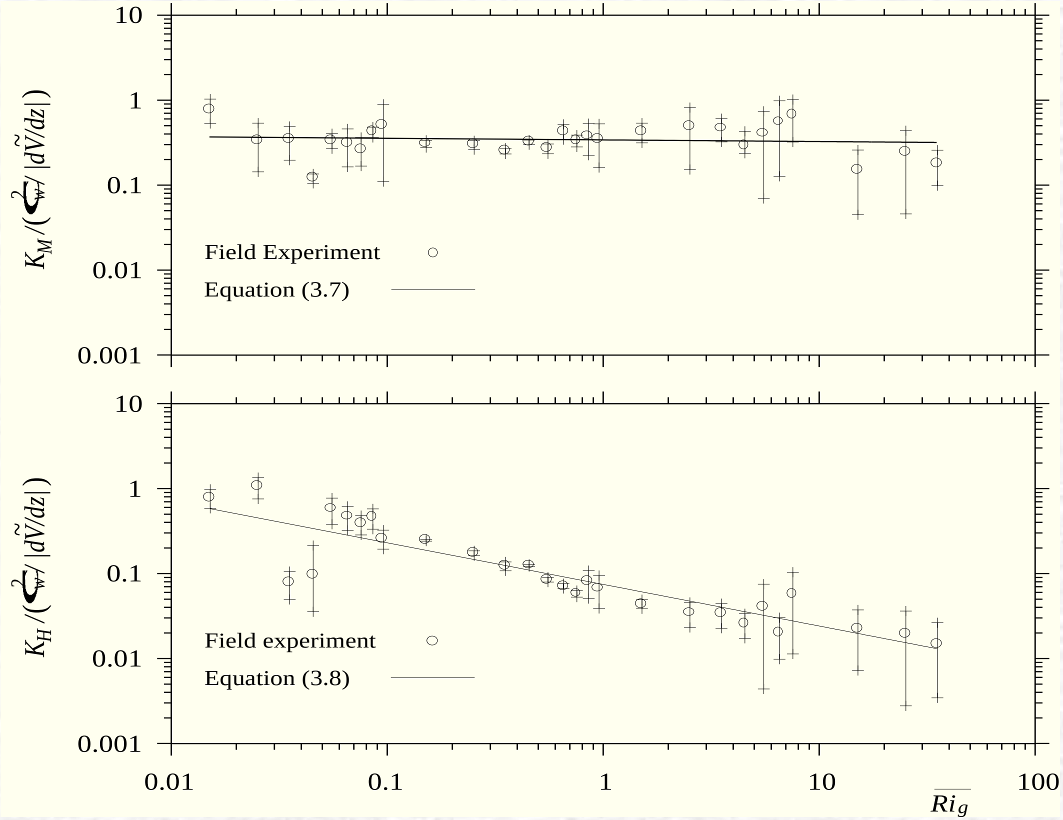
$$-\overline{b u_i} = K_h\left(\frac{\partial \overline{b}}{\partial x_i}\right)$$

Flux versus Gradient Richardson Number



Pardyjak, Monti and Fernando, J. Fluid Mech., 2002

Normalization of the eddy coefficients in the VTMX



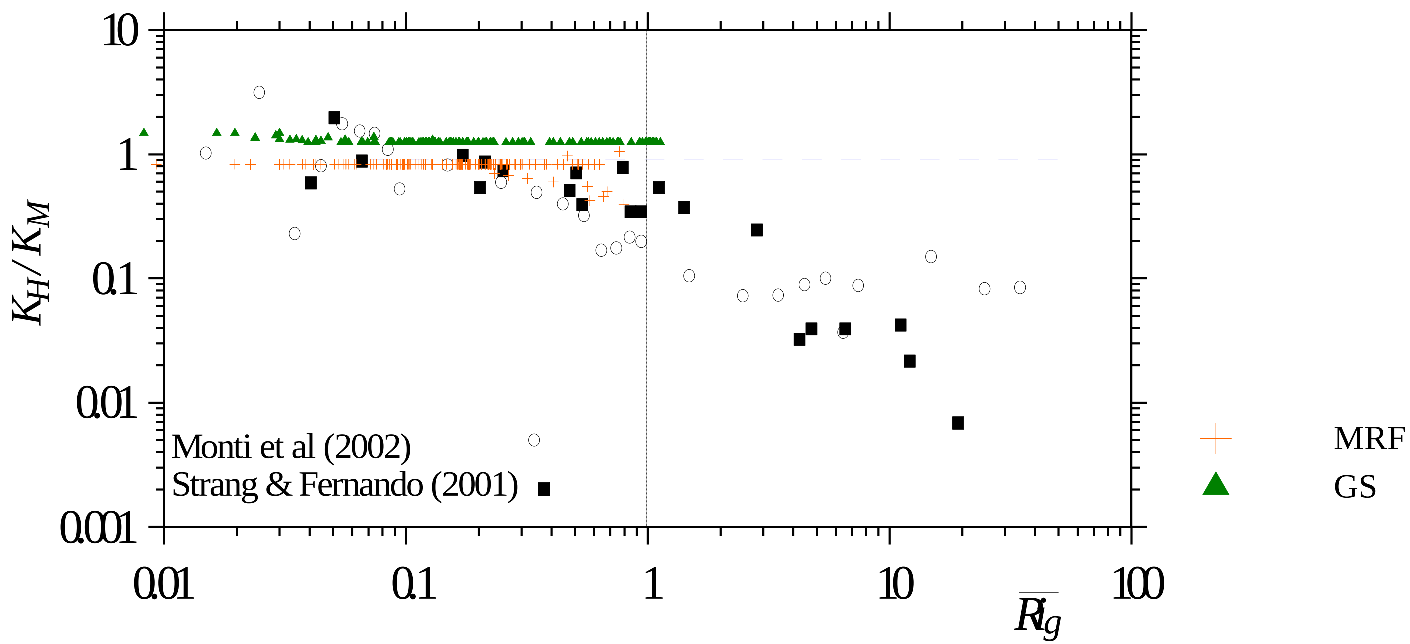
Monti et al., J. Atmos. Sci., 59(17), 2002

Eddy Diffusivity (Semi Empirical)

$$\begin{cases} \dfrac{K_m}{\sigma_w^2 / \left| d\tilde{V} / dz \right|} = (0.34)\overline{Ri_g}^{\,-0.02} \approx 0.34 \\ \dfrac{K_h}{\sigma_w^2 / \left| d\tilde{V} / dz \right|} = (0.08)\overline{Ri_g}^{\,-0.49} \approx (0.08)\overline{Ri_g}^{\,-0.5} \end{cases}$$

Monti et al., *J. Atmos. Sci., 59(17), 2002*

Eddy Diffusivity Ratio



Prandtl Number $Pr_t = \frac{K_m}{K_h}$

• Lee et al. *Boundary layer Meteorology, 119, 2006*

Balance of a Transferable Quantity (a scalar)

$$\frac{\partial \tilde{P}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{P}}{\partial x_j} = D \frac{\partial^2 \tilde{P}}{\partial x_j \partial x_j} + \tilde{F}_p$$

or $$\frac{\partial \overline{P}}{\partial t} + \overline{U}_j \frac{\partial \overline{P}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D \frac{\partial \overline{P}}{\partial x_j} - \overline{p' u'_j}\right) + \overline{F}_p$$

$$-\overline{p' u'_i} = K_\theta \frac{\partial \overline{\theta}}{\partial x_i}$$

Fluctuations

$$\frac{\partial p'}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{P}u_j' + \overline{U}_j p' + u_j p' - \overline{p'u_j'}\right) = D\frac{\partial^2 p'}{\partial x_j \partial x_j}$$

$$\frac{\partial \left(\overline{p'^2}/2\right)}{\partial t} + \overline{U}_j \frac{\partial \left(\overline{p'^2}/2\right)}{\partial x_j} = -\overline{u_j' p'}\frac{\partial \overline{P}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{p'^2 u_j'/2}\right) + D\overline{p'\frac{\partial p'}{\partial x_j \partial x_j}}$$

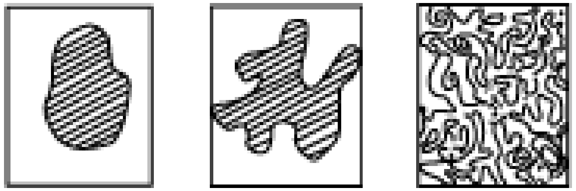
(A) (B) (C) (D) (E)

The term (E) is of special interest

$$D\overline{\frac{p'\partial^2 p'}{\partial x_j \partial x_j}} = D\frac{\partial^2 \frac{\overline{p'^2}}{2}}{\partial x_j \partial x_j} - D\overline{\frac{\partial p}{\partial x_j}\frac{\partial p}{\partial x_j}}$$

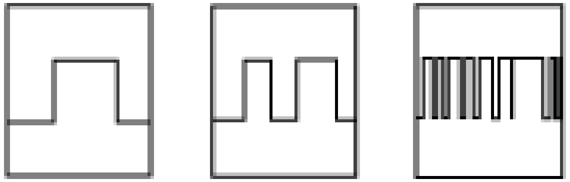
(E_1) (E_2)

(a) Binary distribution in space.



t_0 $t_1 > t_0$ $t_2 > t_1$

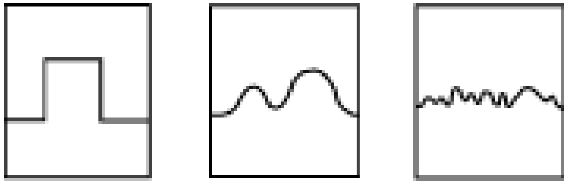
(b) Amplitude on a cut.



t_0 $t_1 > t_0$ $t_2 > t_1$

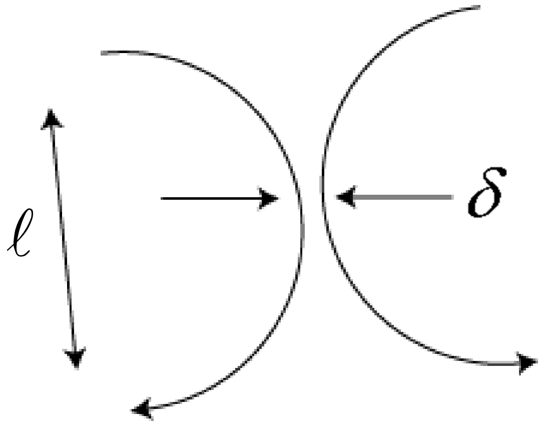
Schematic sketch of turbulent mixing of a contaminant, both without and with molecular diffusion. These represent a small sample from a large statistically homogeneous field.

(c) Effect of molecular transport for $t > t_0$.



t_0 $t_1 > t_0$ $t_2 > t_1$

Scalar Dissipation



$u = (\nu\varepsilon)^{1/4}$ or $(\varepsilon\ell)^{1/3}$

$\ell = (\nu^3/\varepsilon)^{1/4}$ or ℓ

Separation Distance	δ
Diffusion Time	δ^2/D
Separation Time	ℓ/u

$$\frac{\delta^2}{D} \sim \frac{l}{u}$$

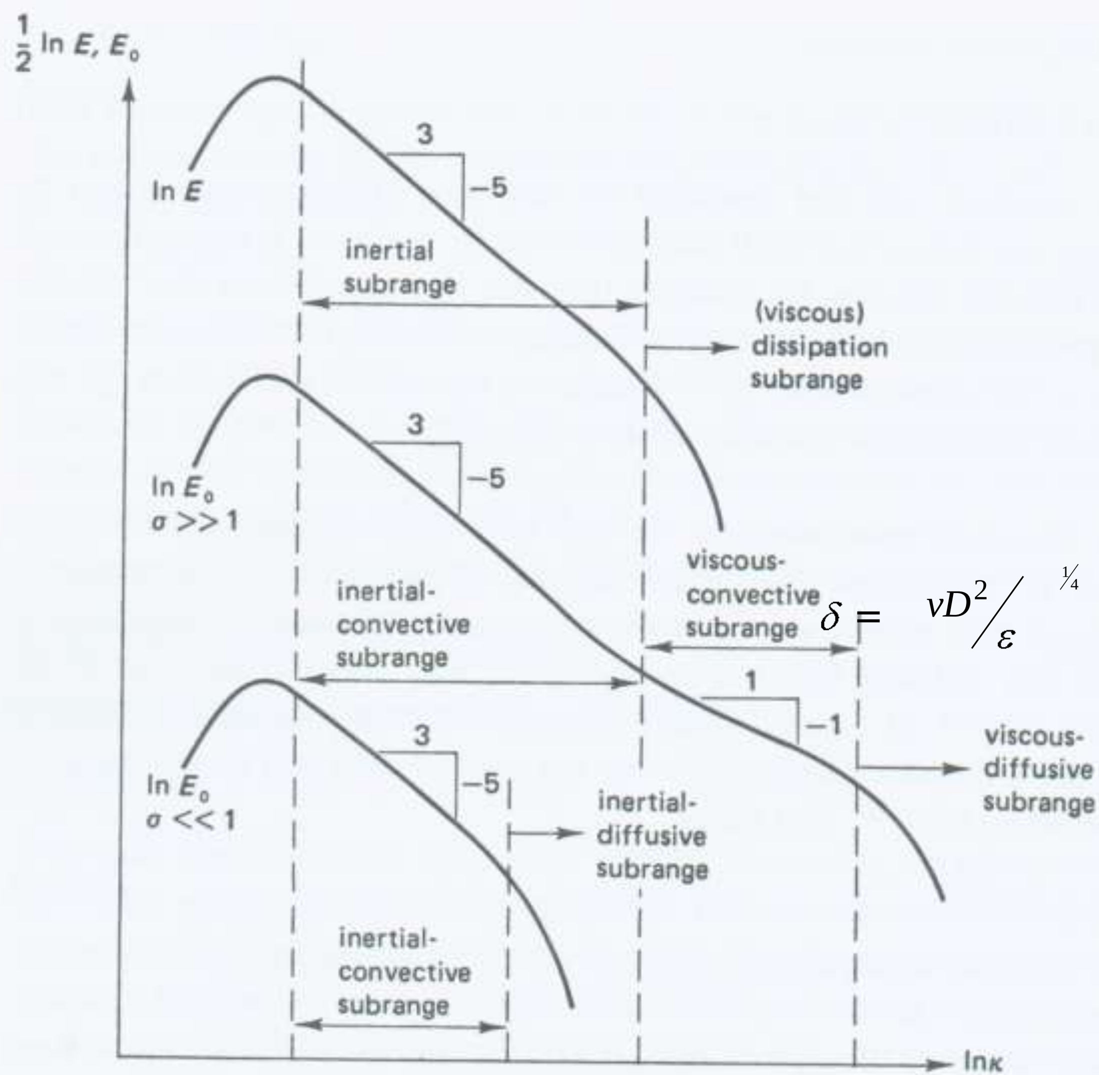
$$\mathit{if}\ \delta \sim l,\ \ u \sim (\varepsilon l)^{1/3} \Rightarrow$$

$$l_{oc} = \left(D^3/\varepsilon\right)^{1/4}\ \mathit{requires}\ \ Pr < 1$$

$$\mathit{if}\ l_{oc} < l_K,\ \ \mathit{or}\ \ Pr > 1$$

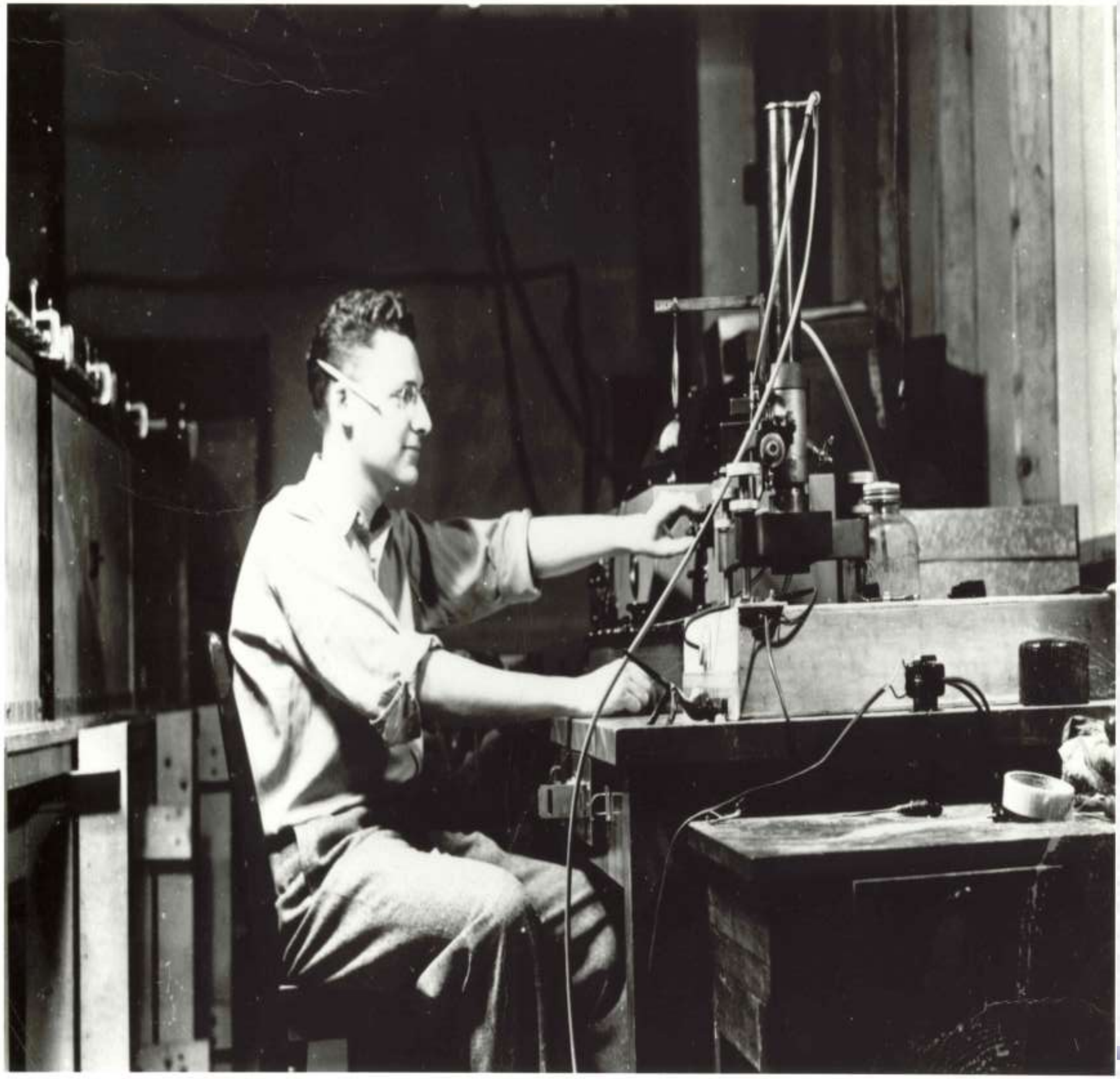
$$l \sim (\nu^3/\varepsilon)^{1/4},\ \ u \sim (\nu\varepsilon)^{1/4}$$

$$l_B = \left(\frac{\nu D^2}{\varepsilon}\right)^{1/4}$$

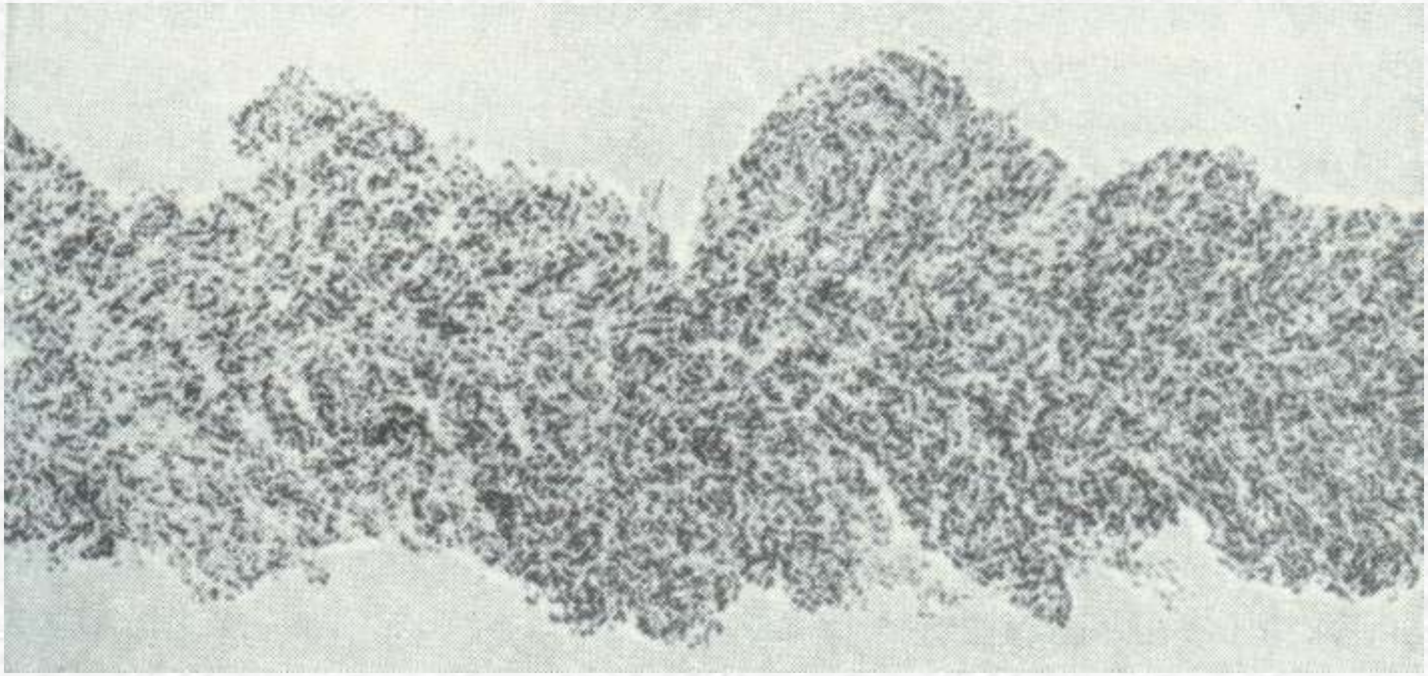


$$\delta = \left(\nu D^2 / \varepsilon \right)^{1/4}$$

$$\delta = \left(D^3 / \varepsilon \right)^{1/4}$$



Incoherent and Coherent Motions



From Monin & Yaglom – wake past a bullet

Brown & Roshko Experiment

Coherent Structures

Double Decomposition

$$\underbrace{}_{} f(\underset{\sim}{x}, t) = \underbrace{f_c(\underset{\sim}{x}, t)}_{\text{coherent}} + \underbrace{f_r(\underset{\sim}{x}, t)}_{\text{incoherent}}$$

