Partial (Incomplete) Thermalization in Turbulence

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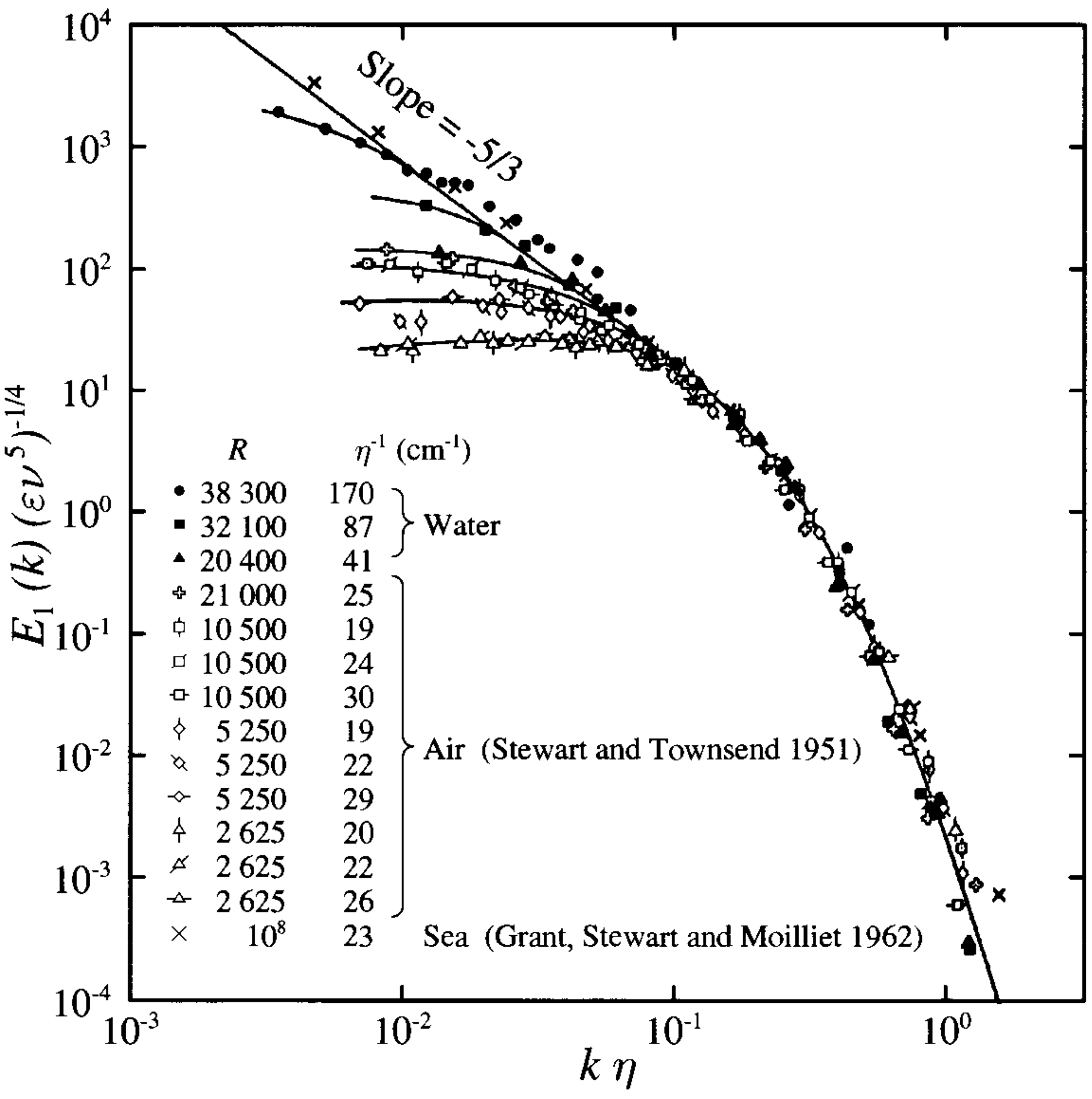
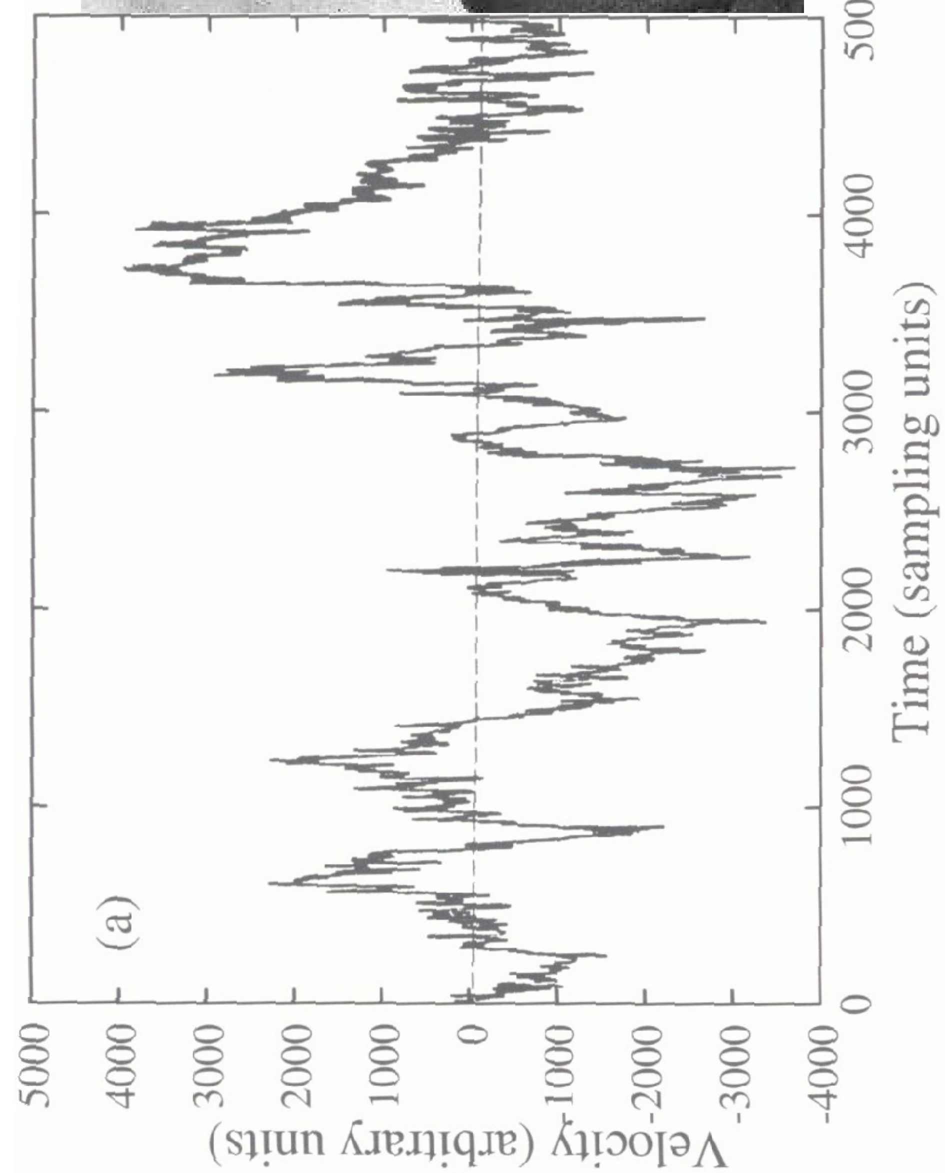
outline