(to explain coherent-incoherent interactions)

Stratified and Rotating Turbulent Flows

…..Discussion

Horizontal Momentum

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + w\frac{\partial u_\alpha}{\partial z} + \varepsilon_{\alpha jk} f\ell_j u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu\left[\left(\frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\alpha} + \frac{\partial^2 u_\alpha}{\partial z^2}\right)\right]$$

$$\left(\frac{L_H}{u_H T_H}\right)\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + w\frac{\partial u_\alpha}{\partial z} + \left(\frac{1}{Ro}\right)\varepsilon_{\alpha jk}\ell_j u_k = -\left(\frac{p_0}{\rho_0 u_H{}^2}\right)\frac{\partial p}{\partial x_\alpha}$$

$$\uparrow$$

$$\tilde{\omega} \times \tilde{u} + \nabla \frac{|\tilde{u}|^2}{2}$$

$$+\left(\frac{1}{Re}\right)\left[\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta} + \left(\frac{L_H}{L_V}\right)^2 \frac{\partial^2 u_\alpha}{\partial z^2}\right]$$

$$L_V \sim L_H \, Re^{-1/2}$$

L_H *and* L_V : *Length Scales*

T_H *and* T_V : *Time Scales*

u_H *and* u_v : *Velocity Scales*

Vertical Momentum

$$\frac{\partial w}{\partial t}+u_\beta\frac{\partial w}{\partial x_\beta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+b+\nu\left[\frac{\partial^2 w}{\partial x_\alpha \partial x_\alpha}+\frac{\partial^2 w}{\partial z^2}\right]$$

$$\left(\frac{L_V}{L_H}\right)^2\left\{\frac{L_H}{u_H T_V}\frac{\partial w}{\partial t}+u_\beta\frac{\partial w}{\partial x_\beta}+w\frac{\partial w}{\partial z}\right\}=-\left(\frac{p_0}{\rho_0 u_H{}^2}\right)\frac{\partial p}{\partial z}$$

$$+\left(\frac{b_0 L_V}{u_H{}^2}\right)\tilde{b}+\left(\frac{\nu}{L_H u_H}\right)\left[\left(\frac{L_V}{L_H}\right)^2\frac{\partial^2 w}{\partial x_\alpha \partial x_\alpha}+\frac{\partial^2 w}{\partial z^2}\right]$$

$$\frac{L_V}{L_H}\to 0$$

$$Re^{-1}\to 0$$

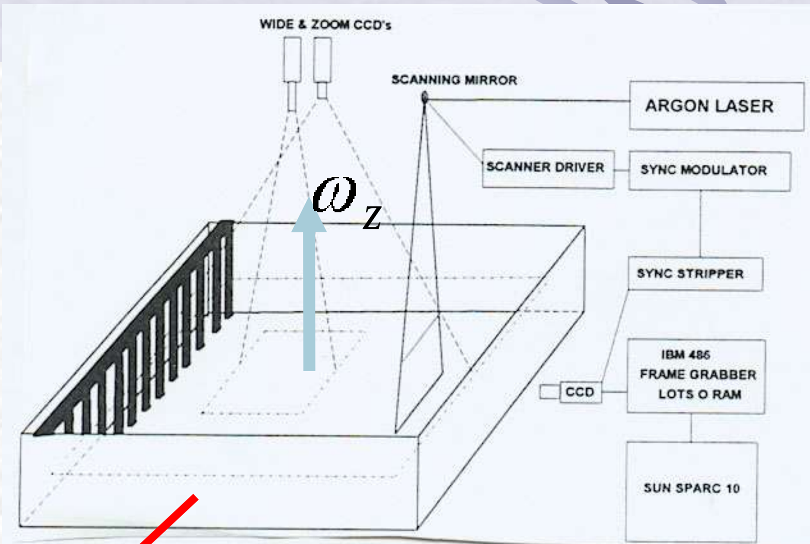
Definitions – 3D Turbulence

Chaotic Motions, where the inertial vortex forces dominates
viscous, Coriolis, buoyancy and all other body forces
+
balance pressure forces

Inertial vortex forces ~ Coriolis → Rotating turbulence

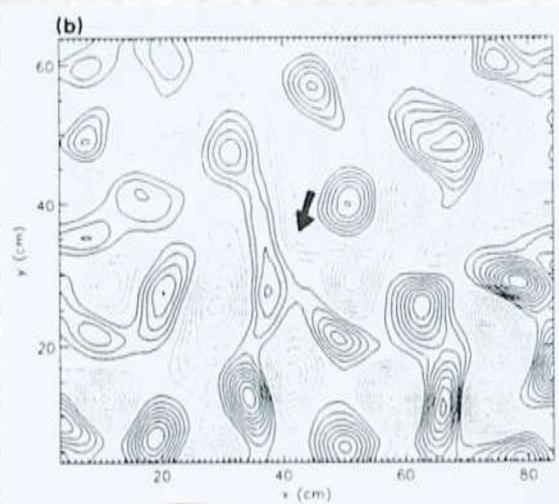
Inertial vortex forces ~ buoyancy forces → stratified turbulence

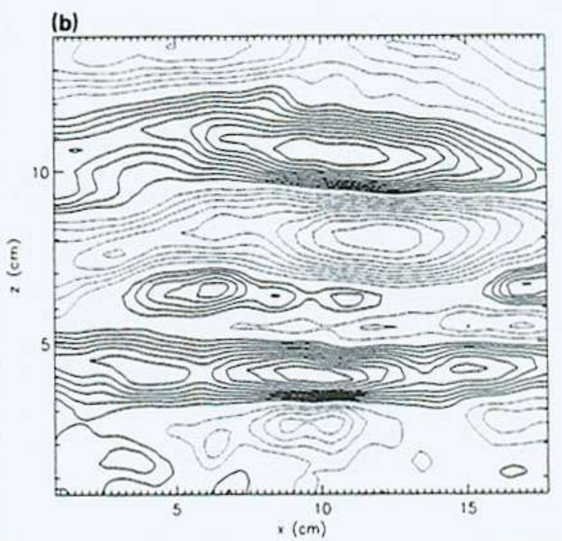
Inertial vortex ~ Coriolis/buoyancy → Geophysical Turbulence



$\varepsilon_y \sim (\partial u / \partial z)^2$

ω_y

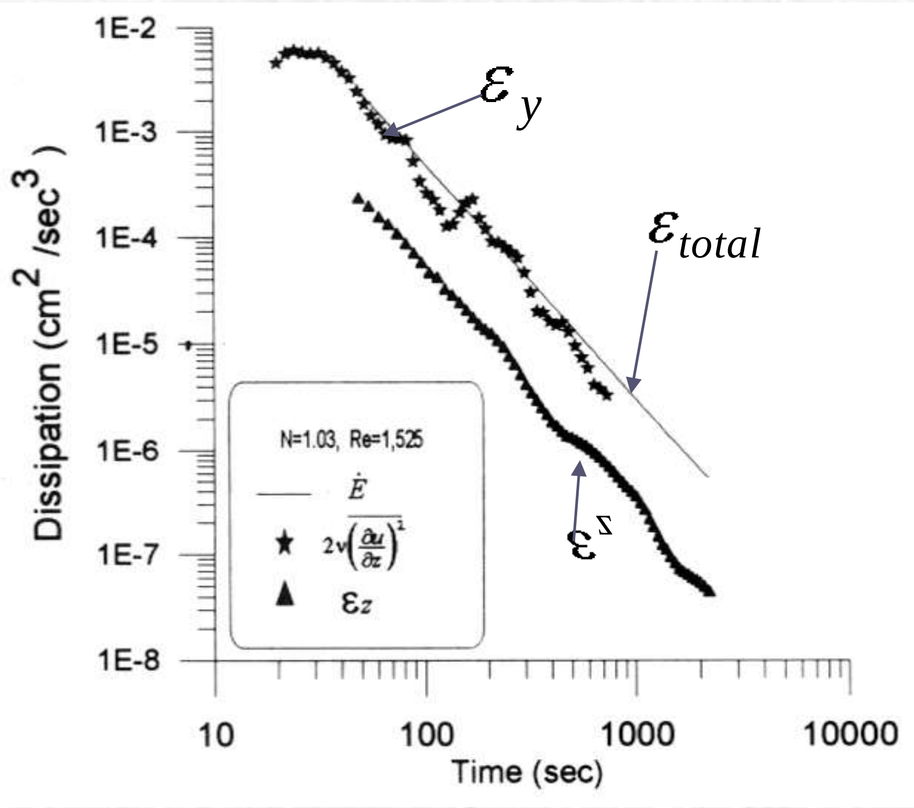




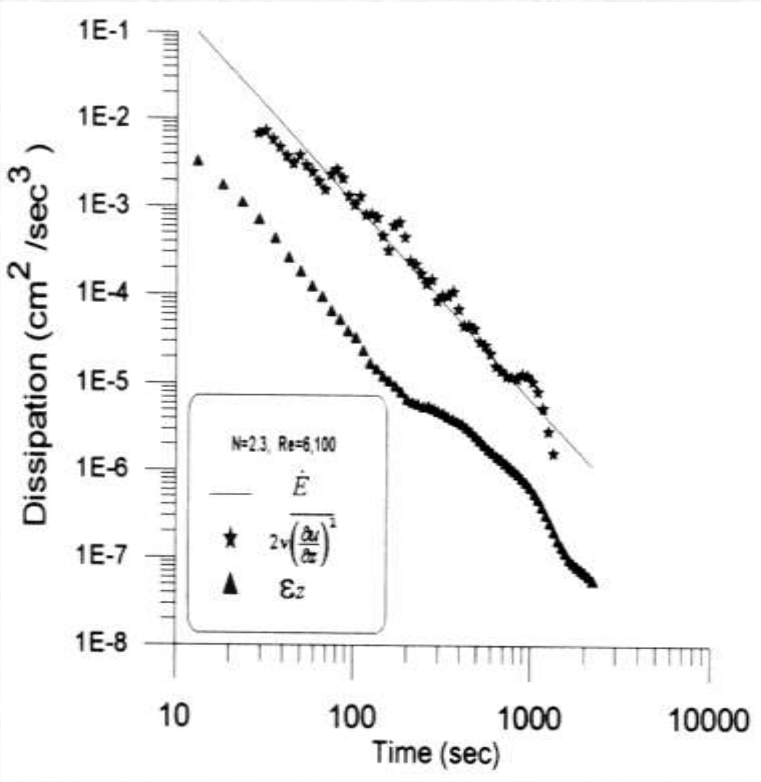
ω_y

ω_z

(Fincham et al 1996)



1E-2
1E-3
1E-4
1E-5
1E-6
1E-7
1E-8
Dissipation (cm^2 /sec^3)
10
100
1000
10000
Time (sec)
ε_y
ε_{total}
ε_z
N=1.03, Re=1,525
$\dot{E}$
$2\nu\overline{\left(\frac{\partial u}{\partial z}\right)^2}$
εz



1E-1
1E-2
1E-3
1E-4
1E-5
1E-6
1E-7
1E-8
Dissipation (cm^2 /sec^3)
10
100
1000
10000
Time (sec)
N=2.3, Re=6,100
$\dot{E}$
$2\nu\overline{\left(\frac{\partial u}{\partial z}\right)^2}$
εz

Buoyancy Equation

$$\left(\frac{b_0}{u_V N^2 T_b}\right)\frac{\partial b}{\partial t}+\left(\frac{b_0}{L_V N^2}\right)u_\beta\frac{\partial b}{\partial x_\beta}+w\frac{\partial b}{\partial z}+w=\left(\frac{\nu}{L_H U_H}\right)\left(\frac{b_0}{L_V N^2}\right)\left(\frac{K}{\nu}\right)\left[\frac{\partial^2 b}{\partial x_\beta \partial x_\beta}+\left(\frac{L_H}{L_V}\right)^2\frac{\partial^2 b}{\partial z^2}\right]$$

$$L_V = L_{Ellison} = \frac{b_o}{N^2}$$

$$\frac{L_V}{L_H} \sim Sc^{-1/2}\,\mathrm{Re}^{-1/2}$$

$$T_b \sim \frac{L_v}{u_v}$$

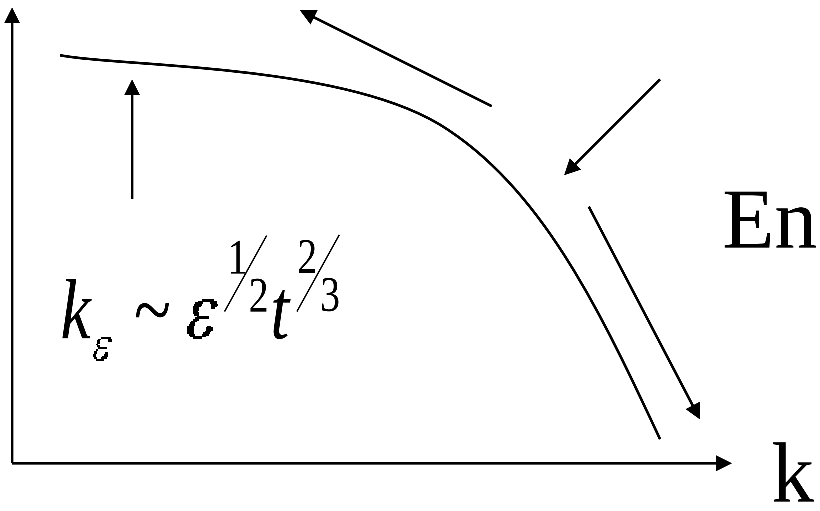
Growth

$$\frac{u_v^{\,2}}{L_V} >> N^2 L_V$$

$$\frac{L_V}{u_v} < N^{-1}$$

$$L_V \sim \frac{u_V}{N} = L_b$$

$E(k)$ Energy flux $E(k) \sim \varepsilon^{2/3} k^{-5/3}$



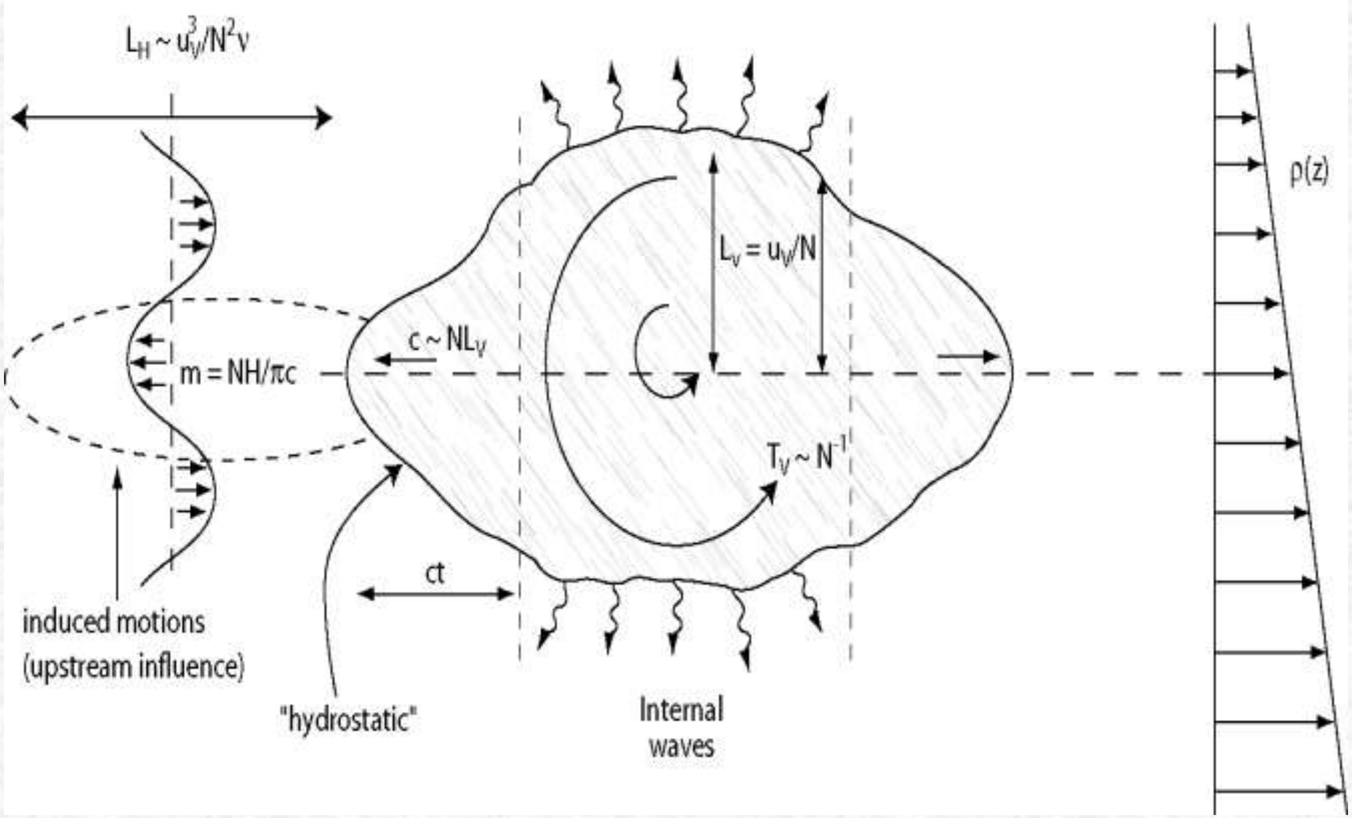
$k_\varepsilon \sim \varepsilon^{1/2} t^{2/3}$

Enstropy Flux β

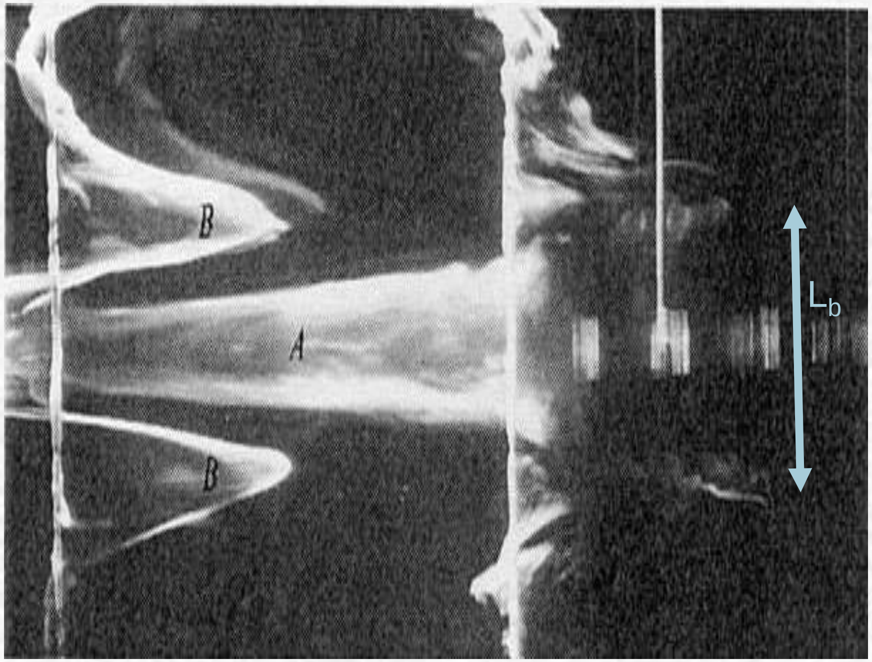
$E(k) \sim \beta^{2/3} k^{-3}$

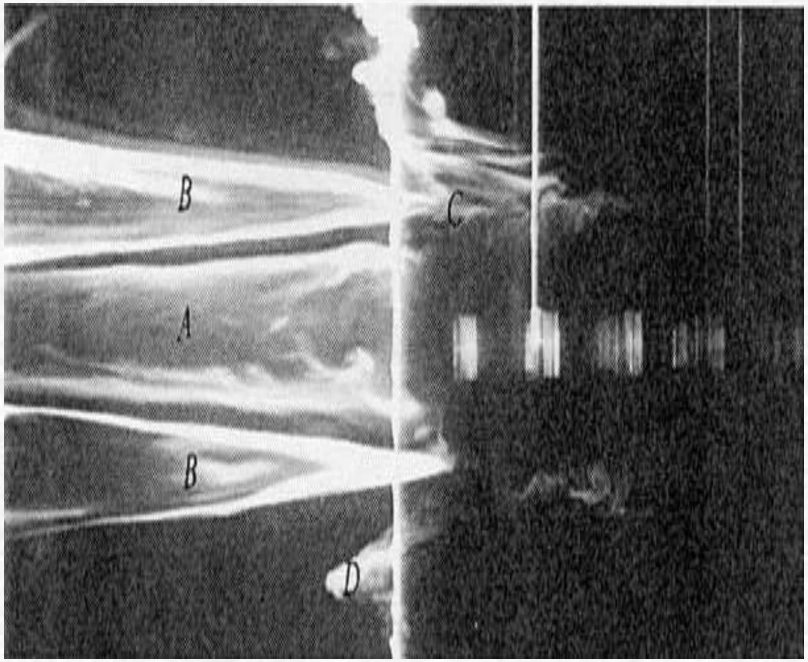
$E(k)$

Evolution of a Turbulent Patch in a Stratified Fluid



Formation of Intrusions





(a) (b)

(A) surrounding a turbulent patch due to the collapse of the patch. Note the setting up of "zero-frequency" modes.
(B) surrounding the patch. (De Silva & Fernando 1998).