- bottleneck phenomena;
- thermalization physics;
- mathematics behind;
- numerical techniques and some results.

bottleneck phenomena

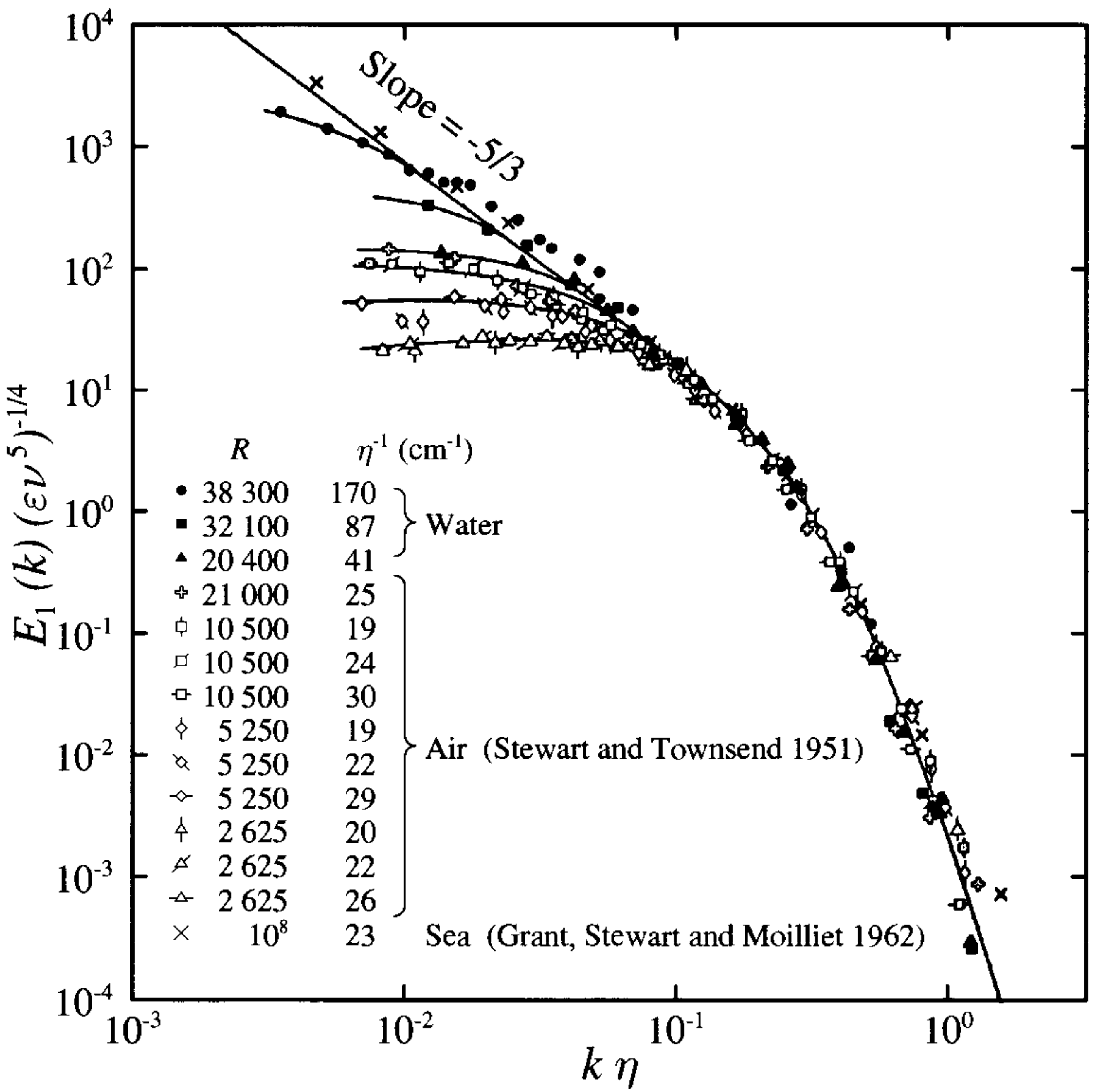


Kolmogorov's first universality assumption



(from Frisch's book CUP1995)