$$\frac{L_H}{U_H T} \sim 1 \sim \frac{p_0}{\rho U_H^2}$$

$$\frac{p_0}{\rho_0 U_H^2} \sim \frac{\Delta b_0 L_v}{U_H^2}$$

$$U_H \sim \left(\Delta b_0 L_v\right)^{1/2}$$

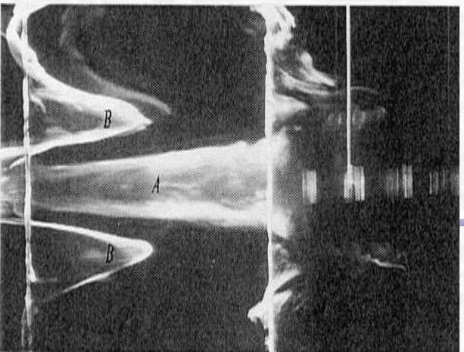
$$L_H \sim U_H T$$

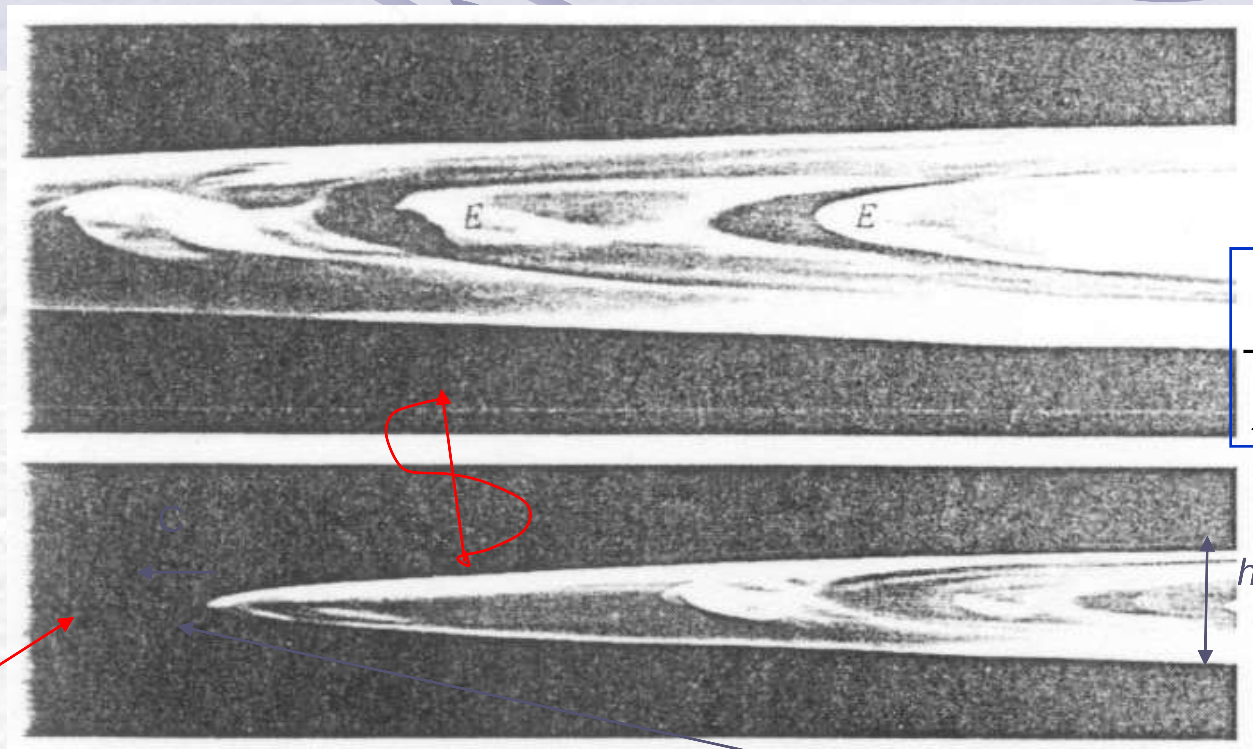
$$\left(\frac{L_H}{u_H T_H}\right)\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + w\frac{\partial u_\alpha}{\partial z} + \left(R_0\right)^{-1} \varepsilon_{\alpha jk}\ell_j u_k = -\left(\frac{p_0}{\rho_0 u_H^2}\right)\frac{\partial p}{\partial x_\alpha}$$

$$+\left(\frac{\nu}{L_H u_H}\right)\left[\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta} + \left(\frac{L_H}{L_V}\right)^2 \frac{\partial^2 u_\alpha}{\partial z^2}\right]$$

$$R_0 \sim 1 \sim \frac{U_H}{\Omega L_H}$$

$$L_H \sim \frac{\left(\Delta b_0 L_v\right)^{1/2}}{f} = R_d$$





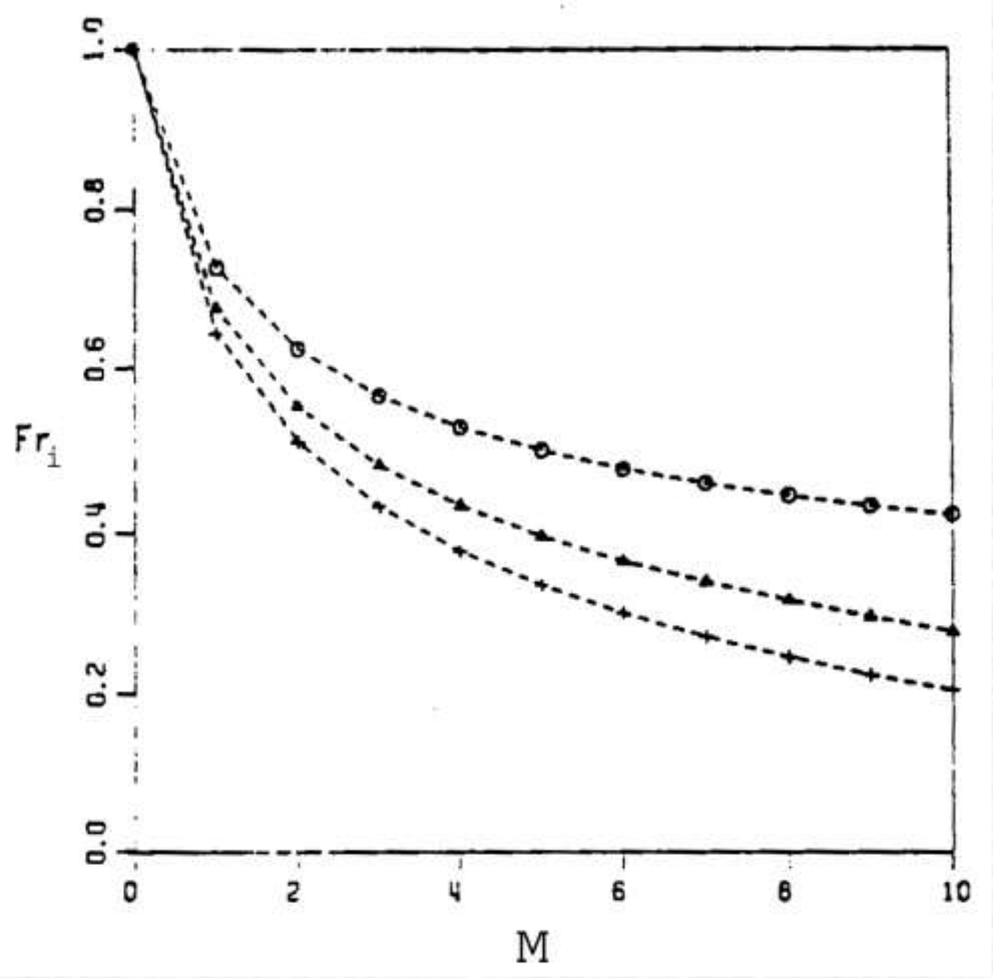
$$\frac{C}{Nh} \sim 0.15$$

$$u.\nabla u \sim \frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$Fr^2 = 1 + \frac{2h}{H} - \sum \frac{2Fr}{m\pi(1 - m\pi\phi_1 Fr)}$$

$$\frac{C^2}{x} \sim \frac{N^2h^2}{x} \Rightarrow Fr = \frac{C}{Nh} = O(1)$$

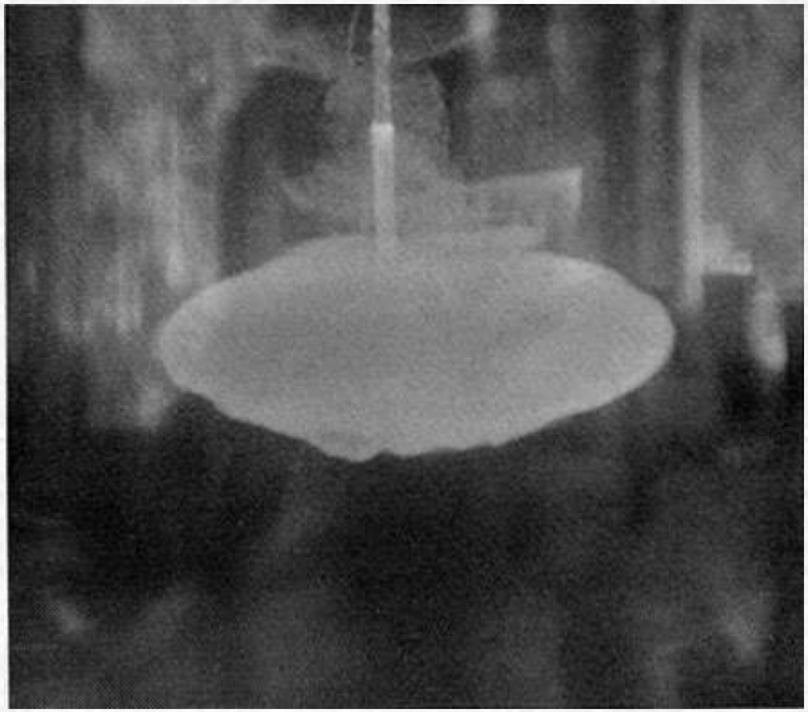
Variation of Fr_i



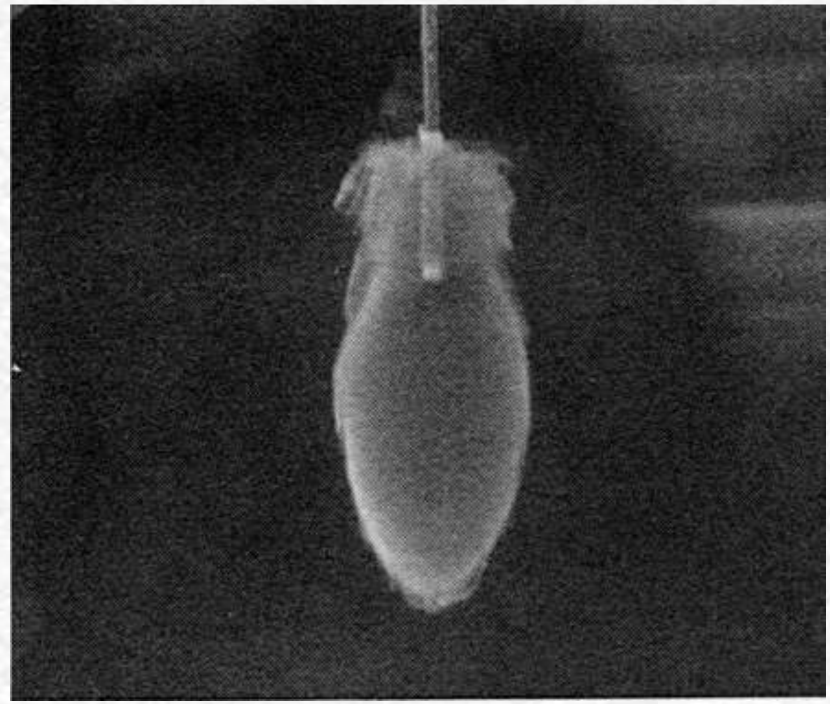
The variation of Fr_i (lower bound) with the number of forward propagating wave modes M. Three aspect ratios φ_1 of the patch (- 0.01; Δ- 0.1; and +- 0.15) are shown.

$$R_D = (\Delta b_0 L_V)^2 / f \sim N L_V / f$$

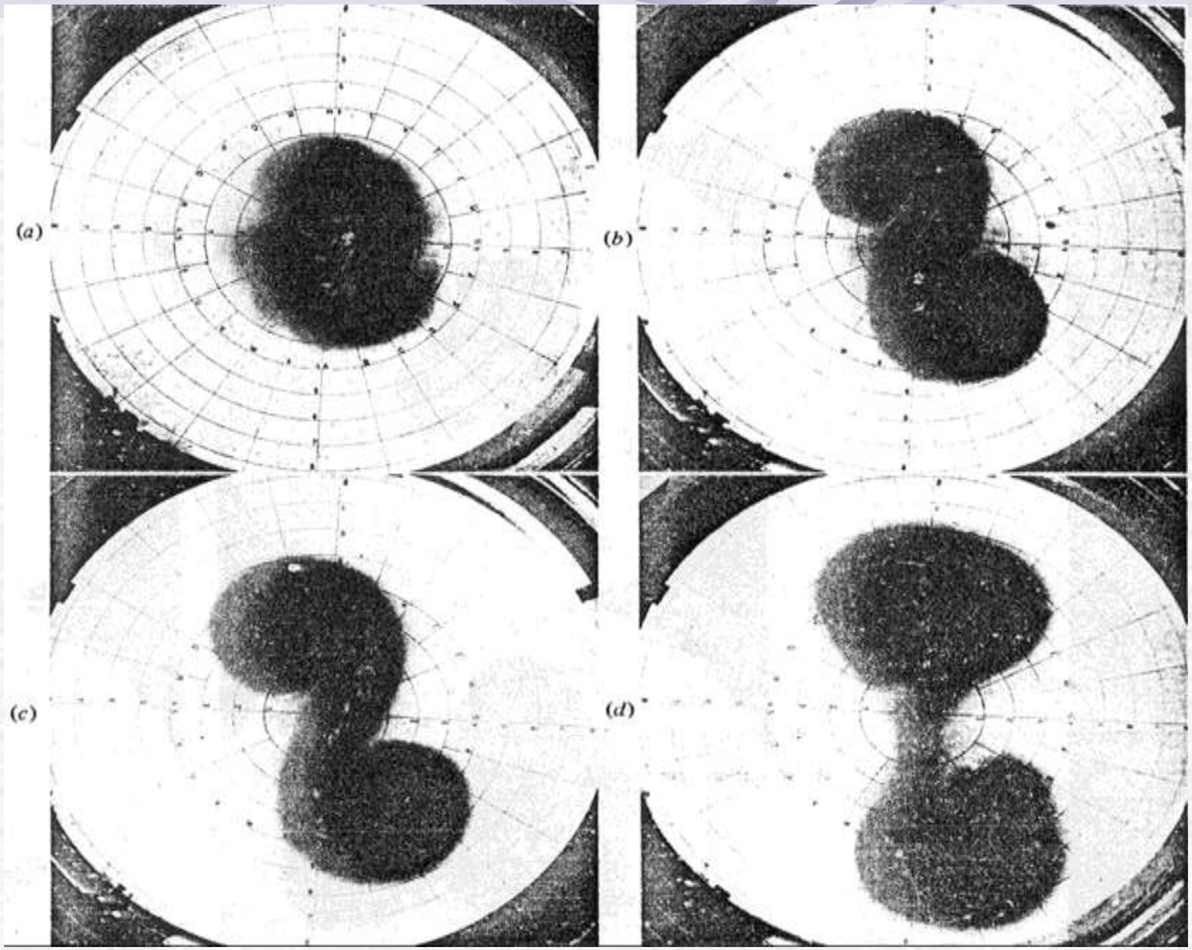
$$\frac{L_H}{L_V} \approx \frac{N}{f}$$



$N/f = 1.3$
(Hedstrom & Armi 1988).

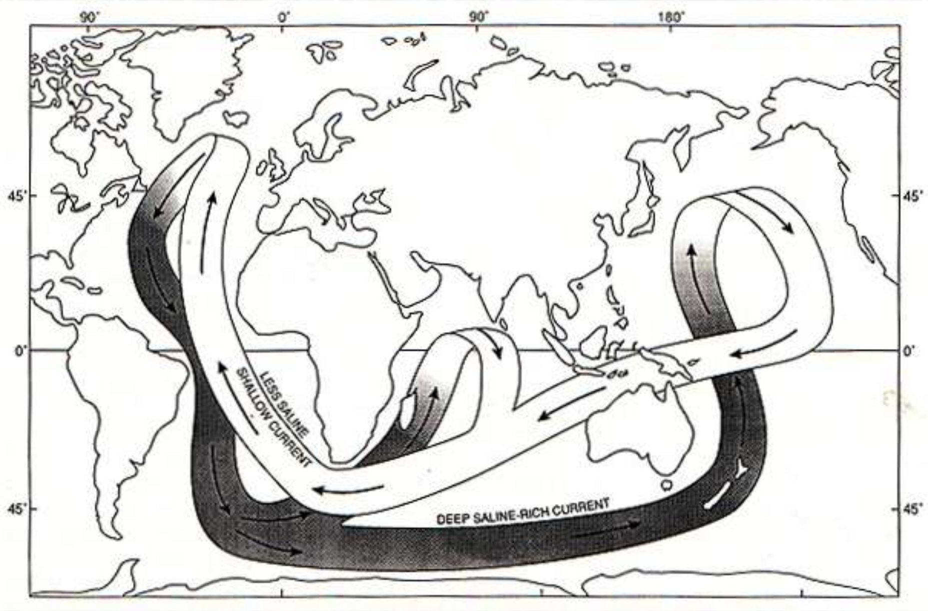


$N/f = 0.18$

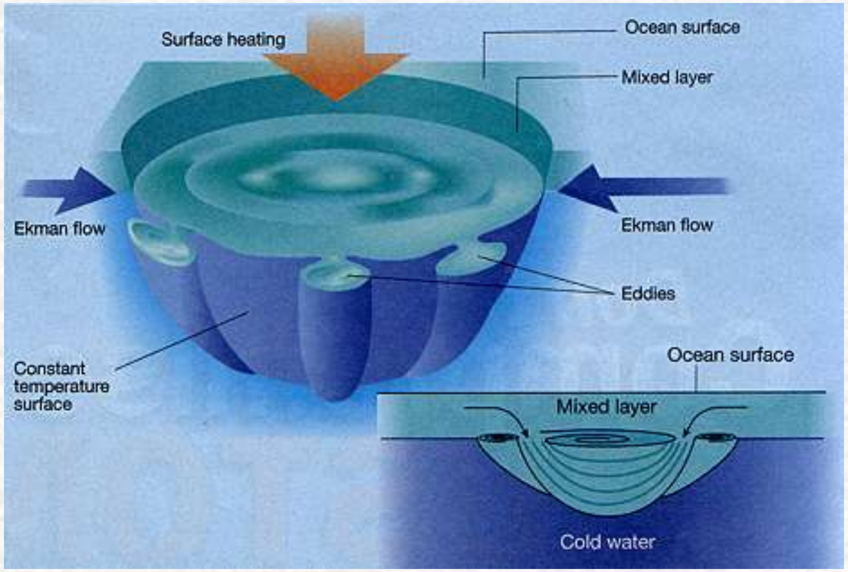


Baroclinic instabilities in a two-layer fluid (Ivey 1987)

Thermohaline Circulation in Oceans



(a) The thermohaline conveyor belt in the oceans. Dark bands show flow of deep, cold, and salty water; light bands show return surface flow. (From Broecker 1981).