bottleneck phenomena



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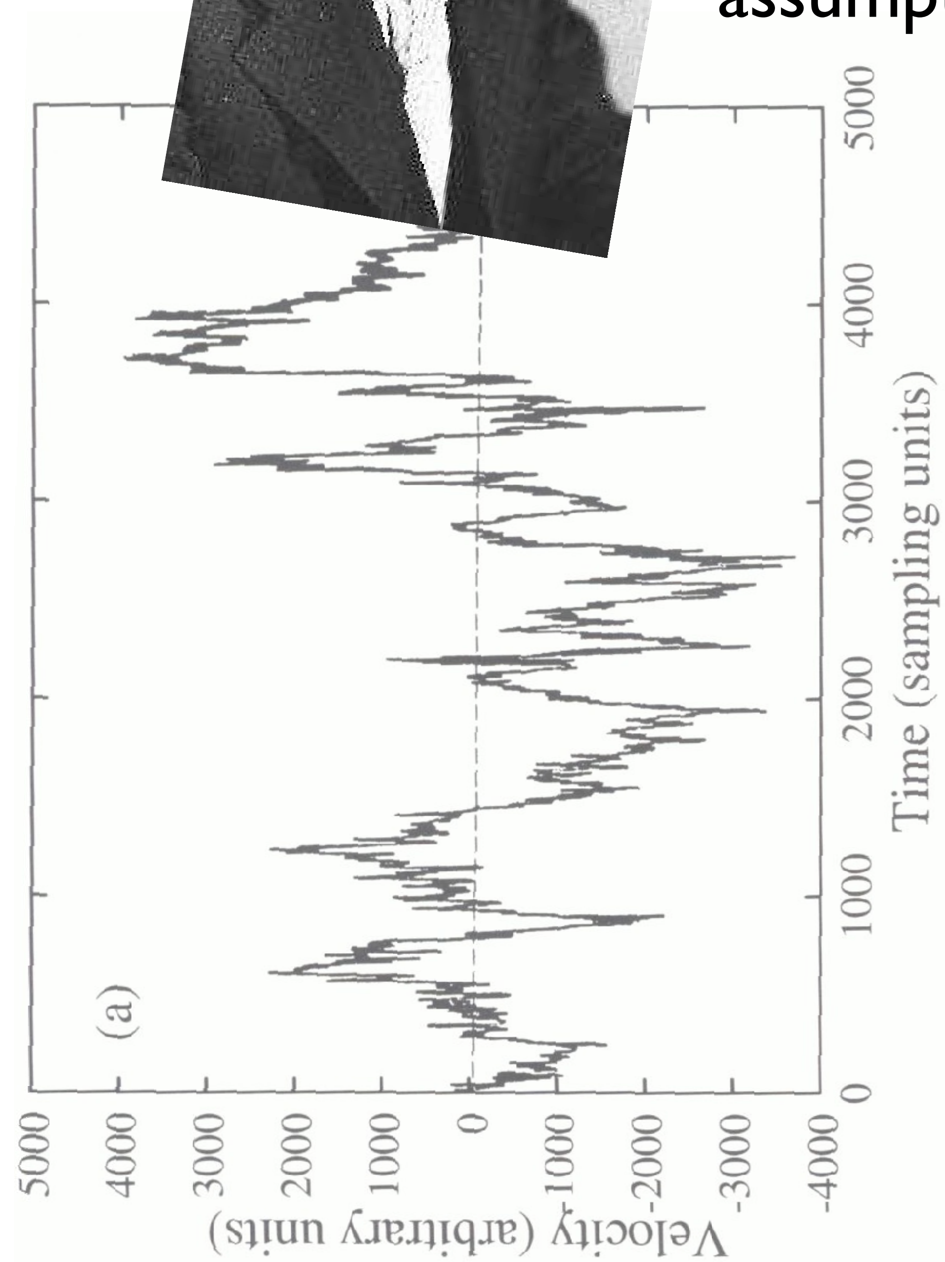

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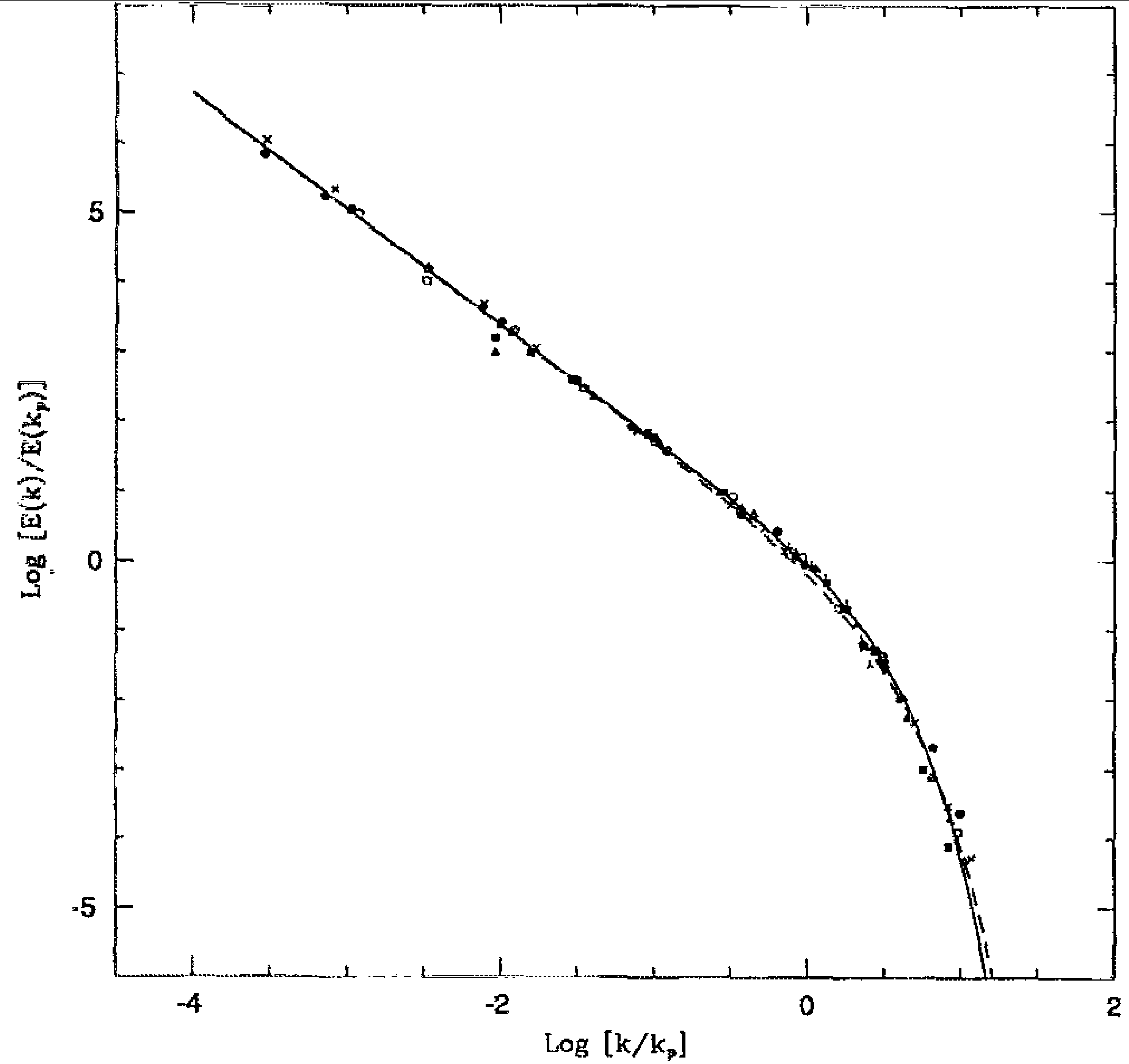
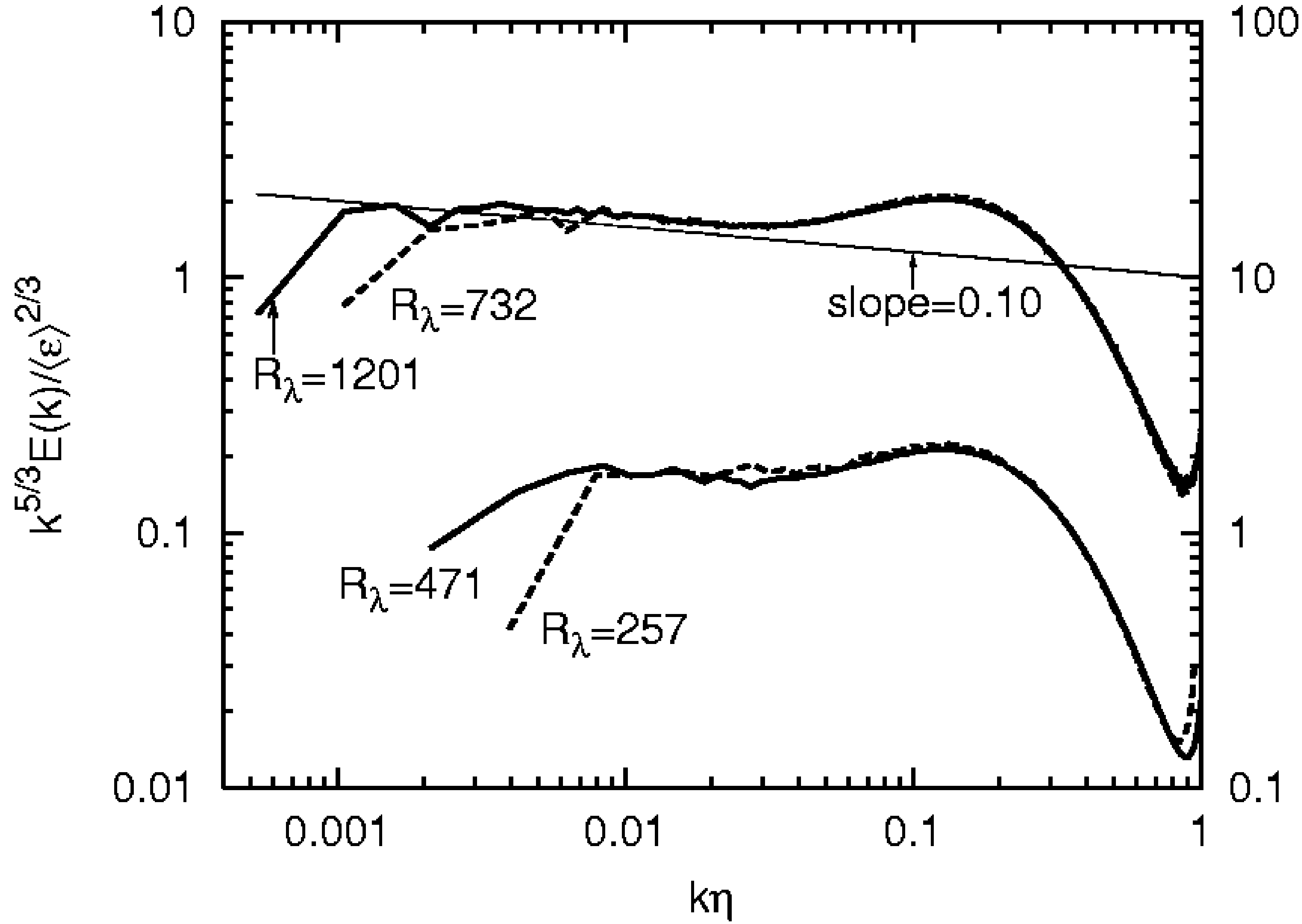