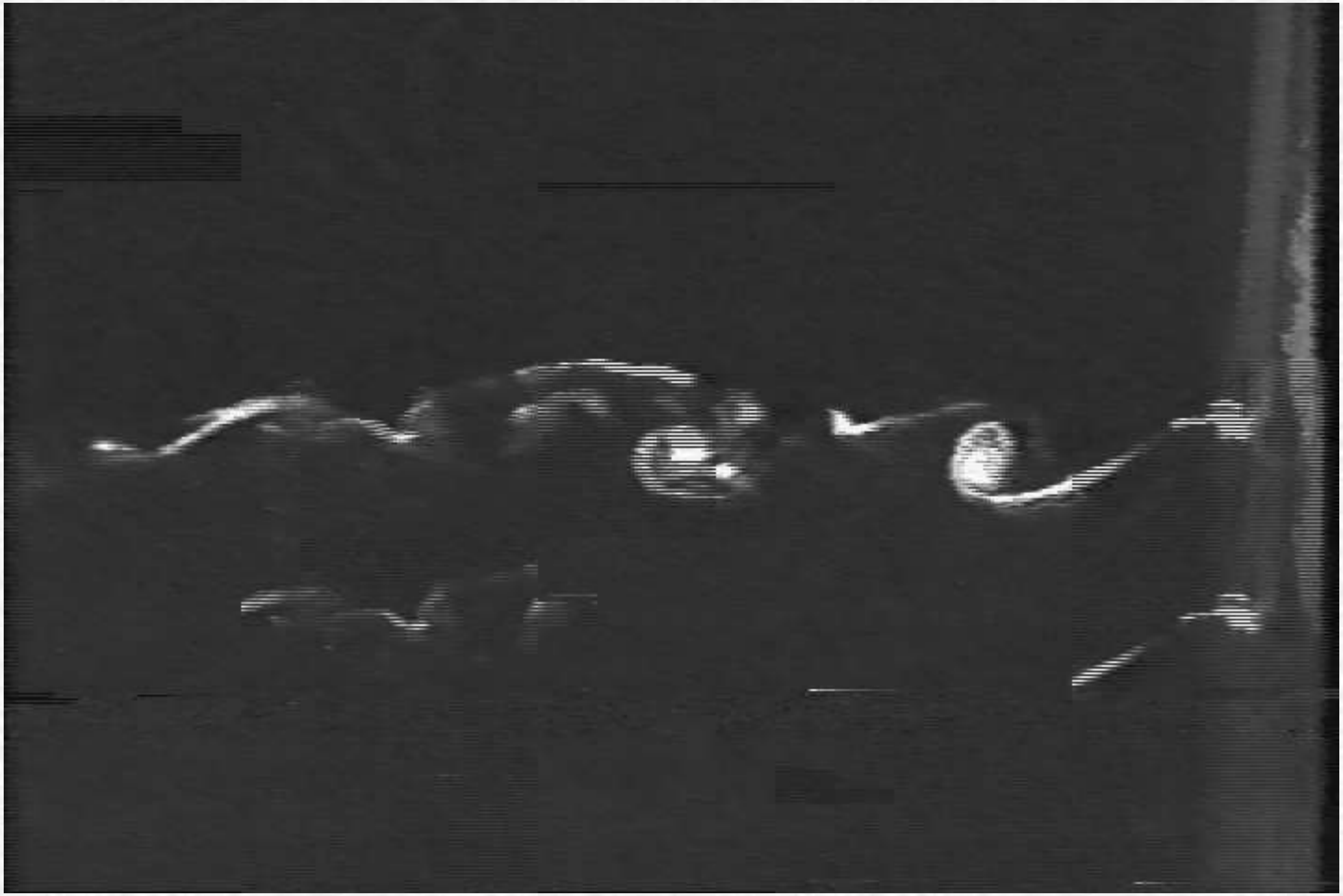


(b) An oceanographic mixing bowl. The bowl represents a surface on which the temperature is constant, capped by an Eckman layer in which the wind directly drives water flow and mixing. (From Marshall et. al. 2002)

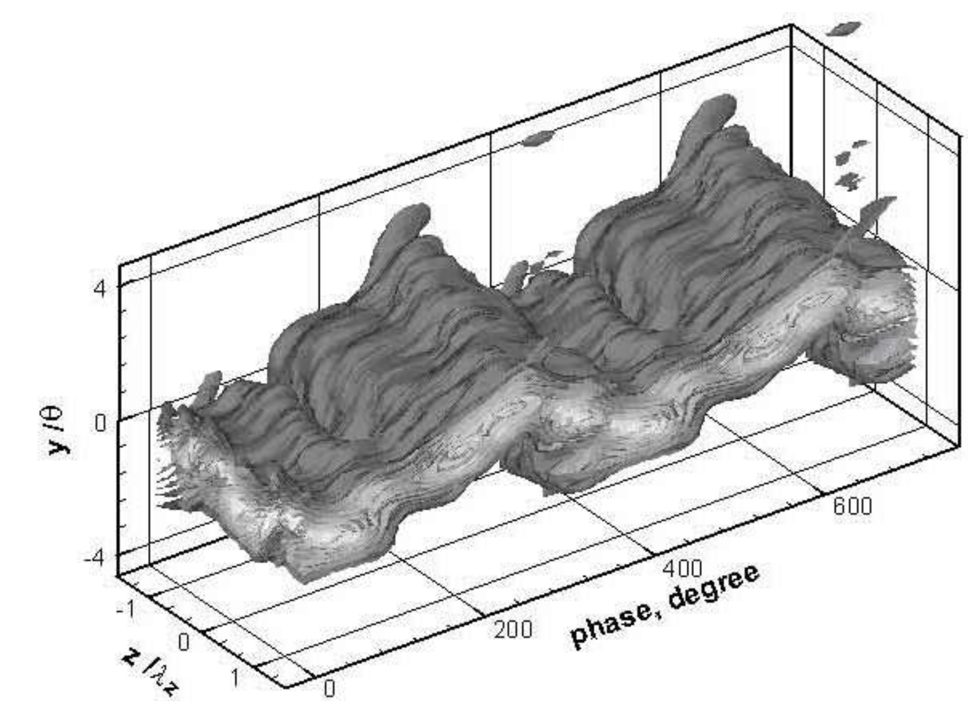
CLASSES OF STRATIFIED TURBULENT FLOWS

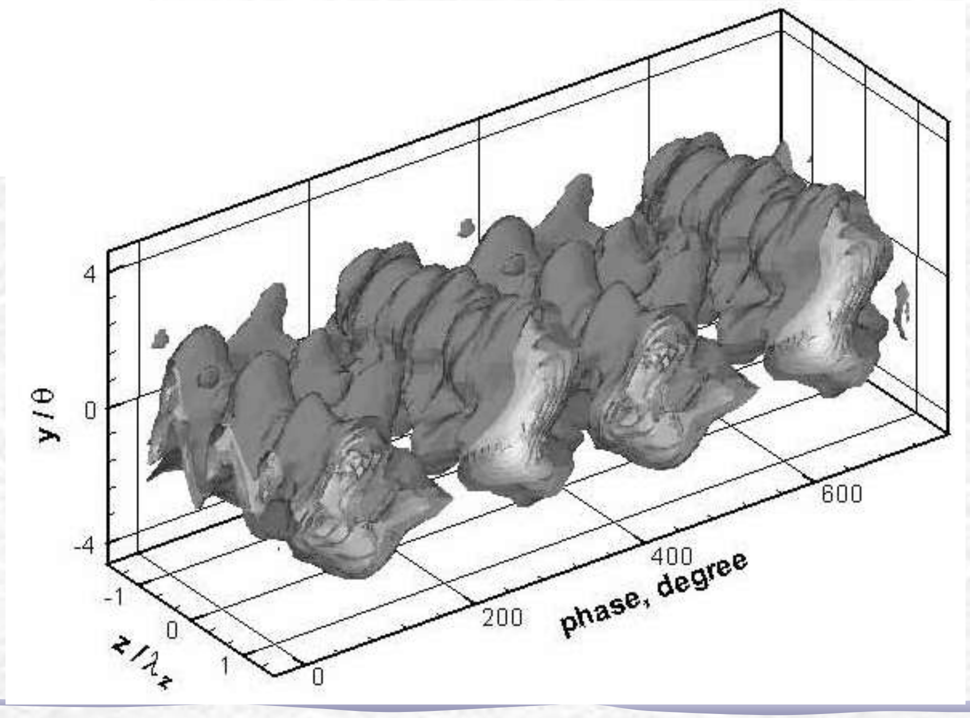
Stratified Shear Layers

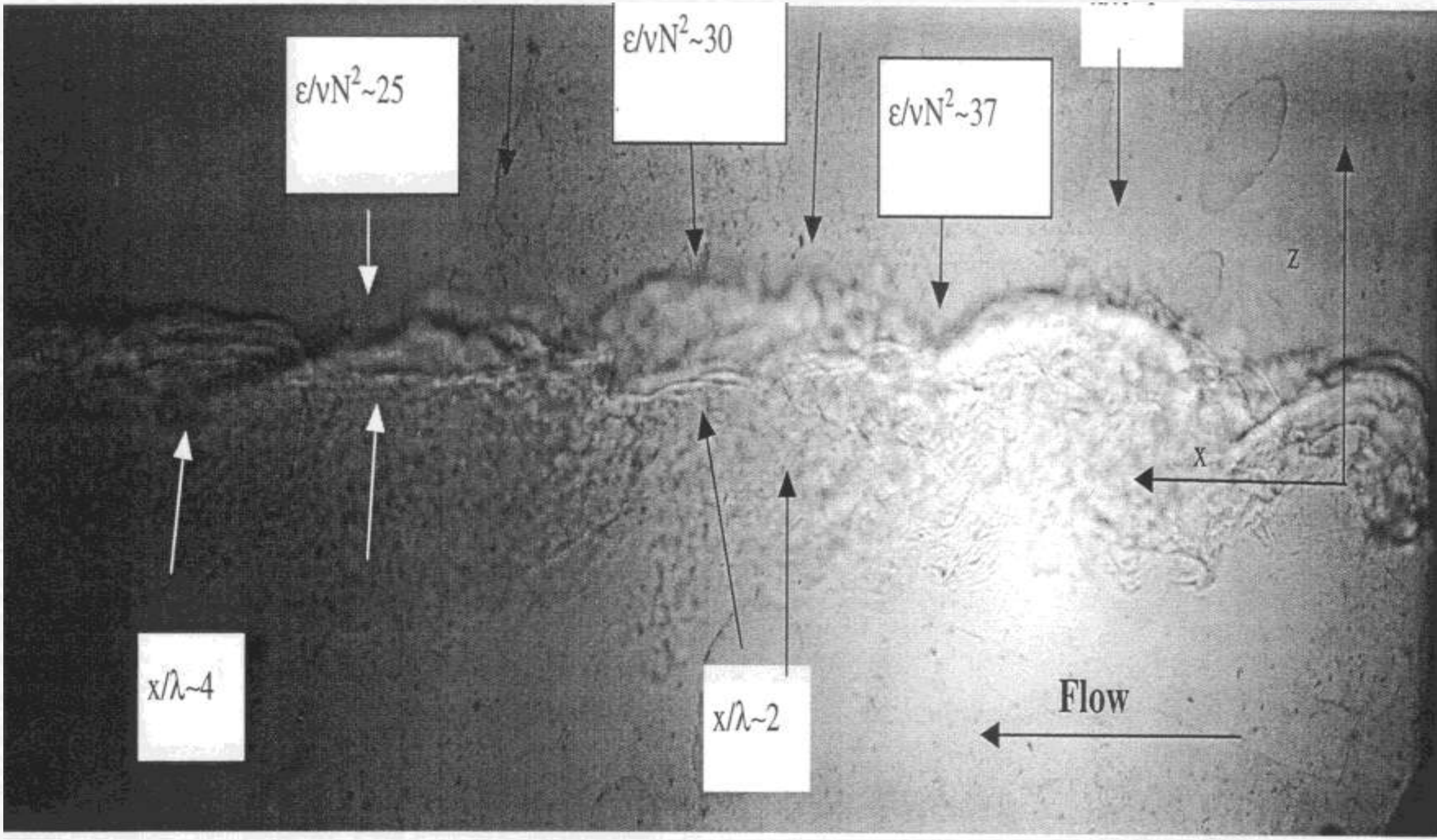
Slow

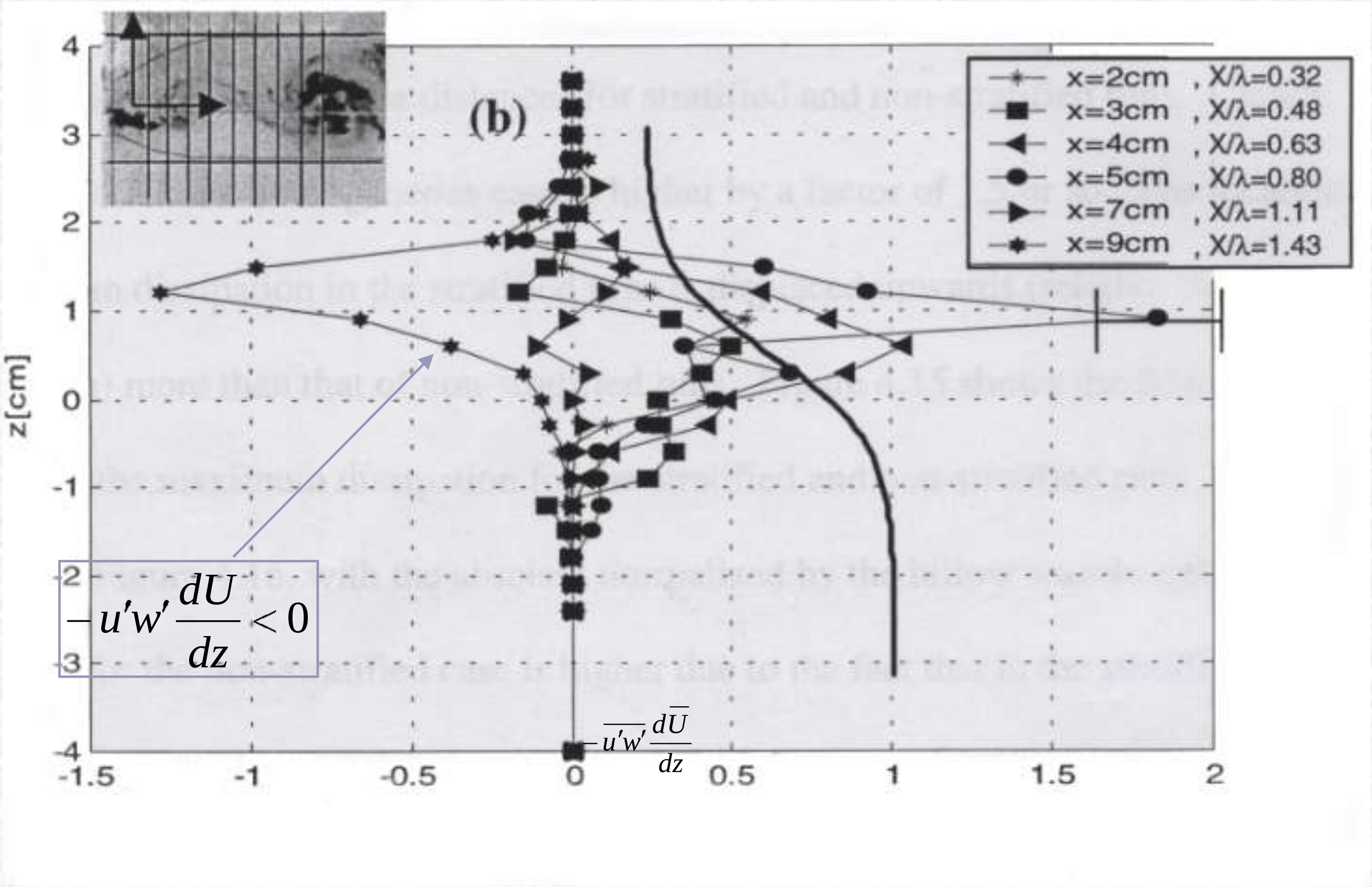


Faster


4
0
-4
y / θ
-1
0
1
z / λz
0
200
400
600
phase, degree


4
0
-4
y / θ
-1
0
1
z / λz
0
200
400
600
phase, degree

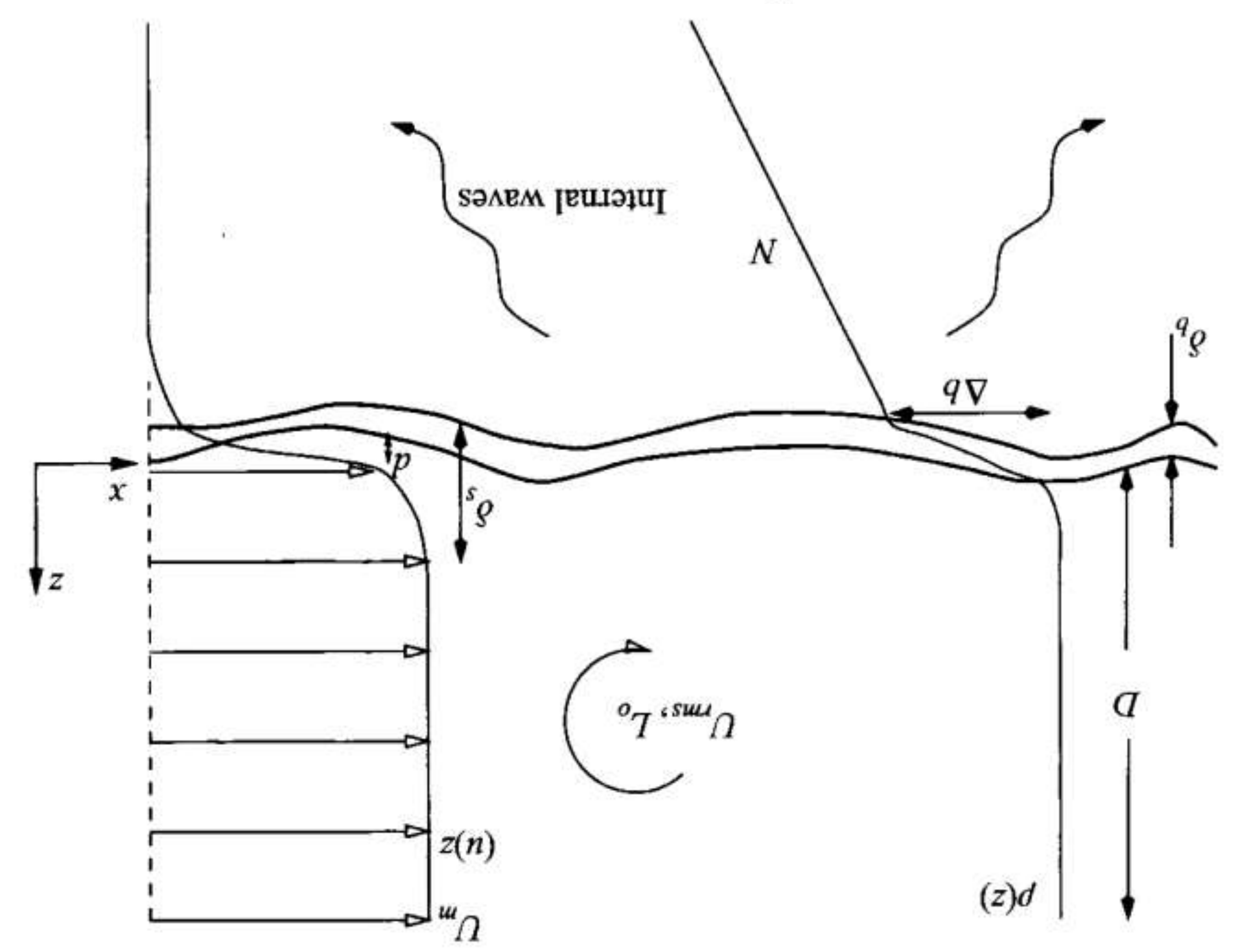

ε/νN²~25
ε/νN²~30
ε/νN²~37
z
x
x/λ~4
x/λ~2
Flow


(b)
x=2cm , X/λ=0.32
x=3cm , X/λ=0.48
x=4cm , X/λ=0.63
x=5cm , X/λ=0.80
x=7cm , X/λ=1.11
x=9cm , X/λ=1.43
z[cm]
$-u'w'\frac{dU}{dz}<0$
$-\overline{u'w'}\frac{d\overline{U}}{dz}$

Theory/Laboratory Profiles

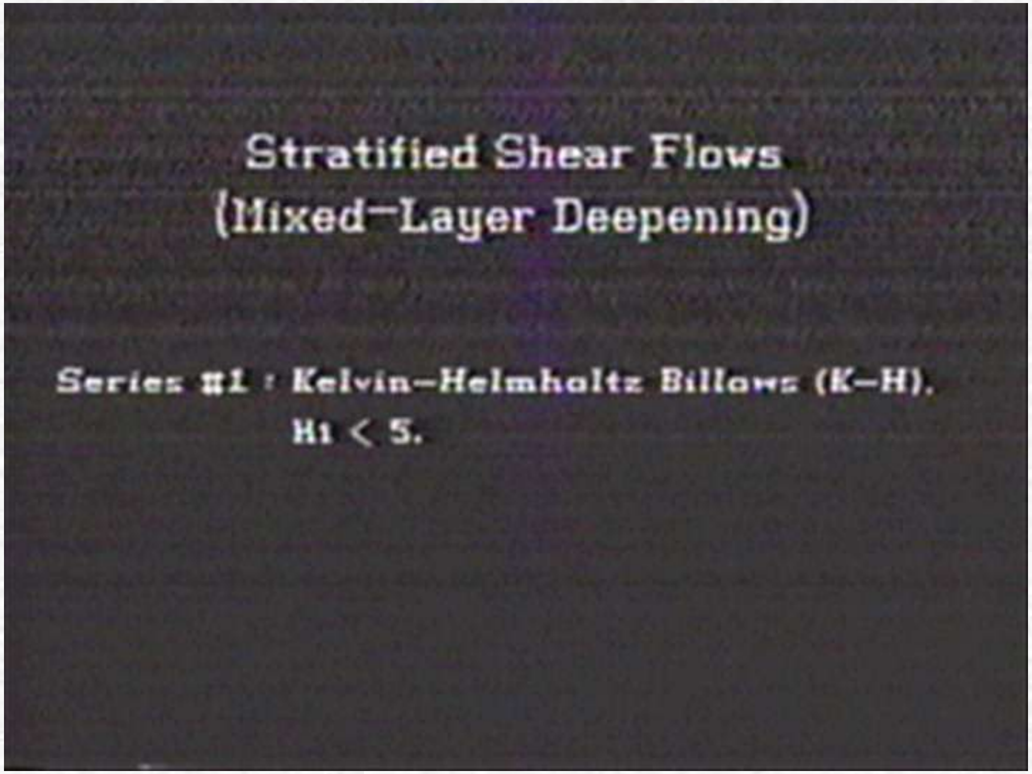
$$Ri = \frac{\Delta b D}{U^2}$$

$$\overline{Ri_g} = \frac{N^2}{\left(d\overline{U}/dz\right)^2}$$



Stratified Shear Flow (Lab)

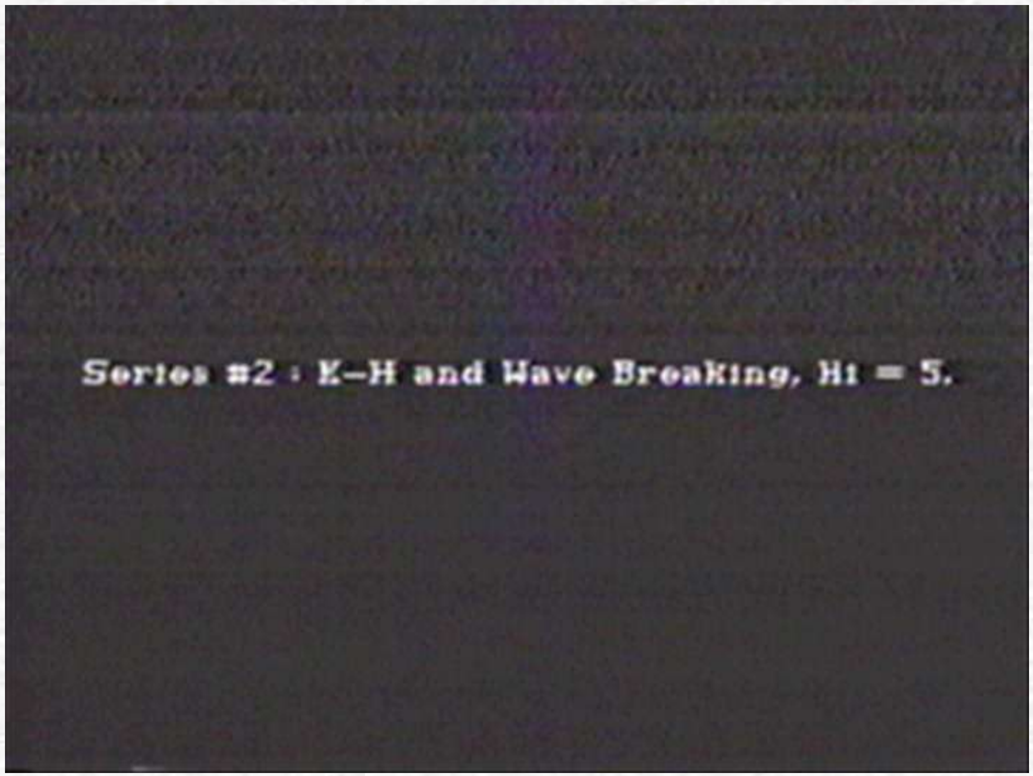
$$Ri_g < 1$$



(Strang & Fernando JFM, 2001)