FIG. 1. Energy spectra from experimental measurements rescaled according to the maximum dissipation wave number k_p and its energy $E(k_p)$. The solid line is a fit by $E(k)=E(k_p)[(k/k_p)^{-5/3}+\alpha(k/k_p)^{-1}]\exp[-\mu(k/k_p)]$ (see the text). The dashed line is a fit with $\alpha=0$ (without the second power-law range).

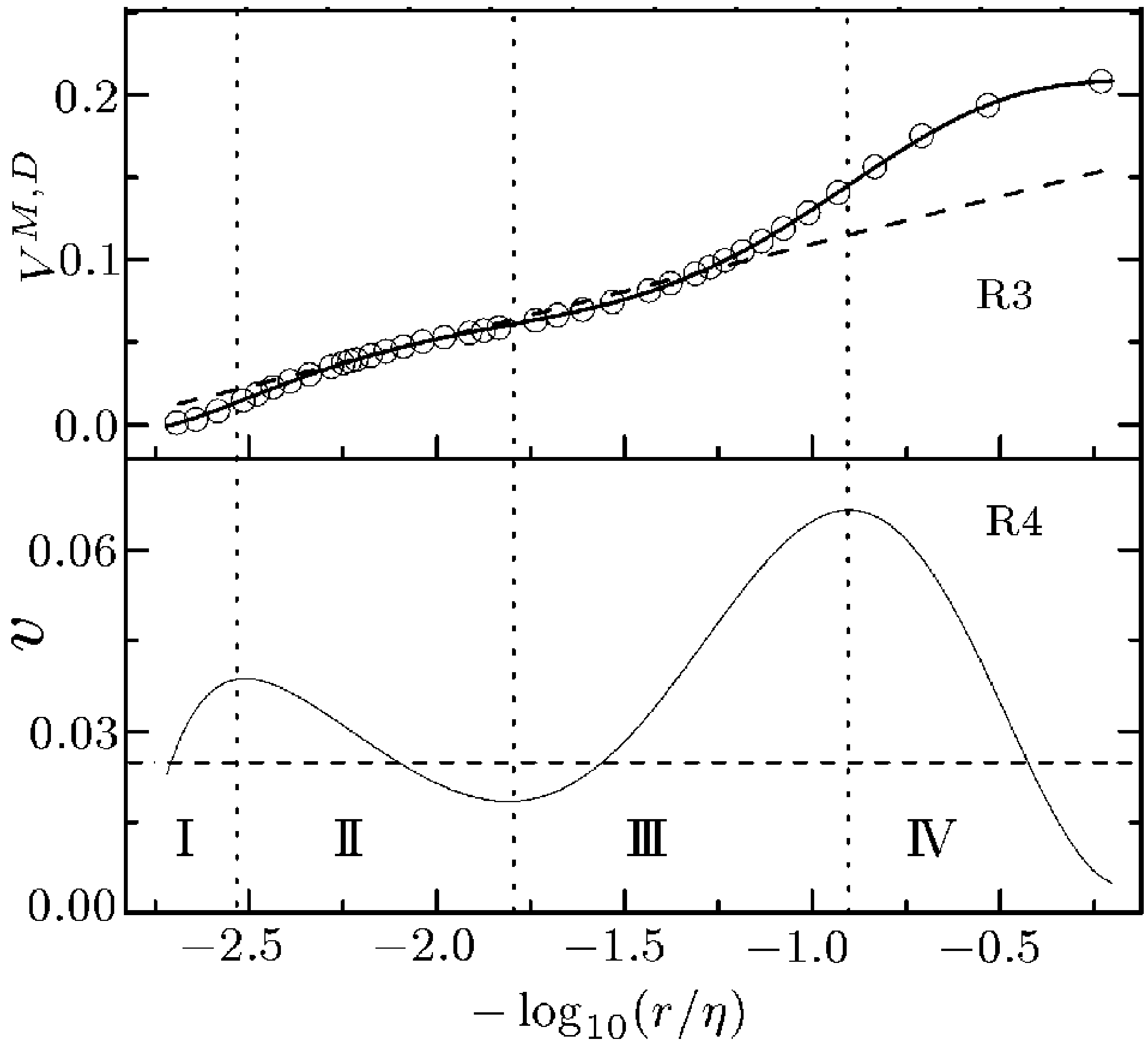
an example of fitting the energy spectrum with a function designating a "bottleneck":
Zhen-Su She and Eric Jackson PFA1993