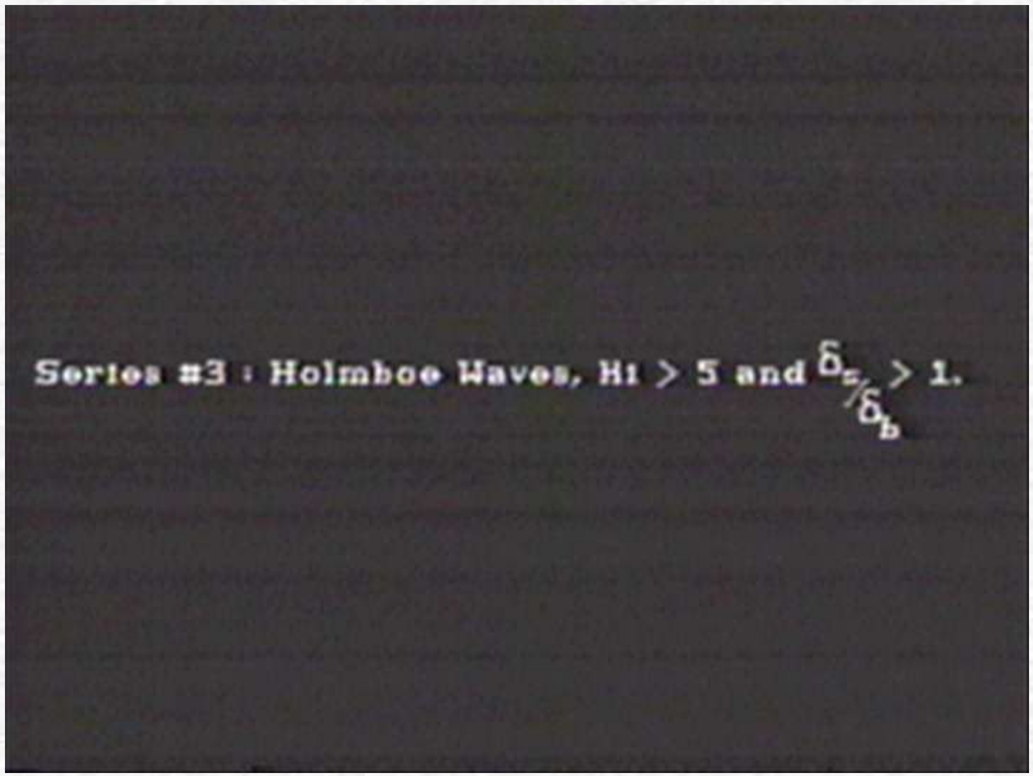
Stratified Shear Flow #2

$Ri_g \approx 1$



Stratified Shear Flow #3

$$Ri_g > 1$$



Mechanisms of Entrainment

(a) $\overline{Rig} = 0.12$

(b) $\overline{Rig} = 0.36$

(c) $\overline{Rig} = 0.83$

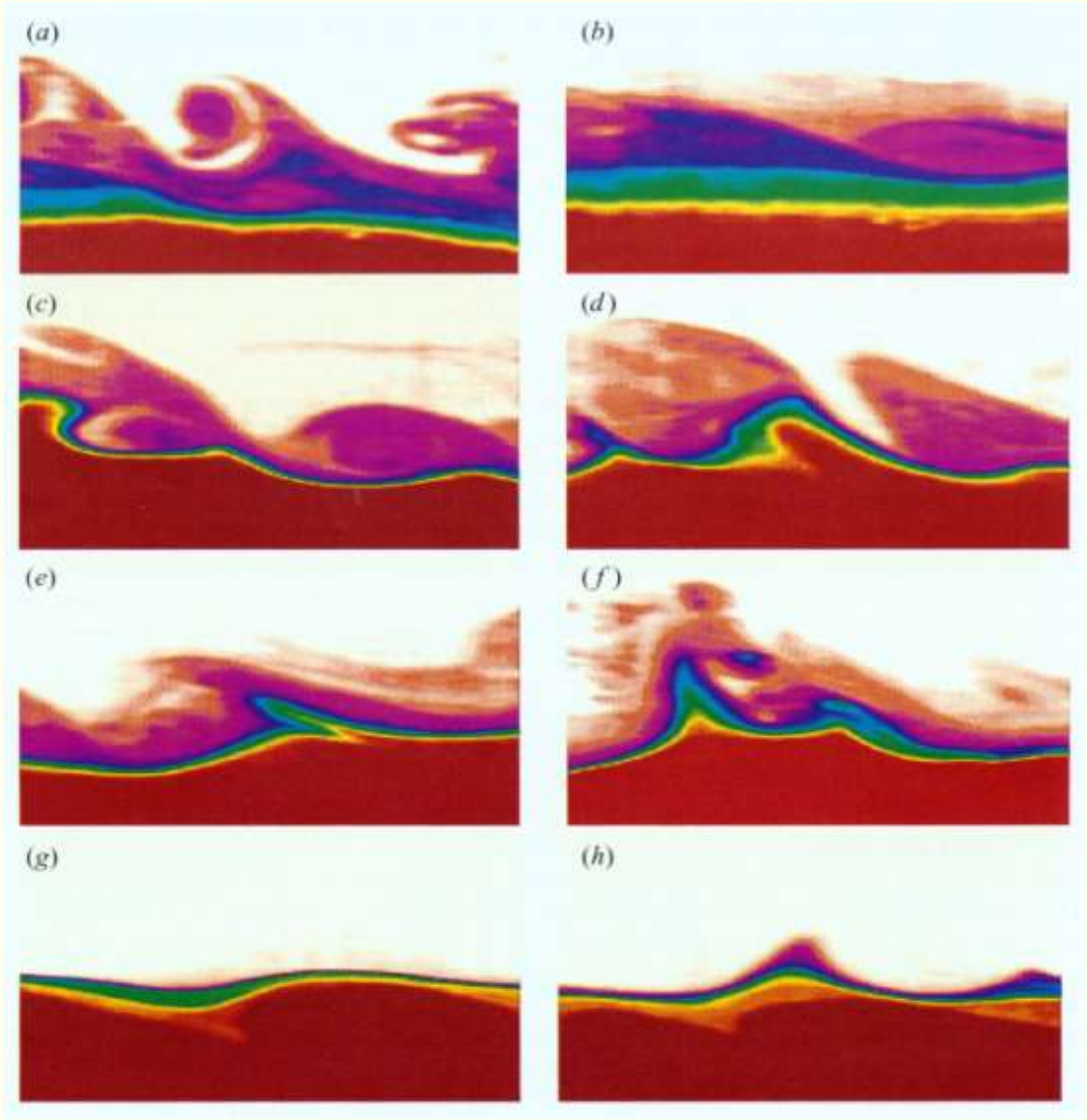
(d) $\overline{Rig} = 1.21$

(e) $\overline{Rig} = 1.21$

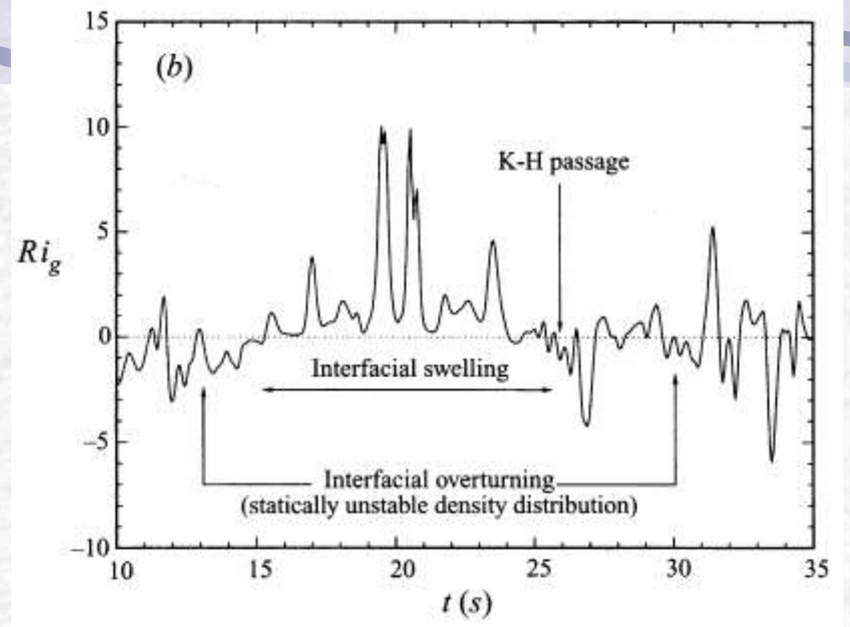
(f) $\overline{Rig} = 1.78$

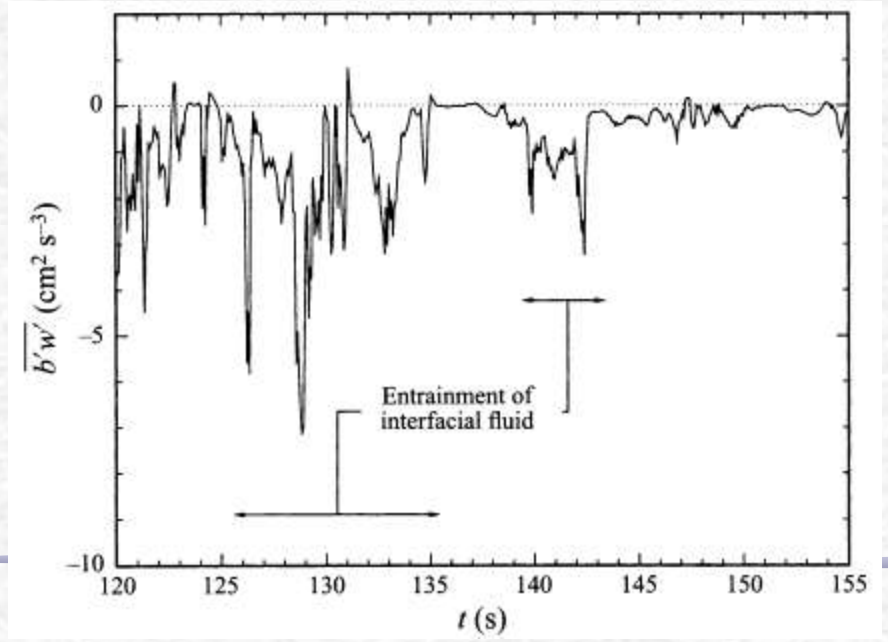
(g) $\overline{Rig} = 2.82$

(h) $\overline{Rig} = 2.82$

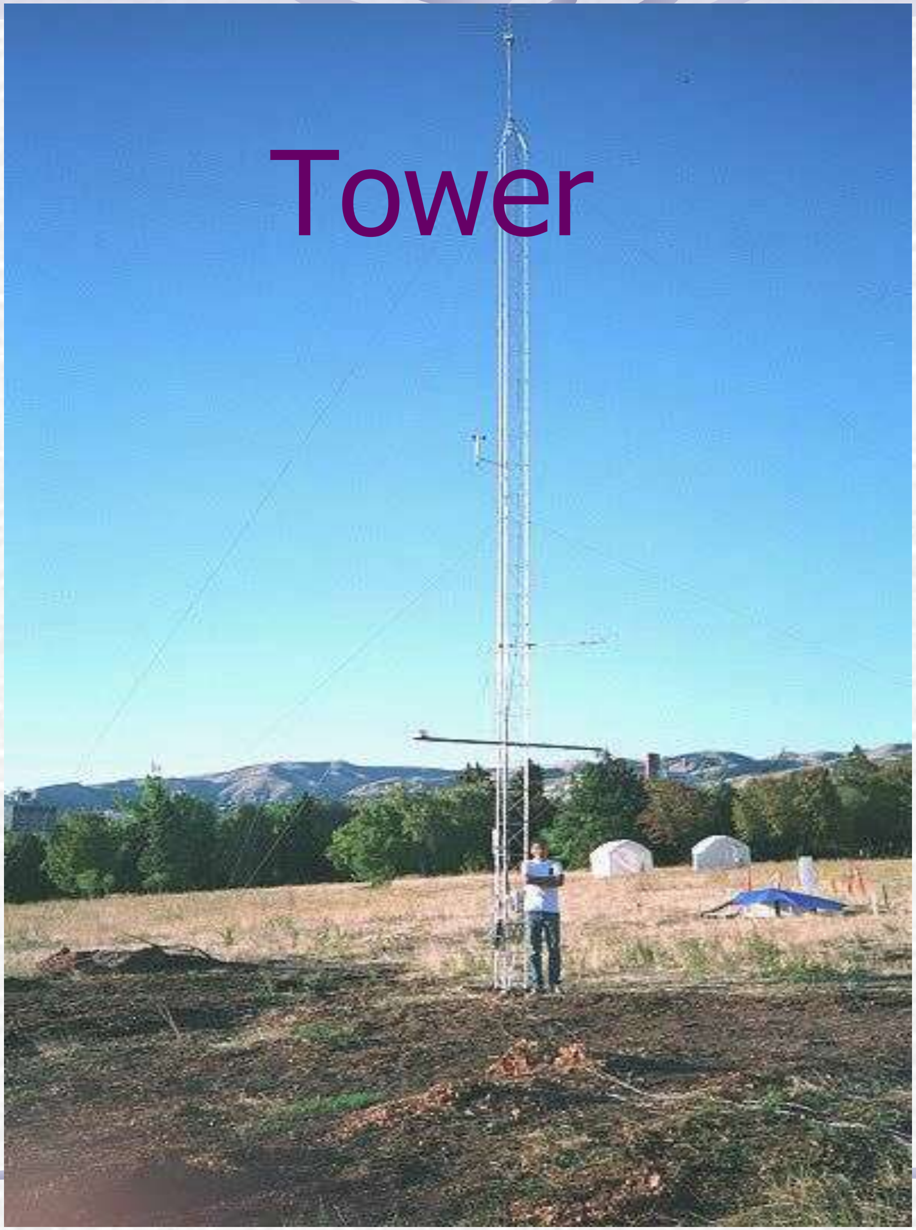


Interfacial Measurements

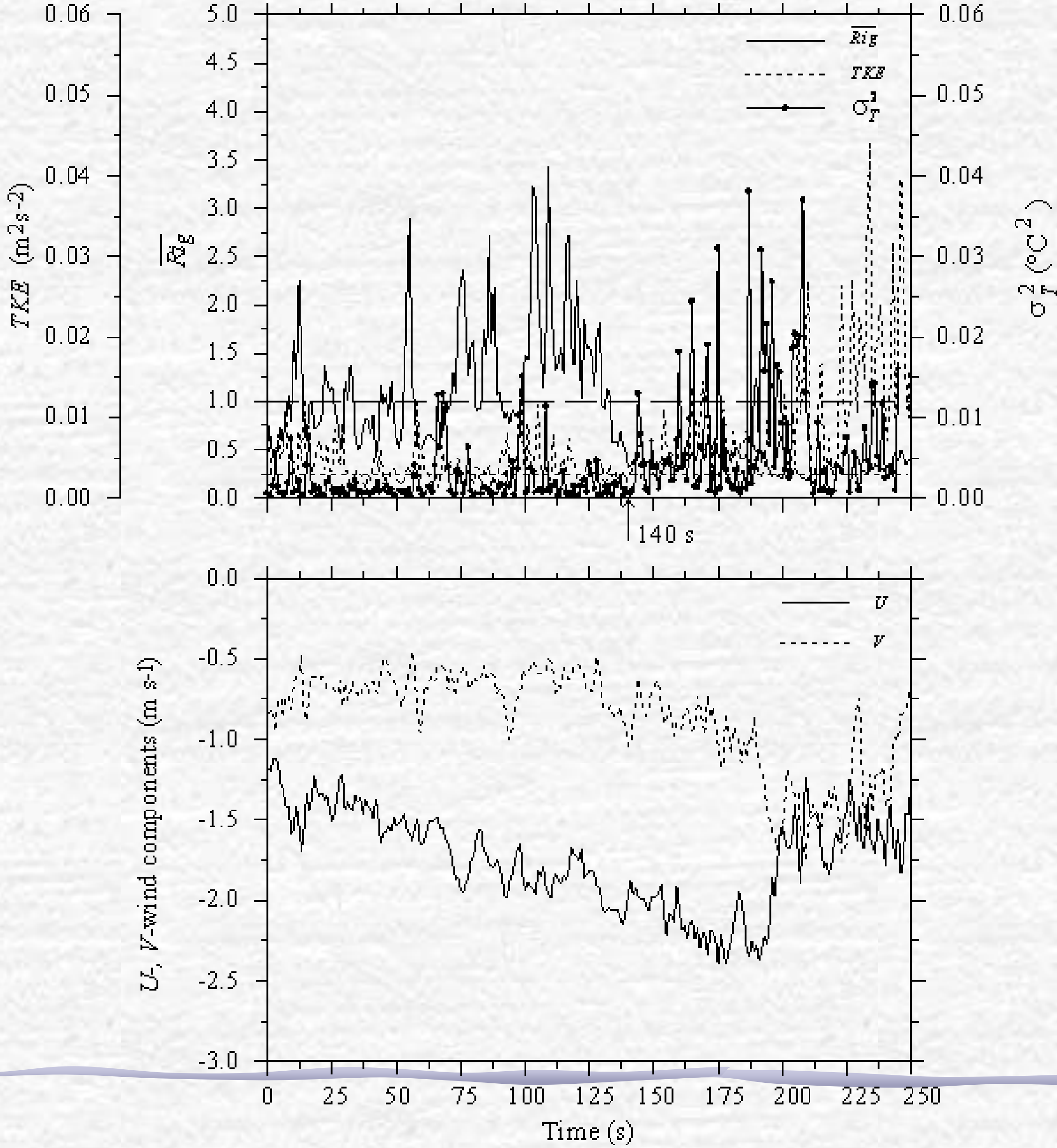




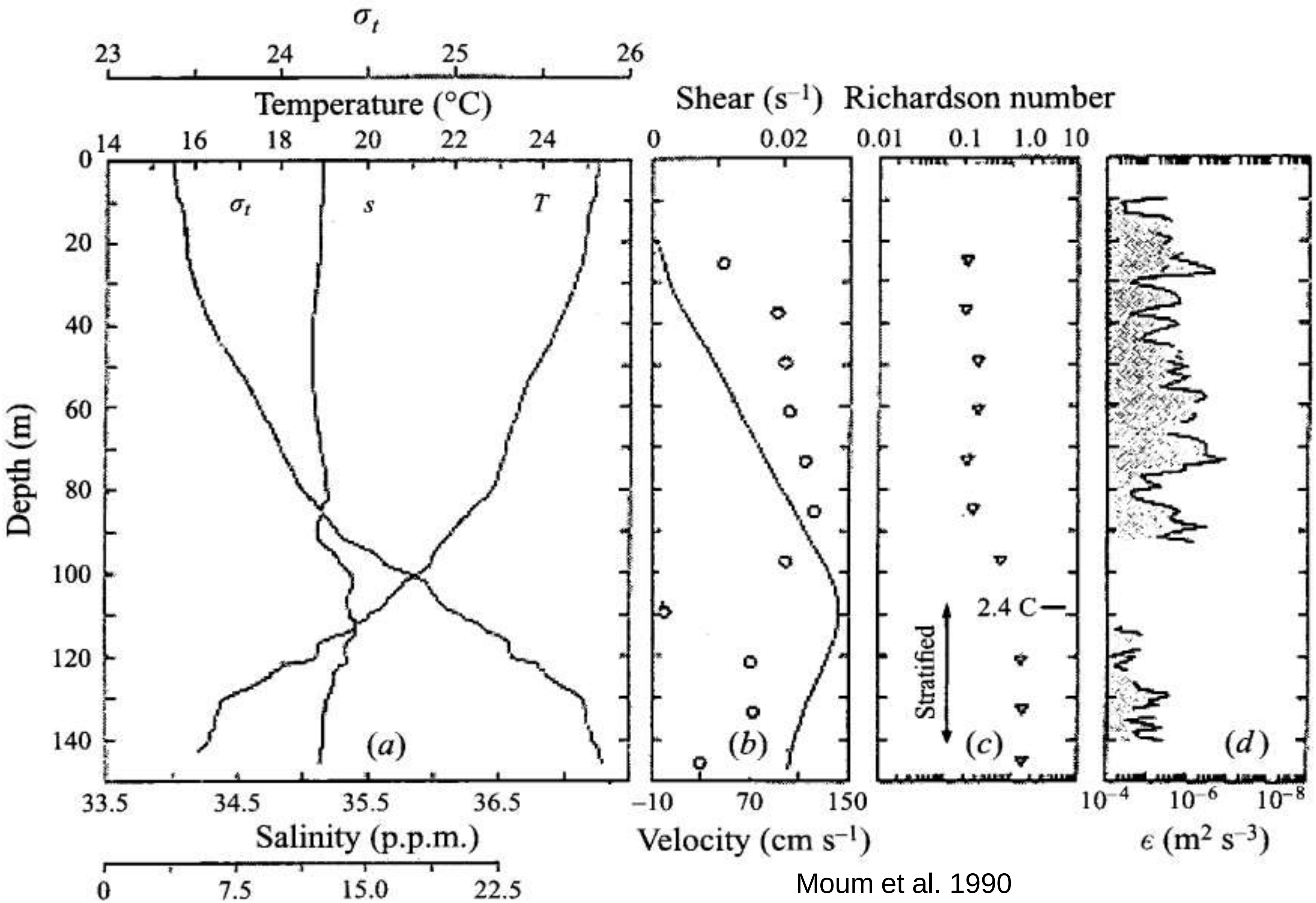
Tower



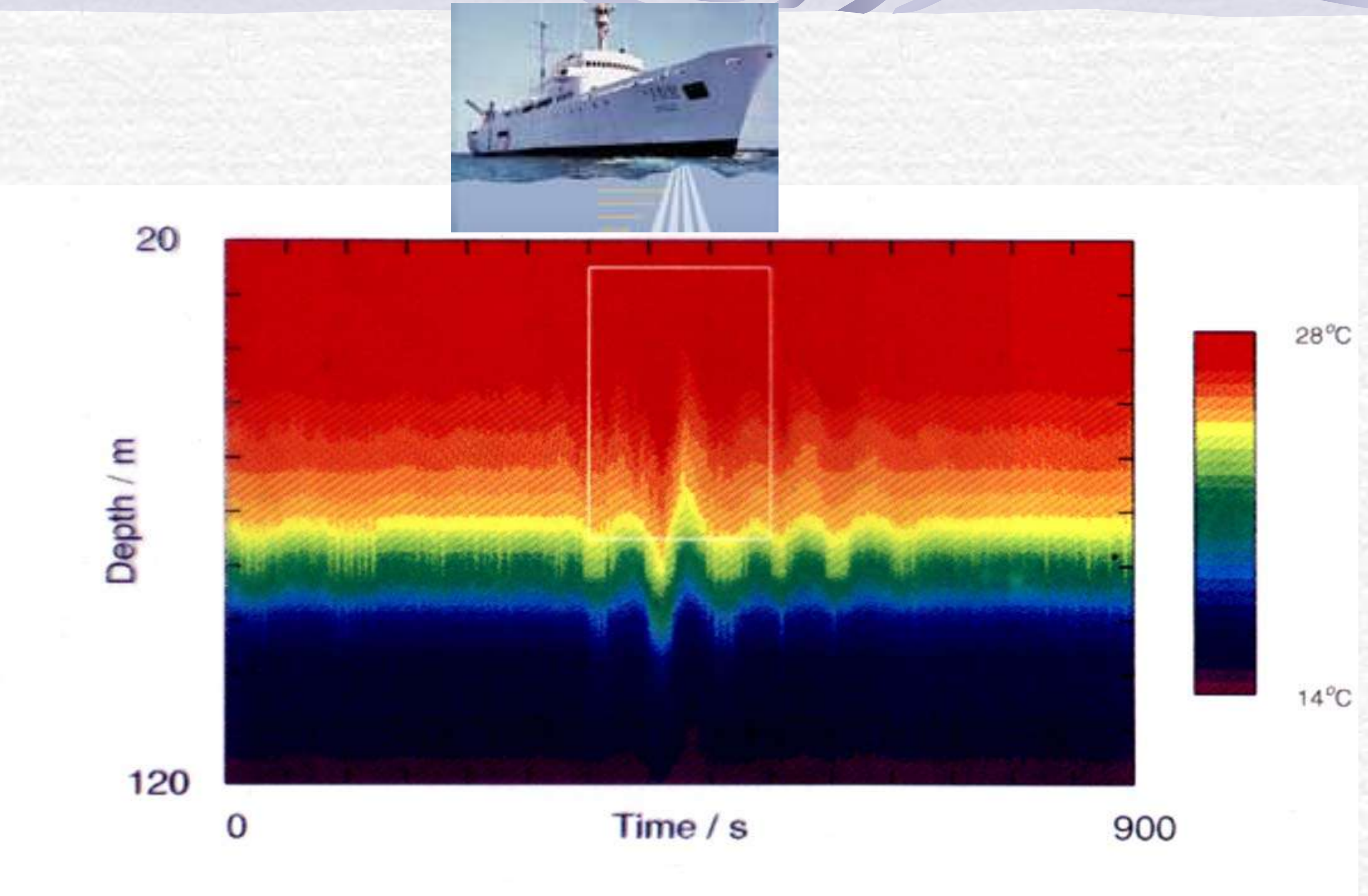
Turbulence in ABL



Daytime Vertical Profile



Moum et al. 1990

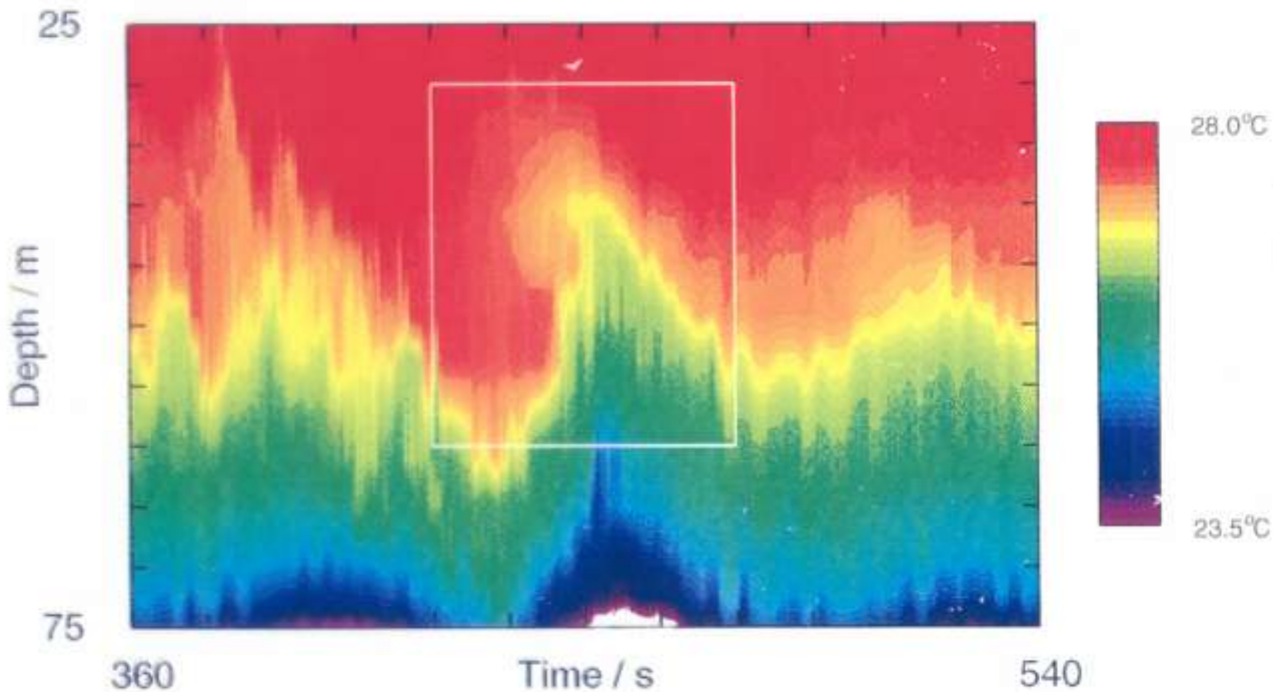


DeSilva, Fernando, Hebert & Eaton, Earth Planetary Sci. Lett. , 1996

$\lambda \approx 175\,m$

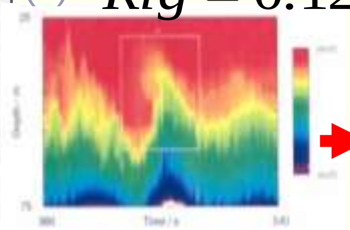
$h \approx 16m$

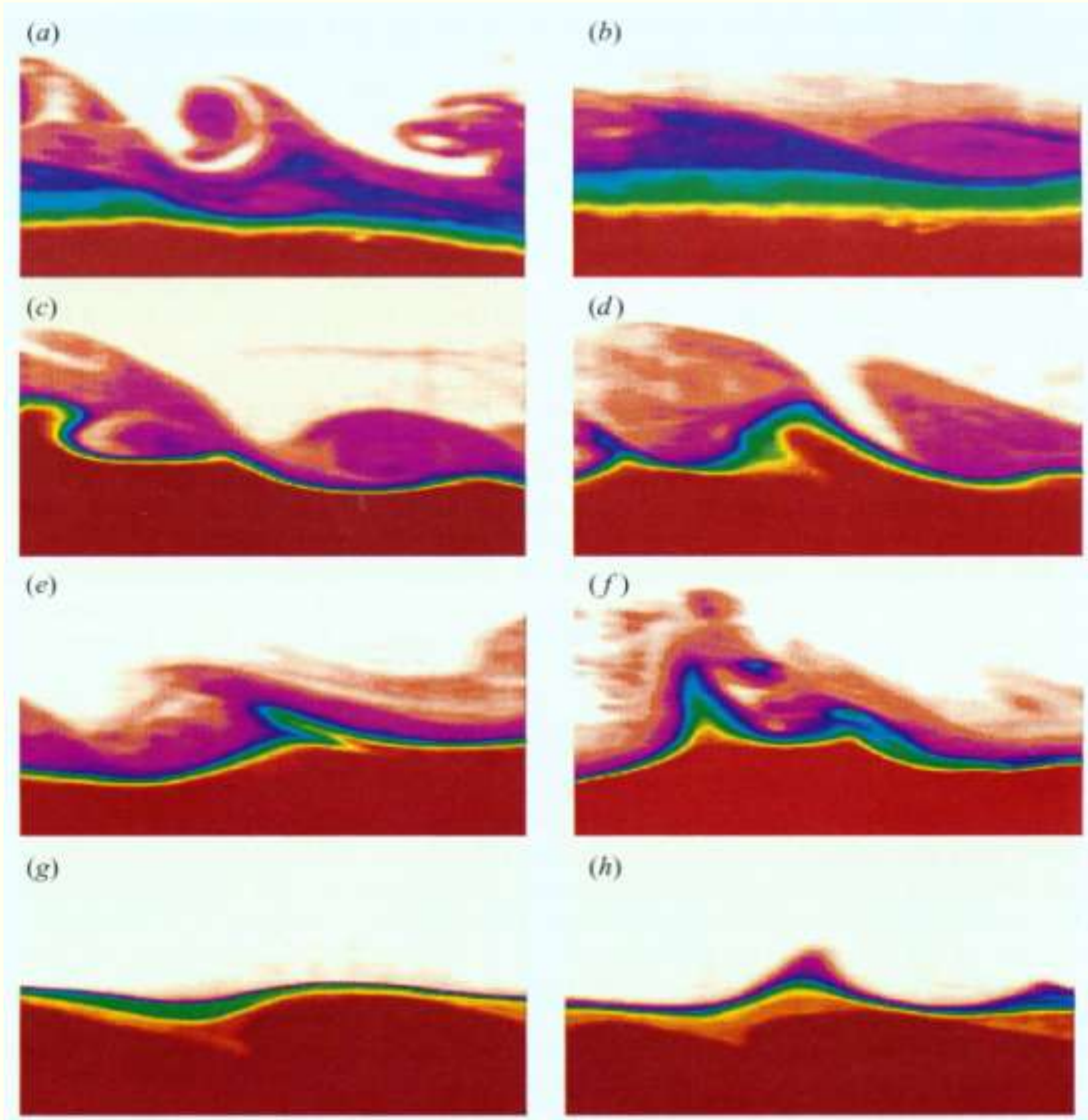
$\frac{h}{\lambda} \approx 0.09$ $\frac{\Delta u t}{\lambda} \approx 1.38$



Mechanisms of Entrainment

(a) $\overline{Rig} = 0.12$





(b) $\overline{Rig} = 0.36$

(c) $\overline{Rig} = 0.83$

(d) $\overline{Rig} = 1.21$

(e) $\overline{Rig} = 1.21$

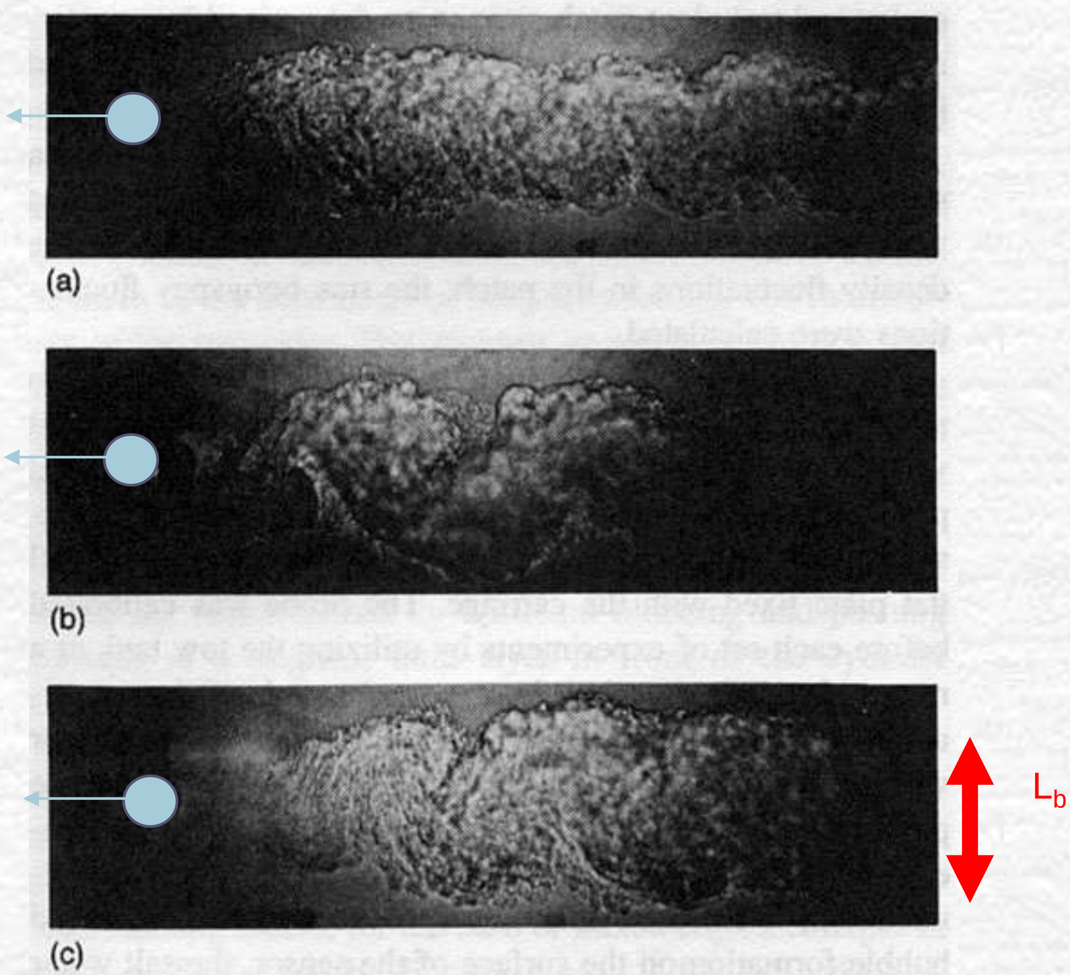
(f) $\overline{Rig} = 1.78$

(g) $\overline{Rig} = 2.82$

(h) $\overline{Rig} = 2.82$

Stratified Wakes

$$Re = \frac{Ud}{\nu} \qquad Fr = \frac{U}{Nd}$$



Re = 200, *Fr* = 2.24

Re = 3040, *Fr* = 1.5

Re = 3800, *Fr* = 3.37

Scales

$$L_b = \frac{U_v}{N}$$

$$\varepsilon \sim \frac{U_v^{\,3}}{L_b}$$

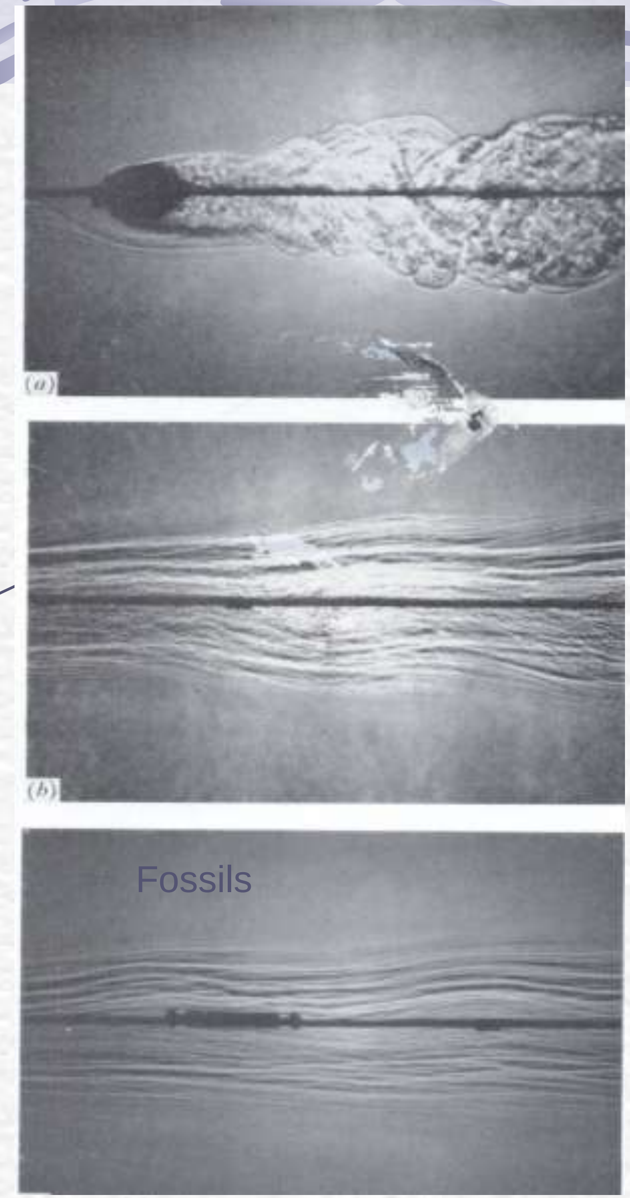
$$L_b \sim \left(\frac{\varepsilon}{N^3}\right)^{1/2} = L_R$$

Ozmidov

$$L_R \sim L_b \sim L_K$$

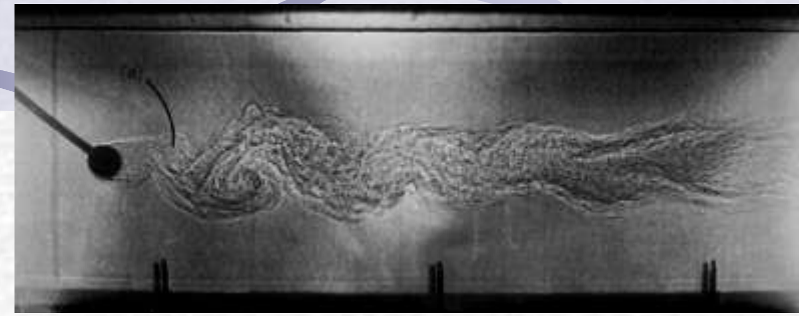
Gibson

$$\frac{\varepsilon}{\nu N^2} = (10 - 30) \approx 25$$



H.P. Pao/Boeing

Energy Spectra in a Wake



4D

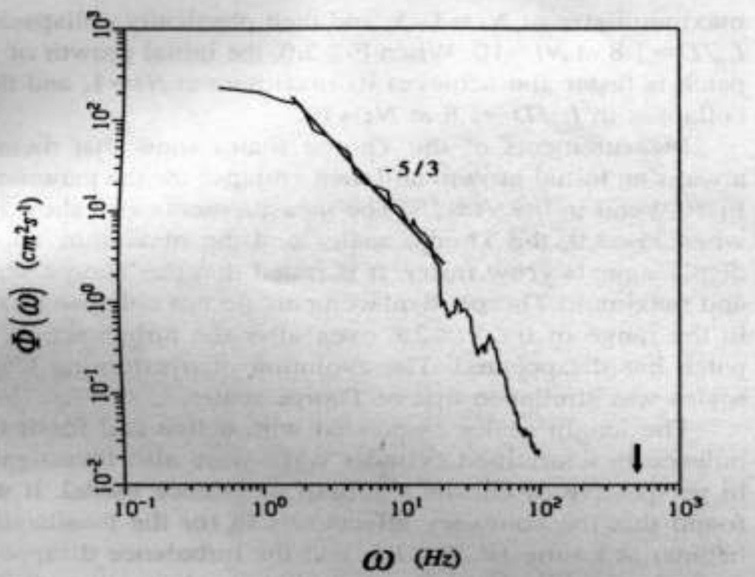
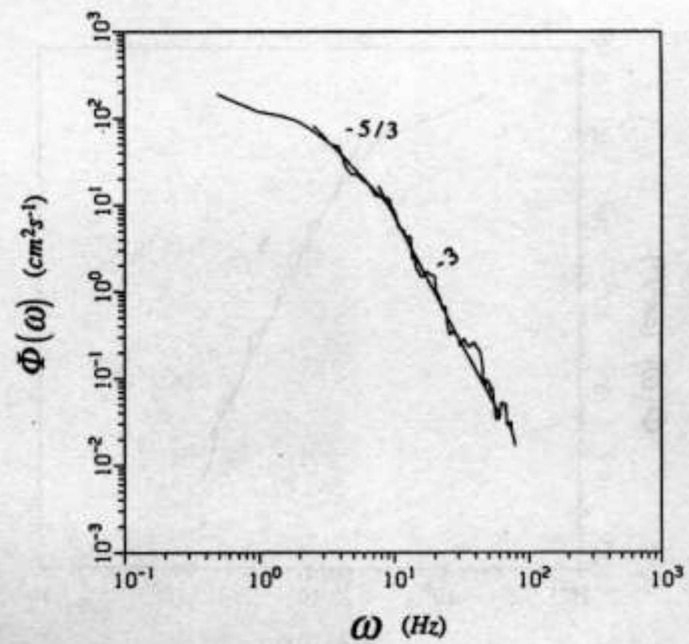


FIG. 15. Frequency spectrum of velocity fluctuations in the turbulent wake $4D$ from the cylinder, with Re=4800, Fi=4.6, and Nt=0.87. The Kolmogorov scale is indicated by an arrow.



15D

FIG. 17. Frequency spectrum of velocity fluctuations in the turbulent wake $15D$ from the cylinder, with Re=4800, Fi=4.6.