experimental data (e.g., Saddoughi and Veeravalli JFM1994)
closures of Navier-Stokes (e.g., Andre and Lesieur JFM1973)
DNS data (e.g., She,Chen,Doolen,Kraichnan,Orszag PRL1993)
some quantitative theories (e.g., Falkovich PoF1994)



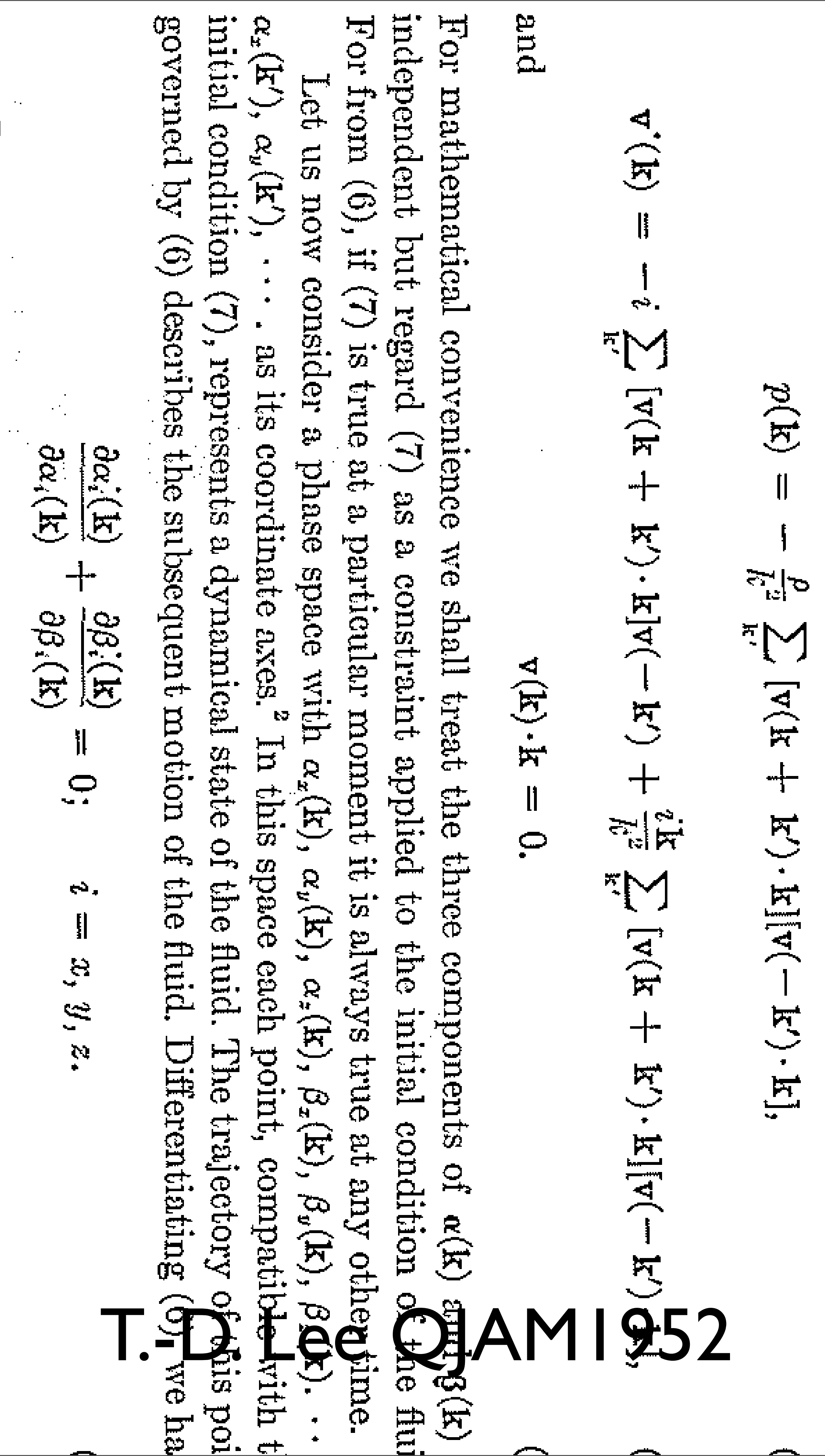
Kaneda et al. PoF2003: compensated energy spectrum

beyond energy: e.g., intermittency growth



JZZ ChinesePhysicsLetters2006

therr sics



$$p(\mathbf{k}) = -\frac{\rho}{k^2}\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}],$$

$$\dot{\mathbf{v}}(\mathbf{k}) = -i\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}') + \frac{i\mathbf{k}}{k^2}\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}],$$

and
$$\mathbf{v}(\mathbf{k})\cdot\mathbf{k} = 0.$$

For mathematical convenience we shall treat the three components of $\boldsymbol{\alpha}(\mathbf{k})$ and $\boldsymbol{\beta}(\mathbf{k})$ independent but regard (7) as a constraint applied to the initial condition of the flui For from (6), if (7) is true at a particular moment it is always true at any other time.