10D

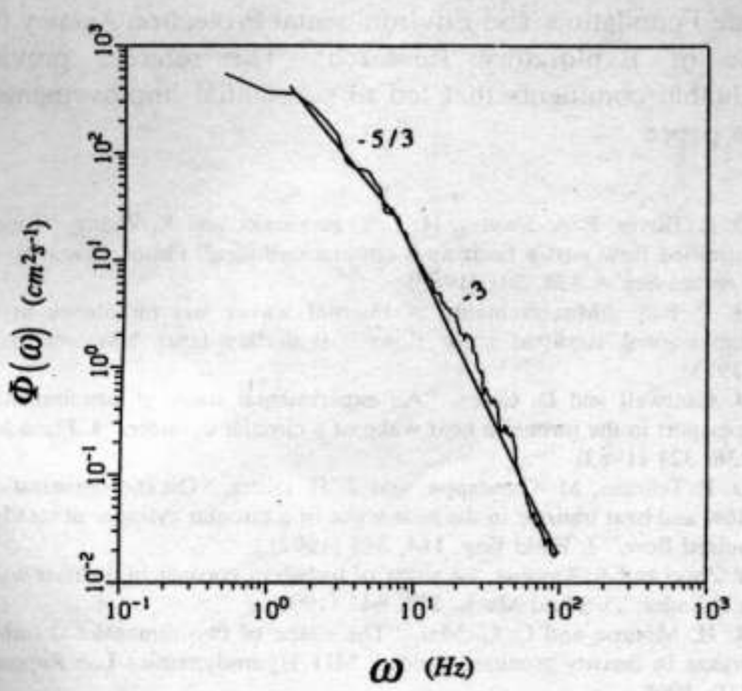


FIG. 16. Frequency spectrum of velocity fluctuations in the turbulent wake 10*D* from the cylinder, with Re=4800, Fi=4.6.

20D

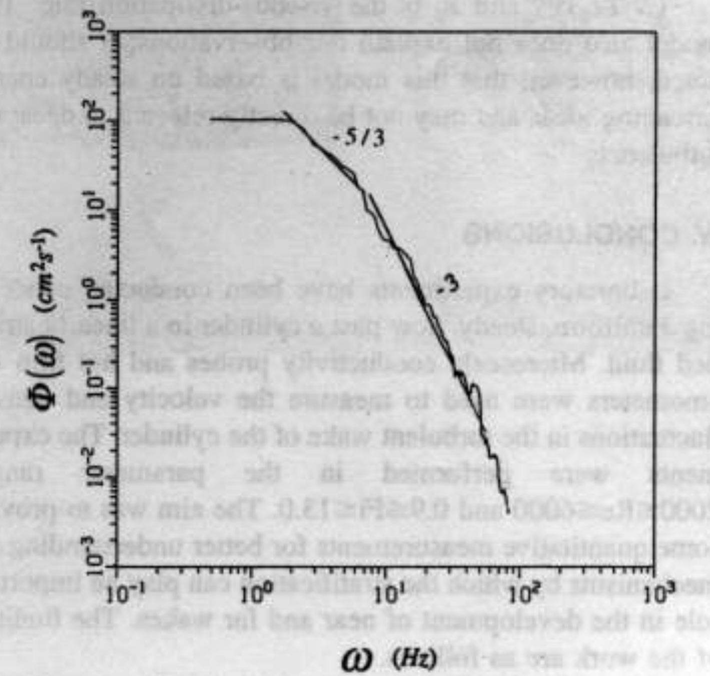
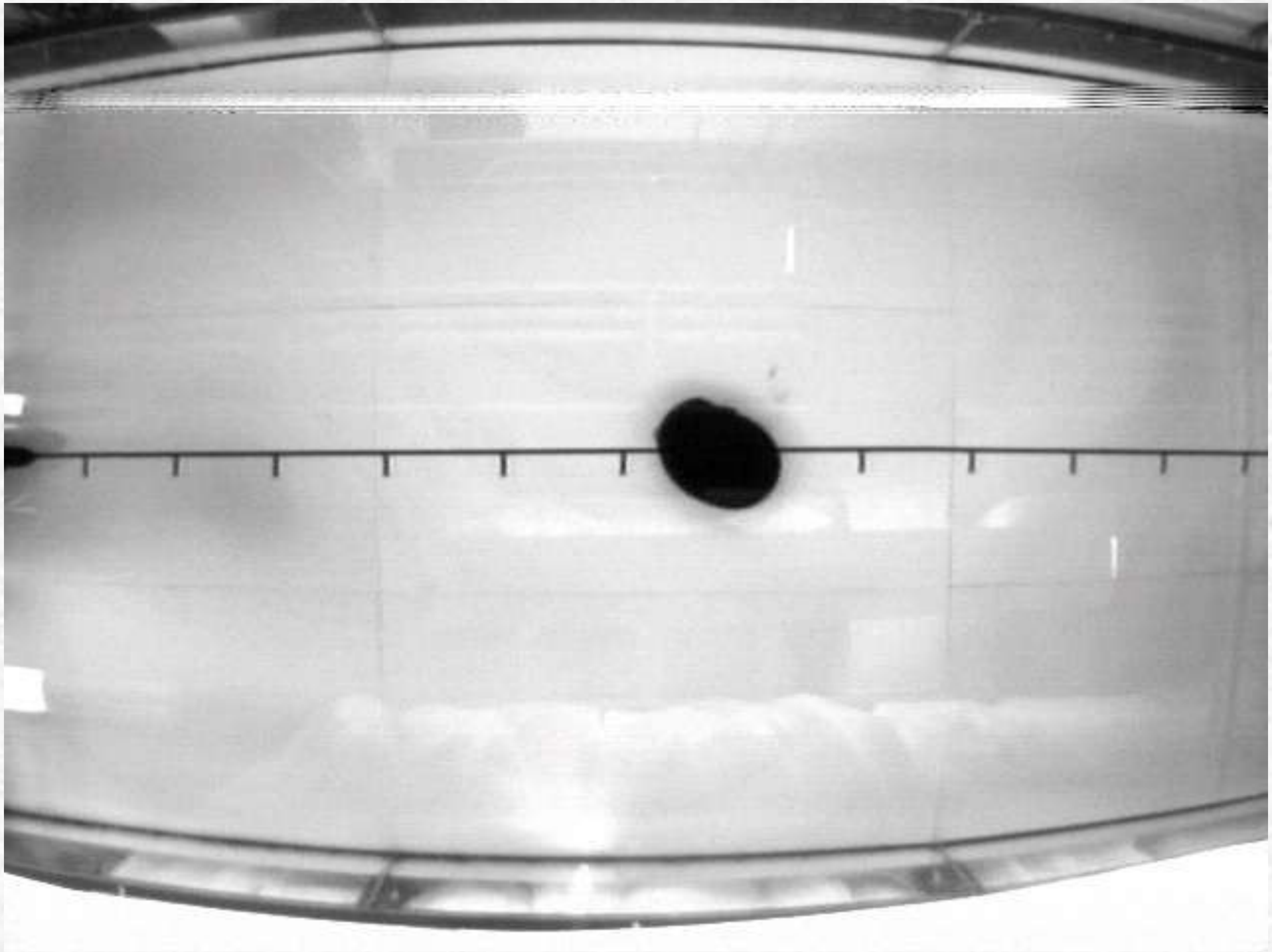


FIG. 18. Frequency spectrum of velocity fluctuations in the turbulent wake 20*D* from the cylinder, with Re=4800, Fi=4.6.

Self-Propelled Bodies

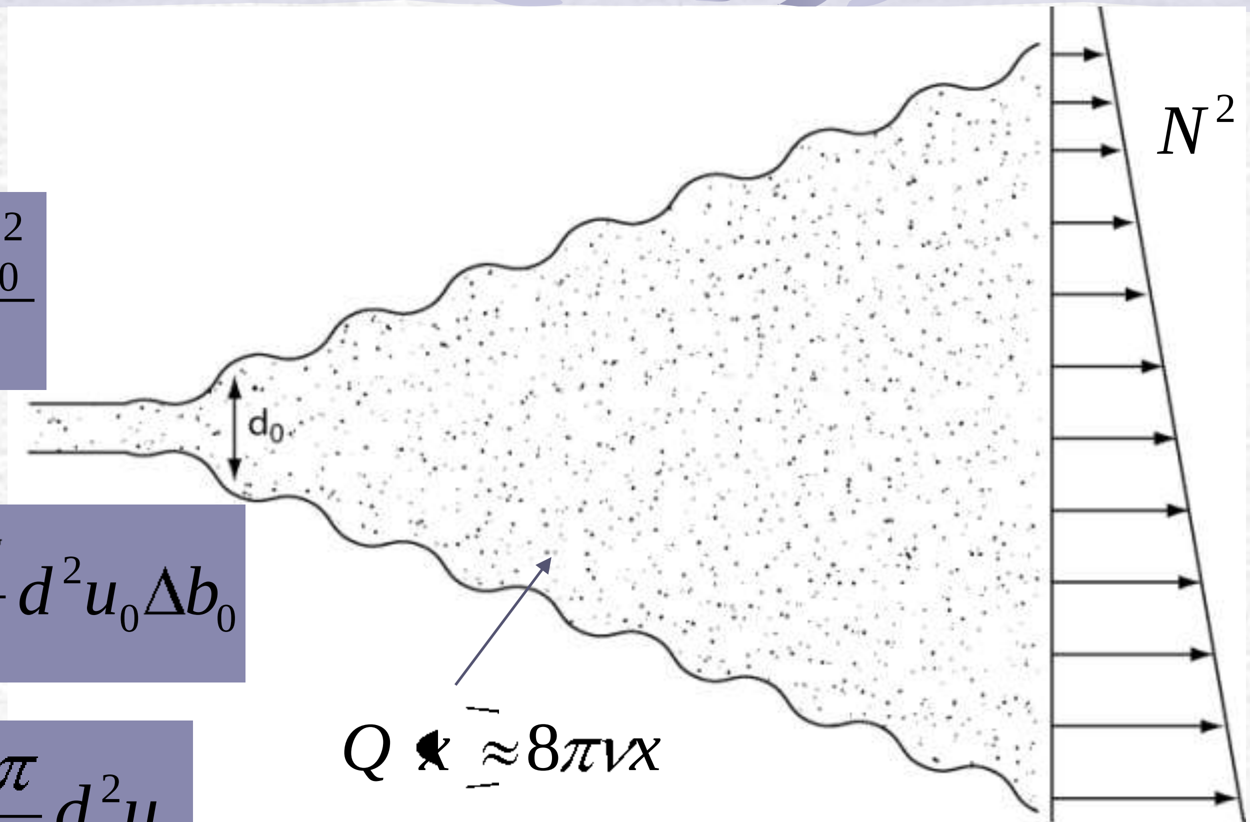


Jets in Stratified & Rotating Fluids

$$J_0 = \frac{\pi d^2 u_0^2}{4}$$

$$B_0 = \frac{\pi}{4} d^2 u_0 \Delta b_0$$

$$Q_0 = \frac{\pi}{4} d^2 u_0$$



$$Q(x) \approx 8\pi\nu x$$

$$u_e \, \alpha \, U$$

$$\mathrm{Re} = \frac{J_0^{1/2}}{\nu}$$

$$Ri_0 = \frac{N^2 L \nu^2}{u_0^2}$$

$$Ro = \frac{u_0}{f L_H}$$

Turbulent Jets

$$Ri \sim \frac{N^2 L_v^2}{U^2}$$

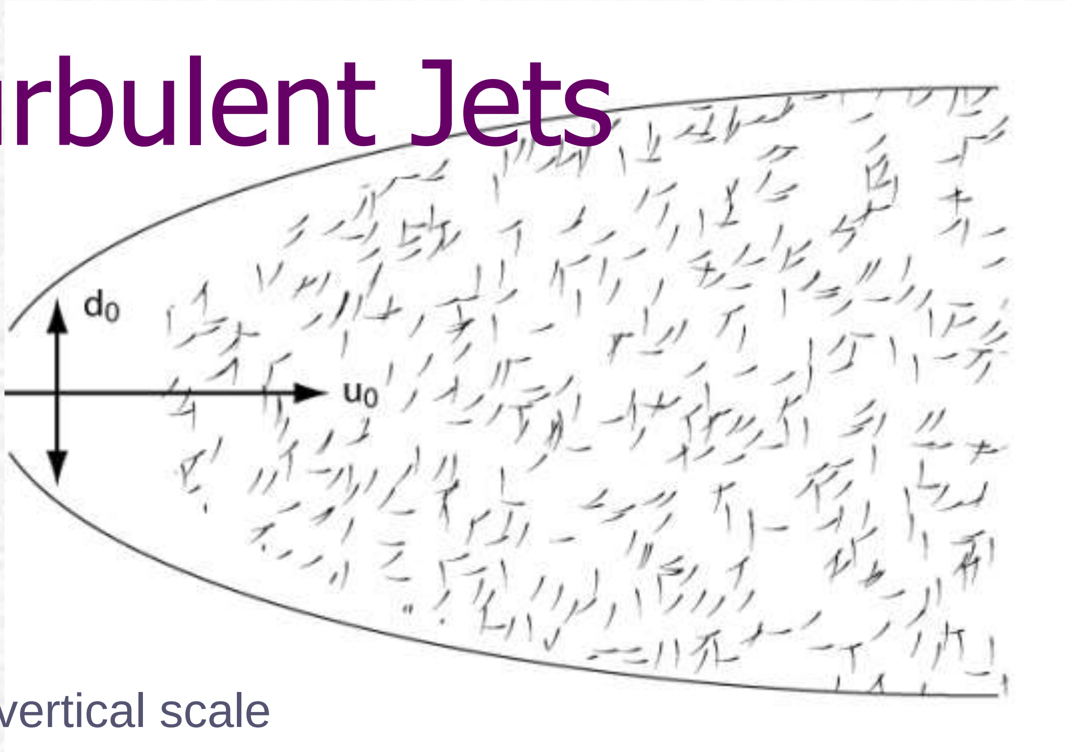
$$U \sim \frac{J_0^{1/2}}{x}$$

$$L_v \sim x$$

$$h_{max} \sim \left(\frac{J_0}{N^2}\right)^{1/4}$$

Limiting vertical scale

$$\frac{h_{max}}{d_0} \sim \left(\frac{u_0}{N d_0}\right)^{1/2} \sim Fr^{1/2}$$



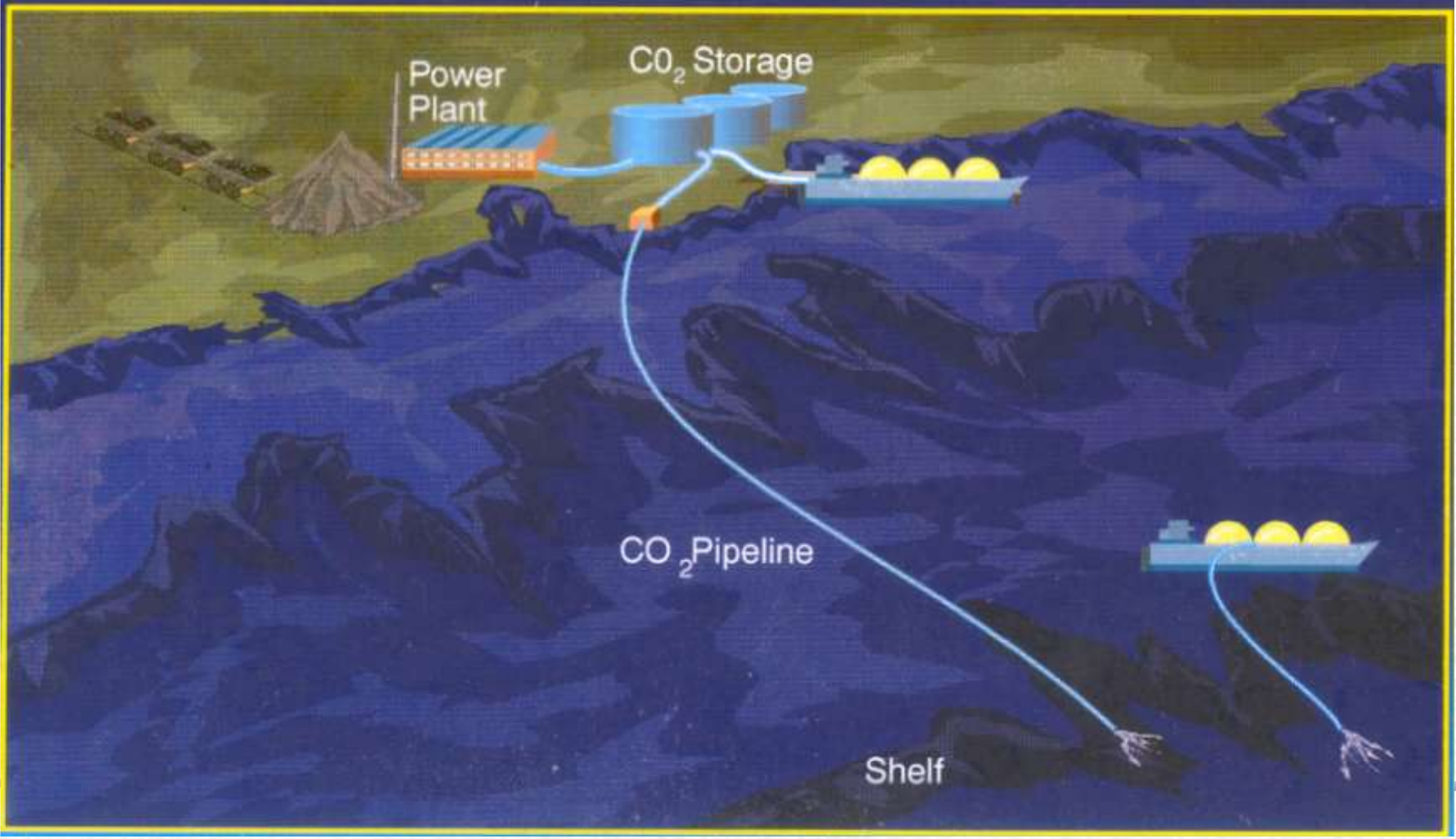
LAMINAR JETS IN ROTATING STRATIFIED FLUIDS

$Re = 85$

$N = 1.2 \ S^{-1}$


A
TURBULENT JET
Re = 10,000
Fr = 1500
N = 1.01 S⁻¹

Integrated Energy System Including Deep Sea Disposal of CO_2



Transport and Dispersion (Material P)

$$\tilde{P}(\underset{\sim}{x},t) = p'(\underset{\sim}{x},t) + \overline{P}(\underset{\sim}{x},t)$$

$$\tilde{U}(\underset{\sim}{x},t) = u'(\underset{\sim}{x},t) + \overline{U}(\underset{\sim}{x},t)$$

$$\frac{\partial \overline{P}}{\partial t} + \overline{U}_j \frac{\partial \overline{p}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D \frac{\partial \overline{P}}{\partial x_j} - \overline{p'u'_j}\right) + \overline{F}_p$$

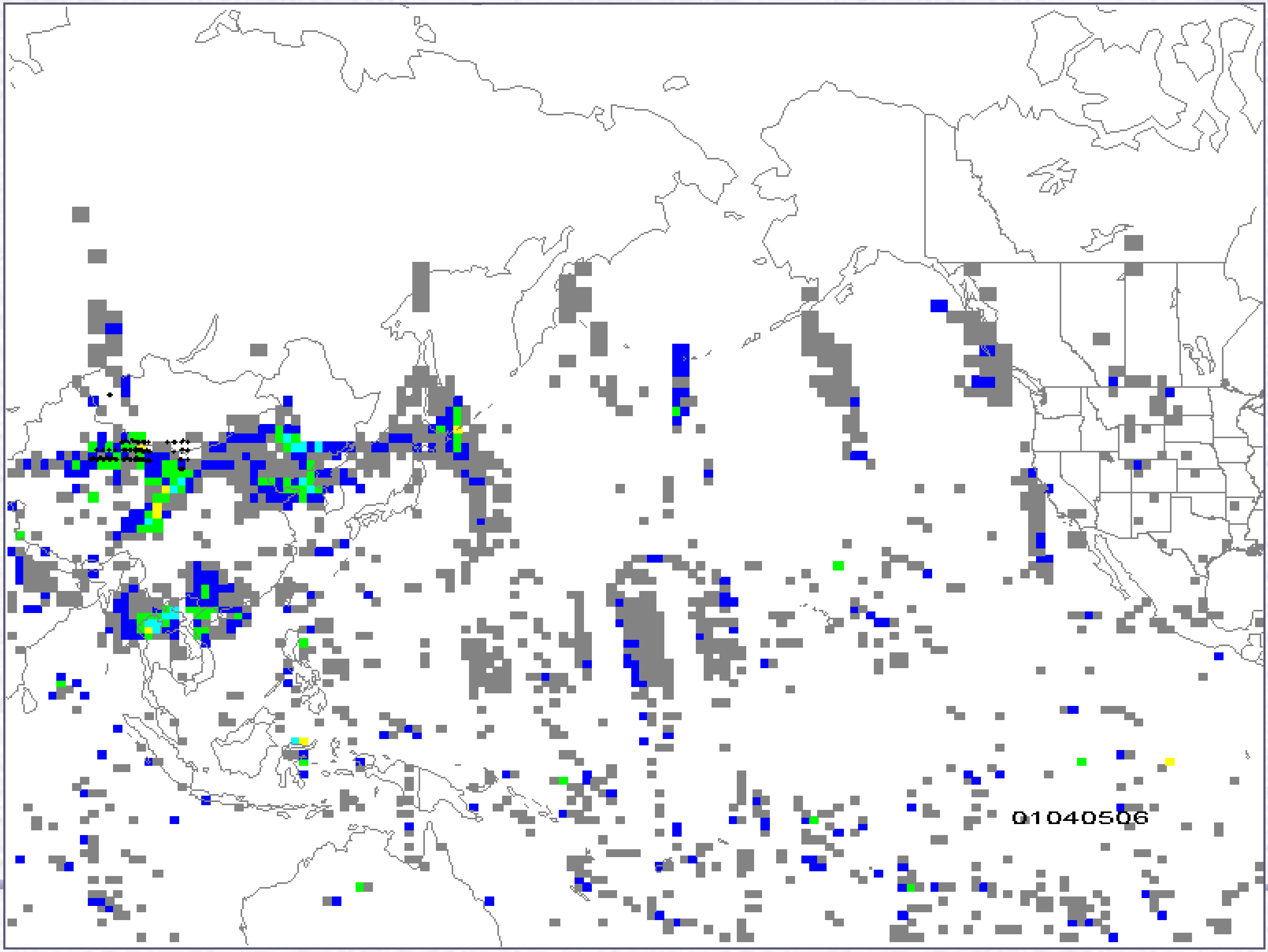
transport

dispersion

source

HYSPLIT verification – Asian dust storm / TOMS data

- 10-day loop, April 2001, http://www.arl.noaa.gov/ss/transport/dust/



Plumes - Convective

Plumes - Stable

Dispersion of Particles in Turbulent Fluids

If a fluid particle moves in successive displacements of Δy_i,

then the resultant displacement after N steps is $y = \sum_1^N \Delta y_i$.

If the field is homogeneous, the average of $y, \overline{y} = 0$,

(over a large number of displacements), then the mean square value is

$$\overline{y^2} = \overline{\left(\sum_{i=1}^{N} \Delta y_i\right)\left(\sum_{j=1}^{N} \Delta y_j\right)} = \sum_{i=1}^{N} \overline{(\Delta y_i)^2} = N\overline{\Delta y^2}.$$

Dispersion

$\frac{dy}{dt} = v_2(t),$ and hence the distance $y(t) = y(0) + \int_0^t v_2(t')\, dt'.$

$$\overline{y^2(t)} = \frac{1}{T}\left(\int_0^t dt' v_2(t_0 + t')\, dt'\right)\left(\int_0^t dt'' v_2(t_0 + t'')\, dt''\right)$$

r.m.s. Dispersion

$$R_L(\tau) = \overline{v_2(t)v_2(t+\tau)}$$

$$\overline{y^2}(t) = 2\overline{v_2'^2}\left(\int_0^t R_L(\tau)d\tau - \int_0^t R_L(\tau)d\tau \right)$$

at large times $t > t^*$, $R_L(\tau)$ is small and negligible

$$\overline{y^2} = 2\overline{v_2'^2}\, t\int_0^{t^*} R_L(\tau)\, d\tau = 2\overline{v_2'^2}\, t\,\tau_L$$

For short times $t << t^*$, $R_L(\tau) \sim 1$

$$\overline{y^2} == 2\overline{v_2'^2}\, t^2$$

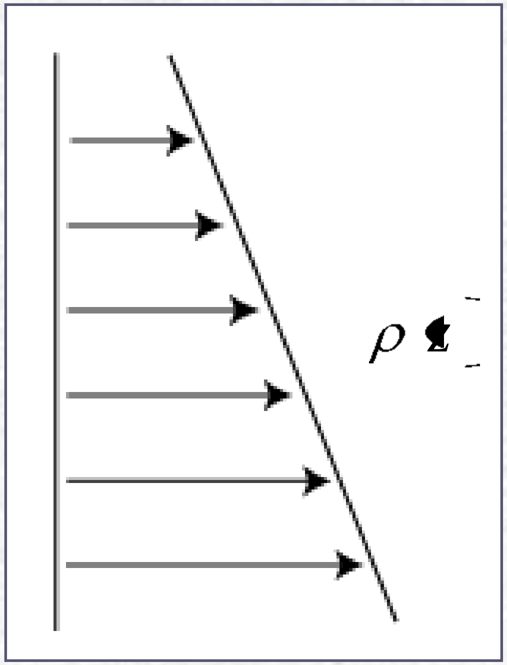
Eddy Diffusivity

$$\overline{y^2} \approx 2\overline{v_2'^2}\, t\, \tau_L = \left(2\overline{v_2'^2}\,\tau_L\right) t$$

$$\left(\overline{y^2}\right)^{1/2} = \left(\sqrt{2\overline{v_2'^2}\,\tau_L}\right)^{1/2} t^{1/2}$$

$$K = 2\overline{v_2'^2}\,\tau_L$$

Stratified Turbulence



$$\overline{y^2}(t) = 2\overline{v_2'^2}\left(t\int_0^t R_L(\tau)\,d\tau - \int_0^t \tau R_L(\tau)\,d\tau \right)$$

Can we use this?

Dynamics of Fluid Parcel Dispersion

Maximum Height

$$\frac{1}{2} N^2 z'^2 \sim \frac{1}{2} \sigma_w^2$$

$$\left(\overline{z'^2}\right)^{1/2} \sim \frac{\sigma_w}{N} : \; L_b = \frac{\sigma_w}{N}$$

Homogeneous/Stationary (Taylor 1921)

$$\overline{z'^2} = 2\sigma_w^2 \left[tT_L - \int_0^\infty \tau R_L(\tau)\, d\tau \right]$$

Very small!

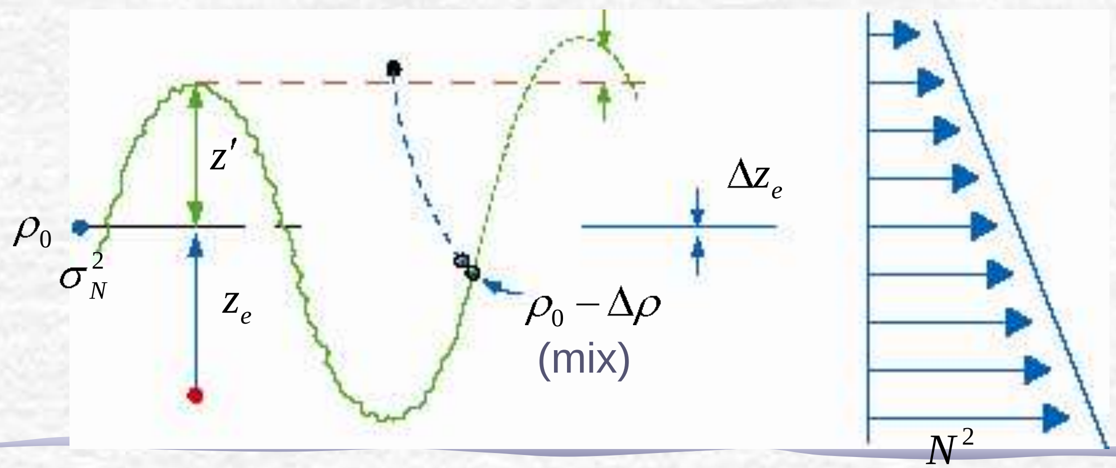
No Mixing

$$\frac{d\overline{z'^2}}{dt} = 0; \quad \overline{z'^2} = \text{const}$$

$$z = Z_e + z'$$

$$\frac{d\overline{z^2}}{dt} = \frac{d\overline{Z_e^2}}{dt} + \frac{d\overline{z'^2}}{dt} \approx \frac{d\overline{Z_e^2}}{dt} \;!$$

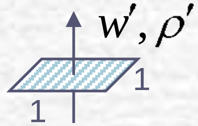
Small!!!

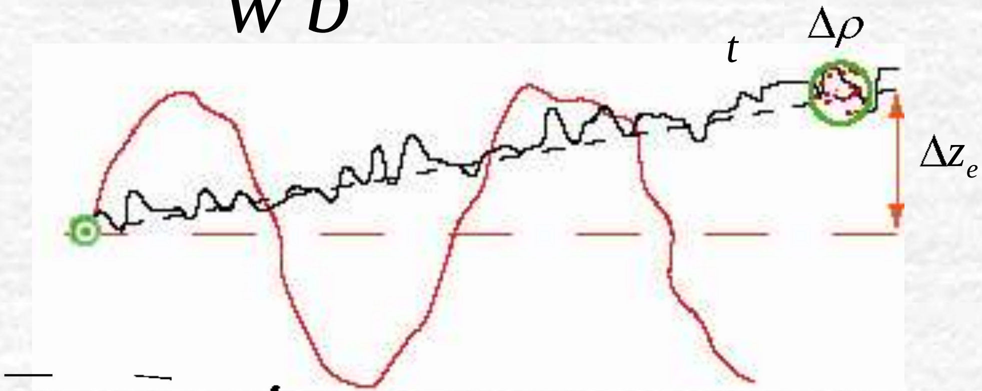


Fluxes of Buoyancy

$$\overline{w'b'}$$

$$\overline{w'\rho'}$$





$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j}; \quad \rho = \overline{\rho}(z) + \rho'$$

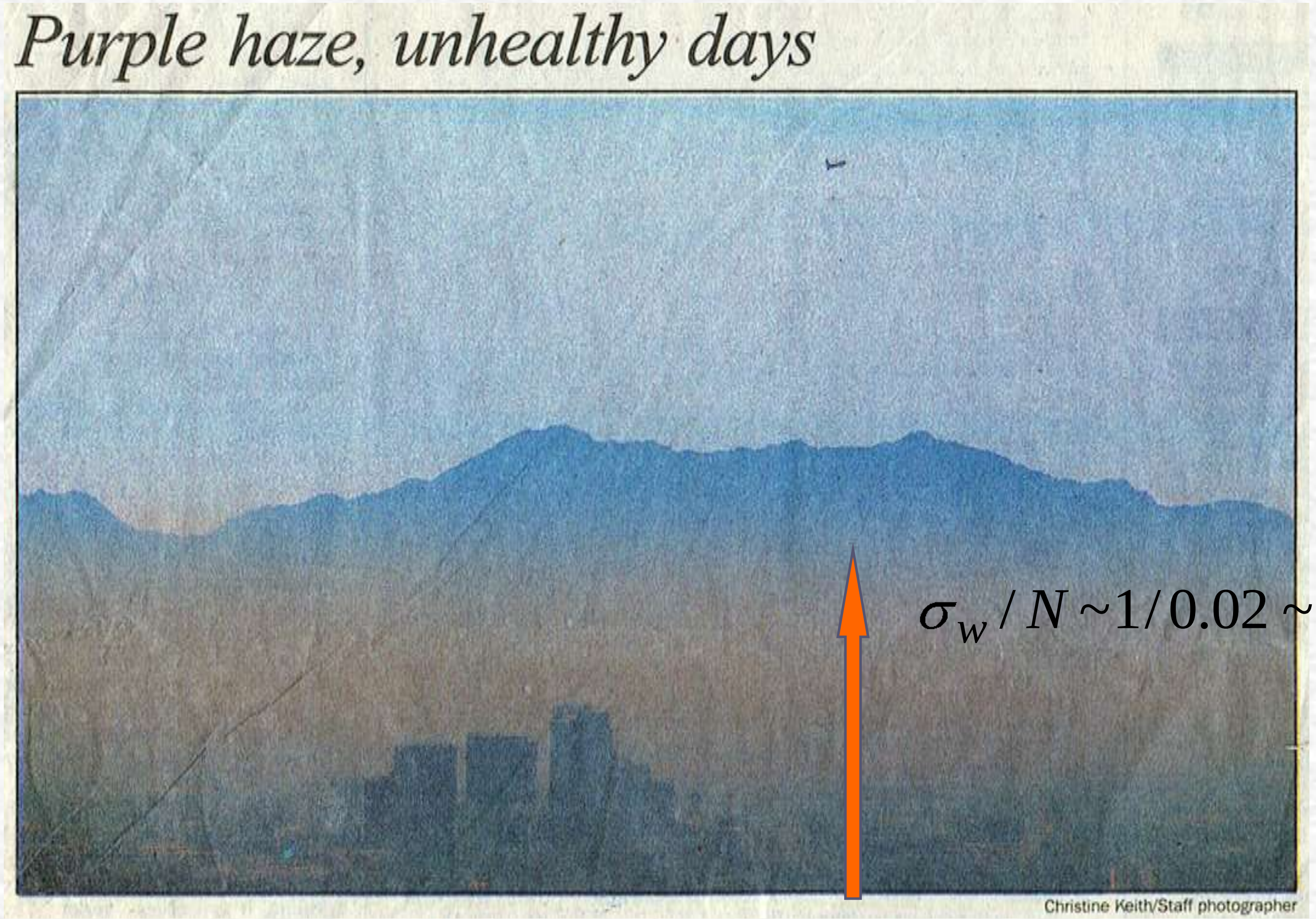
$$= \frac{\partial \rho'}{\partial t} + u_j \frac{\partial \rho'}{\partial x_j} + w \frac{\partial \overline{\rho}}{\partial z}$$

$$\frac{d\rho'}{dt}$$

$$\rho' = \Delta\rho - \left(\frac{\partial \overline{\rho}}{\partial Z}\right) \int_0^t w(t')\, dt'$$

$$\overline{w'\rho'} = -\frac{1}{2}\left(\frac{\partial \overline{\rho}}{\partial Z}\right)\frac{\partial \overline{z'^2}}{\partial t} + \overline{\Delta\rho w(t)}$$

STIR MIX



$$\sigma_w / N \sim 1/0.02 \sim 50\,m$$

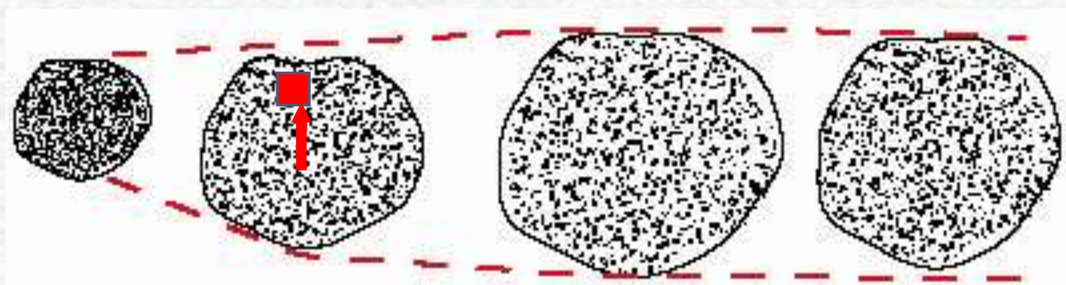
From the *Arizona Republic*

$$\overline{w'\rho'} = -\frac{1}{2}\left(\frac{\partial\bar{\rho}}{dZ}\right)\frac{d\overline{z'^2}}{dt} + \overline{\Delta\rho w'}$$

$$K = 2\sigma_w{}^2 T_L \quad \longleftarrow \quad \gamma^2\frac{\sigma_\omega{}^2}{N} \quad << \quad \gamma\frac{\sigma_\omega{}^2}{N}$$

Density changes contribute more!

i.e. Changes of mean concentrations have to be considered!

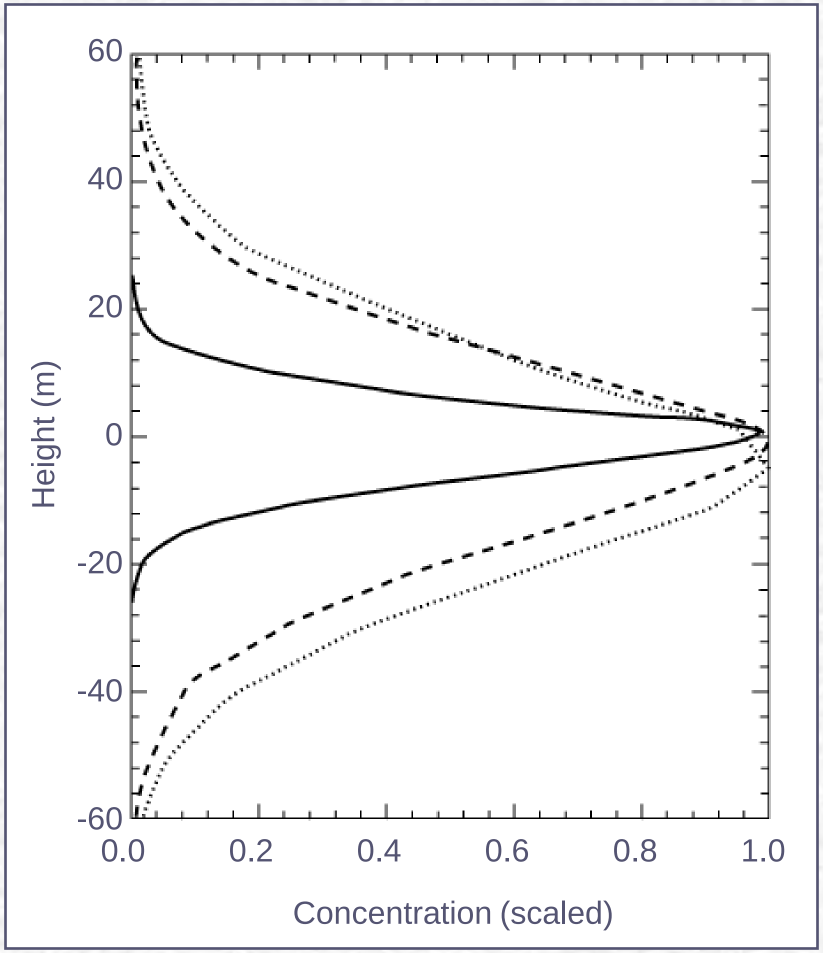


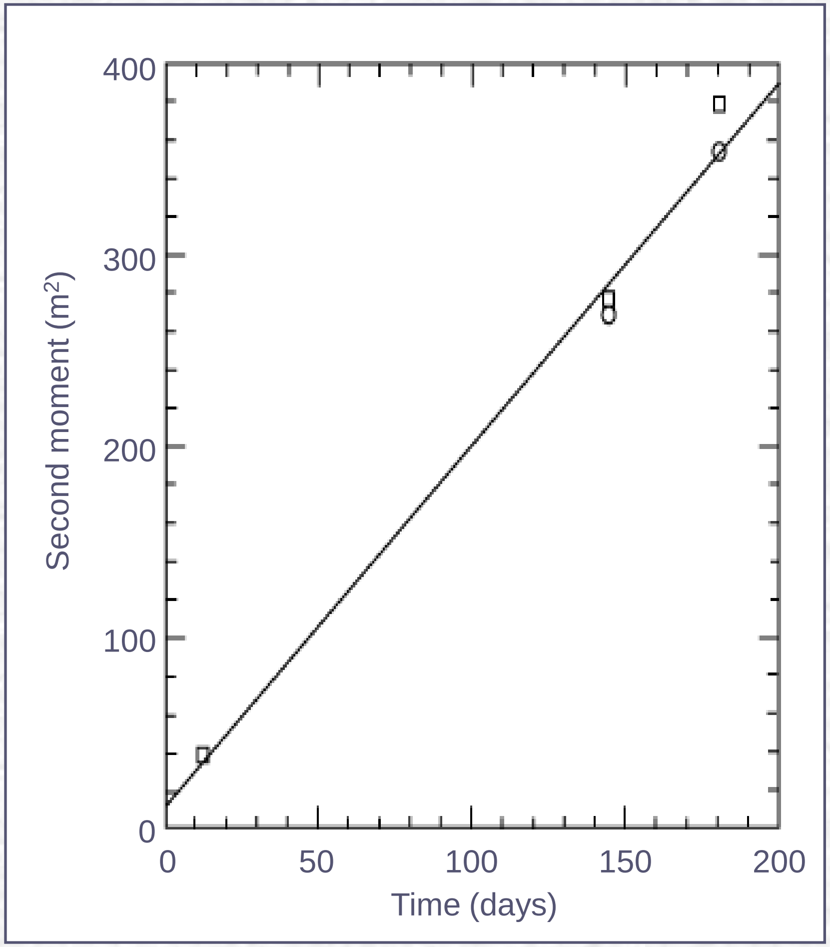
$$\sigma_Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Z - Z_s)^2\,\overline{C}\,dZdy \Bigg/ \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} C\,dZdy$$

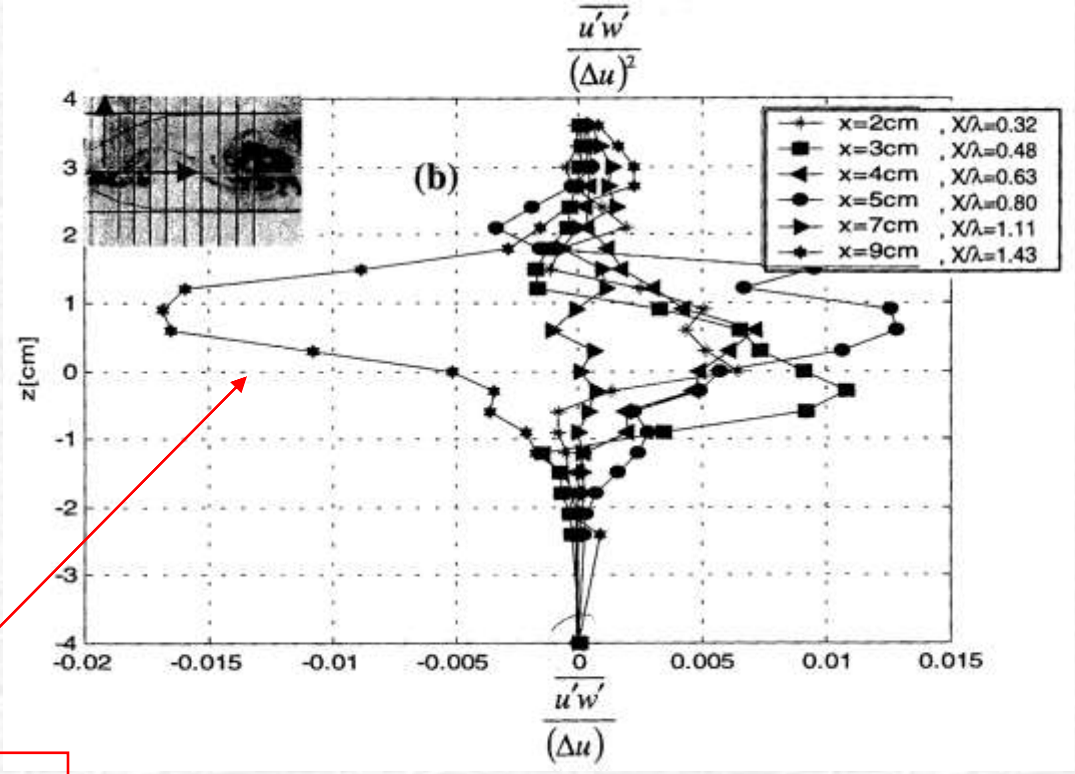
$$\frac{d\sigma_Z{}^2}{dt} = 2K_Z$$

Diffusion of a Substance

$$\frac{d\overline{y_s^2}}{dt} = 2k_c$$







$$-u'w'\frac{dU}{dz} < 0$$