Let us now consider a phase space with $\alpha_x(\mathbf{k})$, $\alpha_y(\mathbf{k})$, $\alpha_z(\mathbf{k})$, $\beta_x(\mathbf{k})$, $\beta_y(\mathbf{k})$, $\beta_z(\mathbf{k})$. $\cdots$ $\alpha_x(\mathbf{k}')$, $\alpha_y(\mathbf{k}')$, $\cdots$. as its coordinate axes.[2] In this space each point, compatible with t initial condition (7), represents a dynamical state of the fluid. The trajectory of this poi governed by (6) describes the subsequent motion of the fluid. Differentiating (6), we ha

$$\frac{\partial\dot{\alpha}_i(\mathbf{k})}{\partial\alpha_i(\mathbf{k})} + \frac{\partial\dot{\beta}_i(\mathbf{k})}{\partial\beta_i(\mathbf{k})} = 0; \qquad i = x, y, z.$$

T.-D. Lee QJAM1952

thermalization physics

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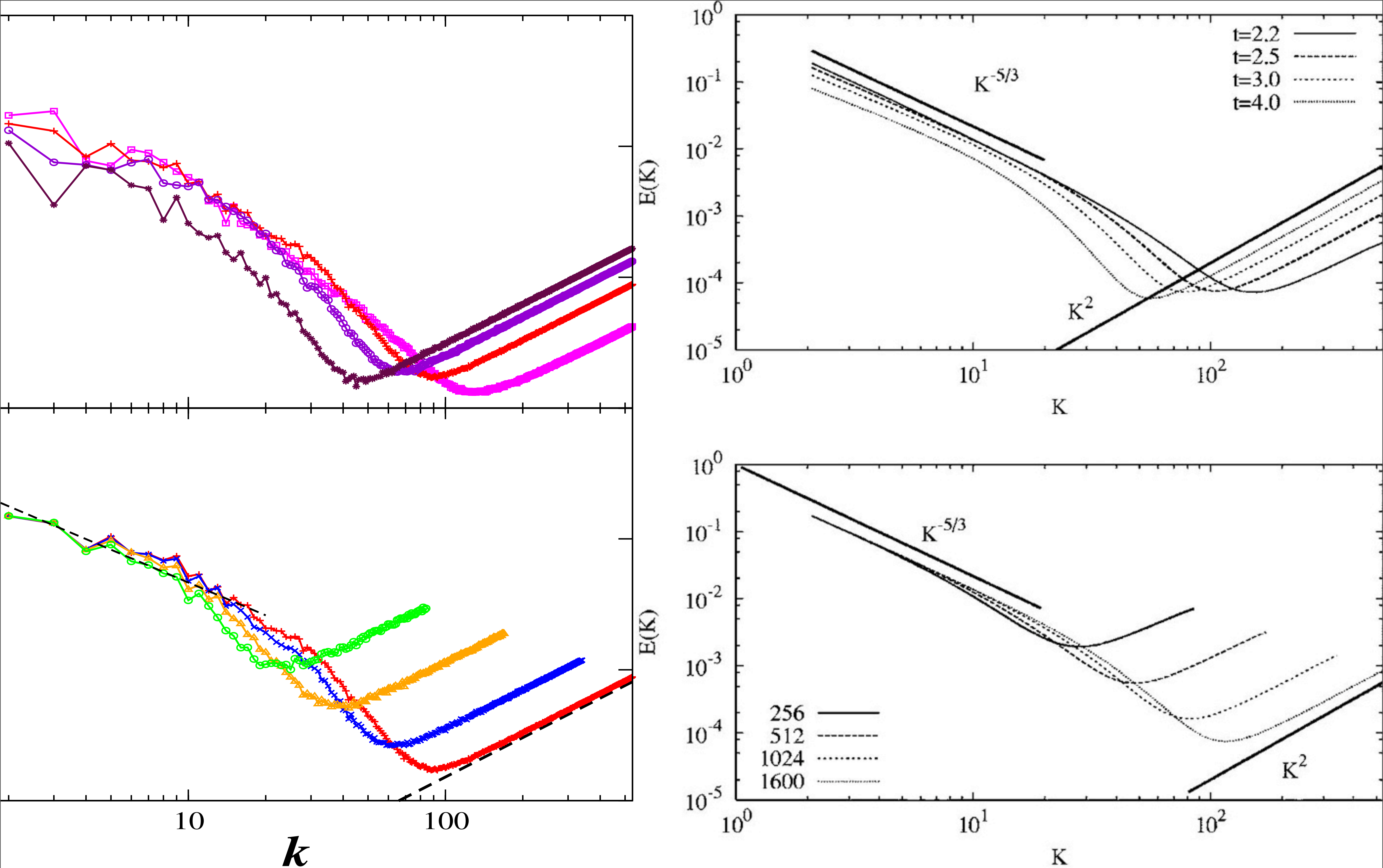
T.-D. Lee QJAM1952



DNS of Cichowlas et al. (PRL2005) "reproduced" by Bos and Bertoglio(PoF2006) with **EDQNM**

mathematics behind

$$\partial_t v + v \cdot \nabla v = -\nabla p - (k_G)^{-2\alpha}(-\nabla^2)^{\alpha} v, \; \nabla \cdot v{=}0$$

$k_G > 0, \quad \alpha = \text{dissipativity}.$ Here $\alpha > 1.$

Dissipation rate $(k/k_G)^{2\alpha} \to 0 \text{ or } \infty \text{ when } \alpha \to \infty$

Abstract form $\partial_t v = B(v, v) + L_\alpha v$

Galerkin truncation $\partial_t u = P_{k_{\mathrm{G}}} B(u, u), \qquad u_o = P_{k_{\mathrm{G}}} v_0$

Projector $P_{k_{\mathrm{G}}}$: low-pass filter at wavenumber k_{G}

ergodicity ... ...

numerical techniques

$$\dot{u} = cu + F(u,t)$$

- problem characteristics: high resolution and stiff
- we want the scheme to have good stability property (acceptable time step size), to be accurate (high order and with small coefficient in the error) and cheap (explicit) ... ...
- so, exact treatment of linear term ("ETD" and "IF": why not) and Runge-Kutta (contrast to multistep method: convenient, smaller error coefficient and larger stability region): ETDRK is much more accurate than IFRK and we finally choose ETD4RK by Cox and Matthews (JCP02)

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$$\begin{aligned}
a_n &= u_n e^{ch/2} + \left(e^{ch/2} - 1\right) F(u_n, t_n)/c,\\
b_n &= u_n e^{ch/2} + \left(e^{ch/2} - 1\right) F(a_n, t_n + h/2)/c,\\
c_n &= a_n e^{ch/2} + \left(e^{ch/2} - 1\right)(2F(b_n, t_n + h/2) - F(u_n, t_n))/c,\\
u_{n+1} &= u_n e^{ch} + \{F(u_n, t_n)[-4 - hc + e^{ch}(4 - 3hc + h^2c^2)]\\
&\quad + 2(F(a_n, t_n + h/2) + F(b_n, t_n + h/2))[2 + hc + e^{ch}(-2 + hc)]\\
&\quad + F(c_n, t_n + h)[-4 - 3hc - h^2c^2 + e^{ch}(4 - hc)]\}/h^2c^3.
\end{aligned}$$

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&\quad + F(c_n, t_n + h)[-4 - 3hc - h^2c^2 + e^{ch}(4 - hc)]\}/h^2c^3.
\end{aligned}$$

$$u \sim -\frac{F}{c} - \frac{1}{c^2}\frac{\mathrm{d}F}{\mathrm{d}t} - \frac{1}{c^3}\frac{\mathrm{d}^2F}{\mathrm{d}t^2} \cdots$$

"fast phase", "slow manifold"

"slaved scheme" (Frisch et al. JFM1986)

numerical results

$$\left(\frac{\partial}{\partial t} + 2\left(\frac{k}{k_G}\right)^{2\alpha} + \left(\frac{k}{k_T}\right)^{\infty}\right) E(k,t) = \iint_{\triangle_k} dpdq\theta_{kpq} b(k,p,q)\frac{k}{pq}E(q,t)\left[k^2E(p,t) - p^2E(k,t)\right]$$

with the third-order correlations relaxation time $\theta_{kpq} = \frac{1-e^{-[\mu_{kpq}+k_G^{-2\alpha}(k^{2\alpha}+p^{2\alpha}+q^{2\alpha})]t}}{\mu_{kpq}+k_G^{-2\alpha}(k^{2\alpha}+p^{2\alpha}+q^{2\alpha})}$,

$\mu_{kpq} = \mu_k + \mu_p + \mu_q$ and $\mu_k = a_1\left[\int_0^k p^2E(p,t)dp\right]^{\frac{1}{2}}$. $b(k,p,q) = \frac{p}{k}(xy + z^3)$, where x, y, and z are the cosines of the interior angles of the triangle facing, respectively, the sides k, p, and q.

"QN" --- Chou(1940), Millionshtchikov(1941): realizability problem

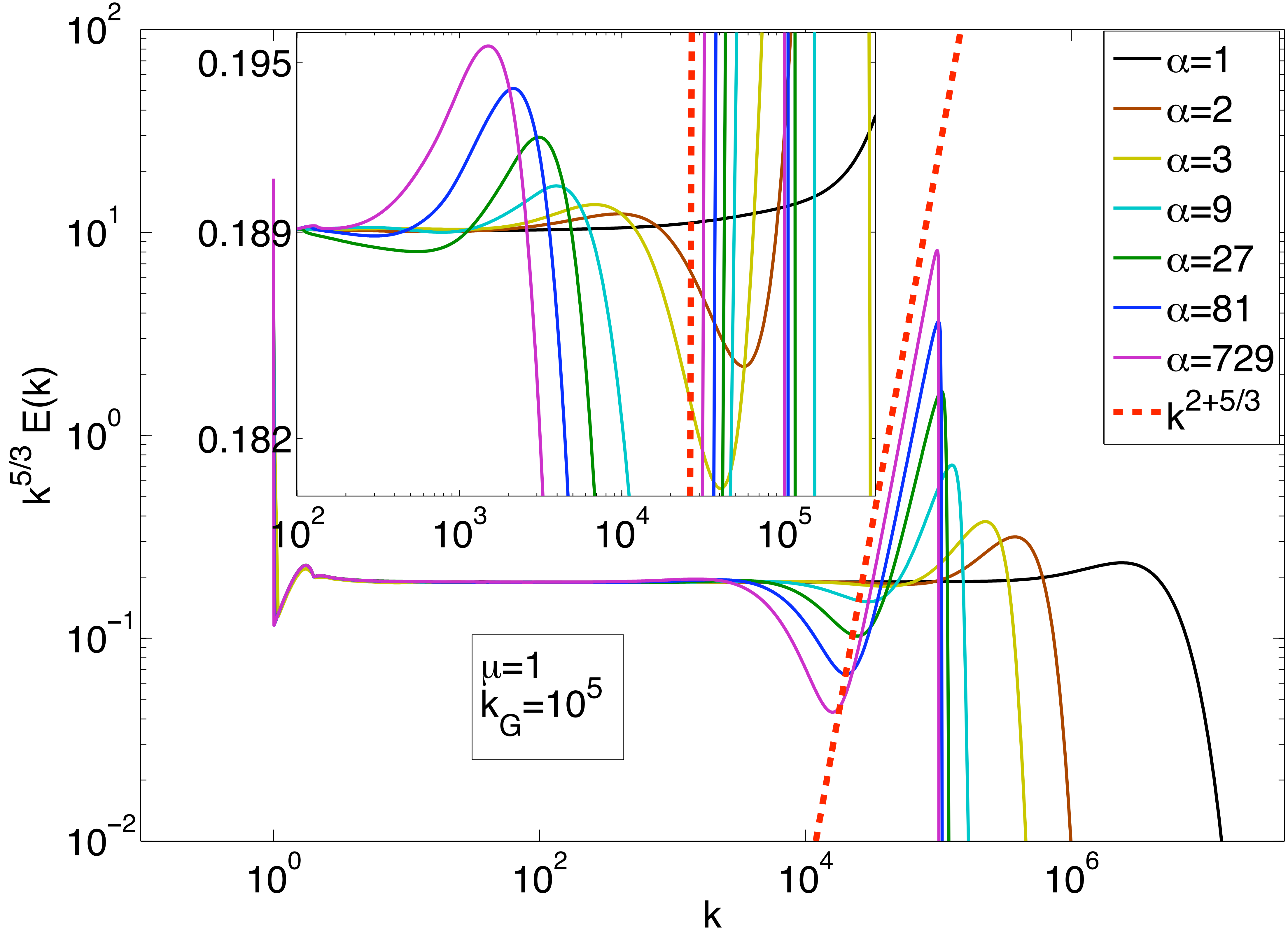
"N" --- Lee (1952), Hopf(1952): statistics of absolute equilibria of truncated Euler

DIA (Kraichnan): tractability problem

"ED","M" --- Orszag(1970, 1977)

Eddy-Damped Quasi-Normal Markovian

Galerkin method with "tophat" bases (energy exactly conserved)



hyperviscous EDQNM:
convergence to Galerkin-truncation and secondary bottleneck resulted from eddy viscosity

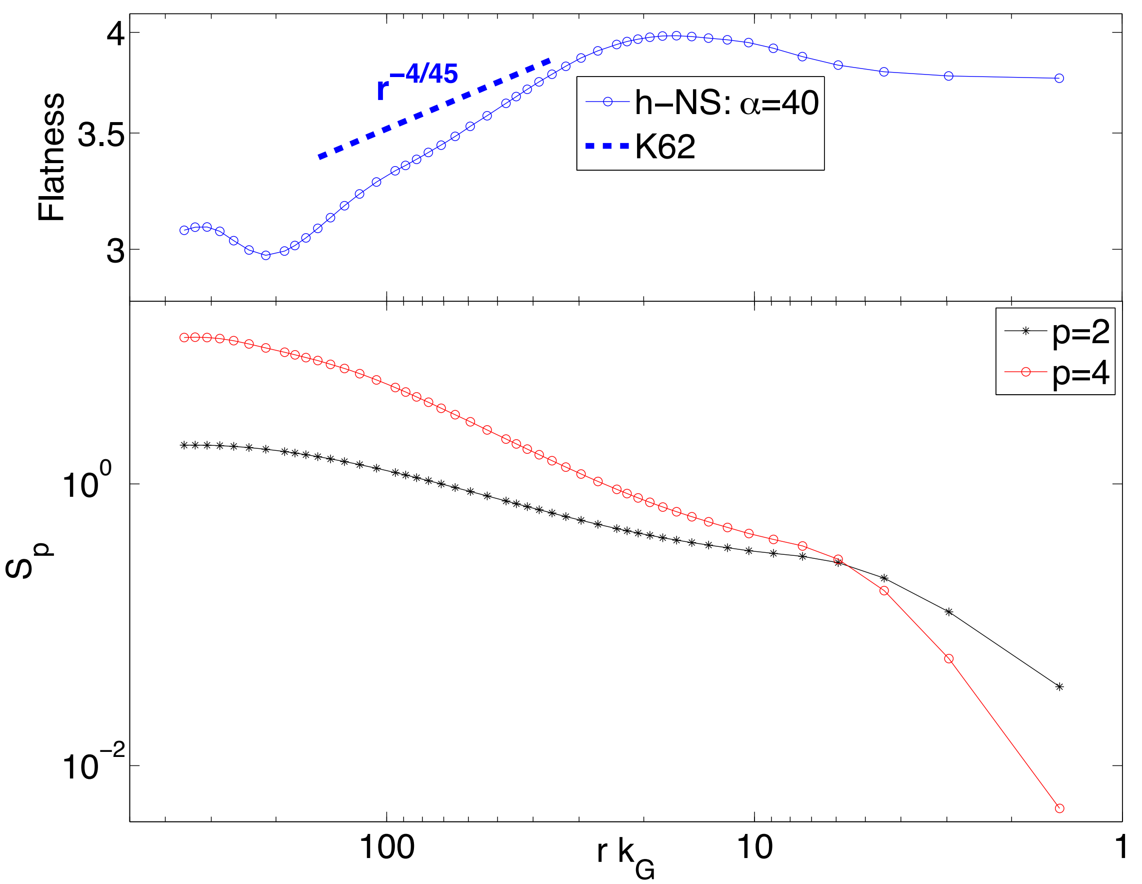


FIG. 1: Flatness and moments of filtered field $\tilde{v}(r)$: The Kolmogorov log-normal model (K62) [8] for intermittency growth is also shown for reference of the numerical hyperviscous Navier-Stokes (h-NS) result.

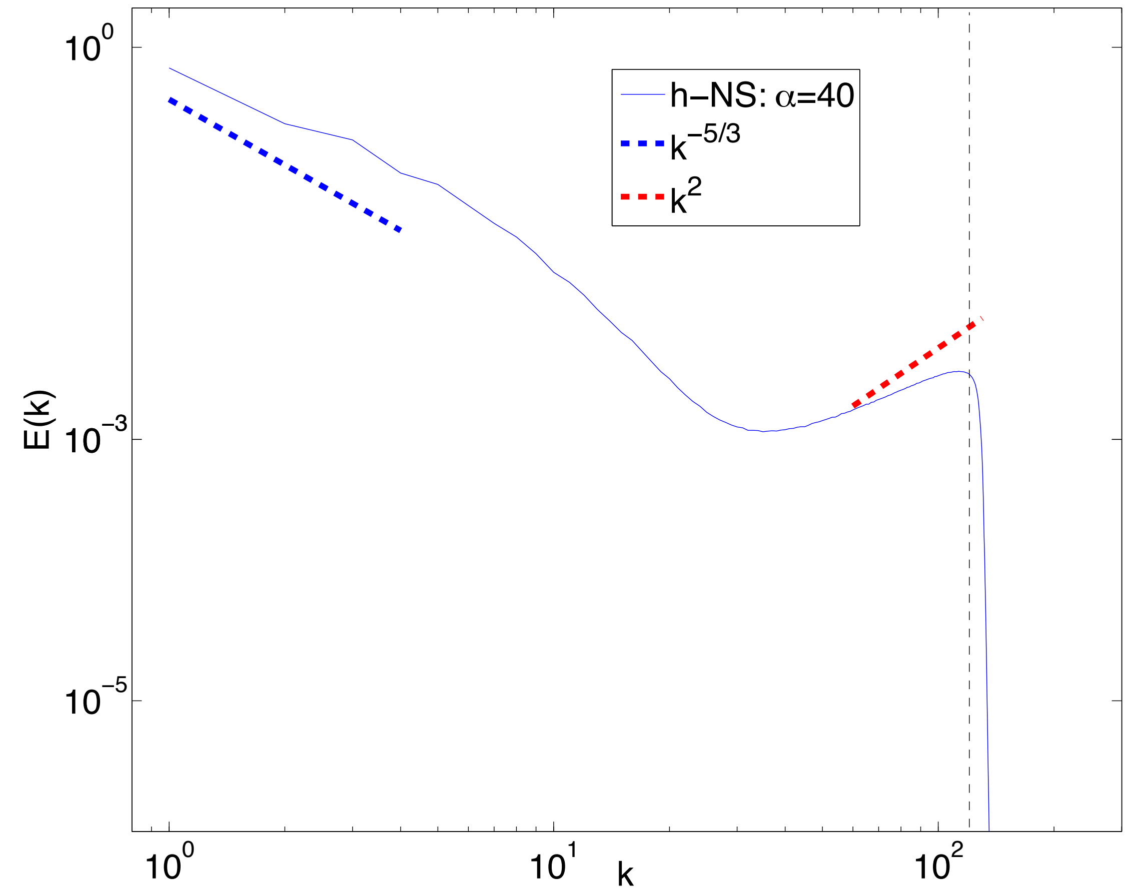
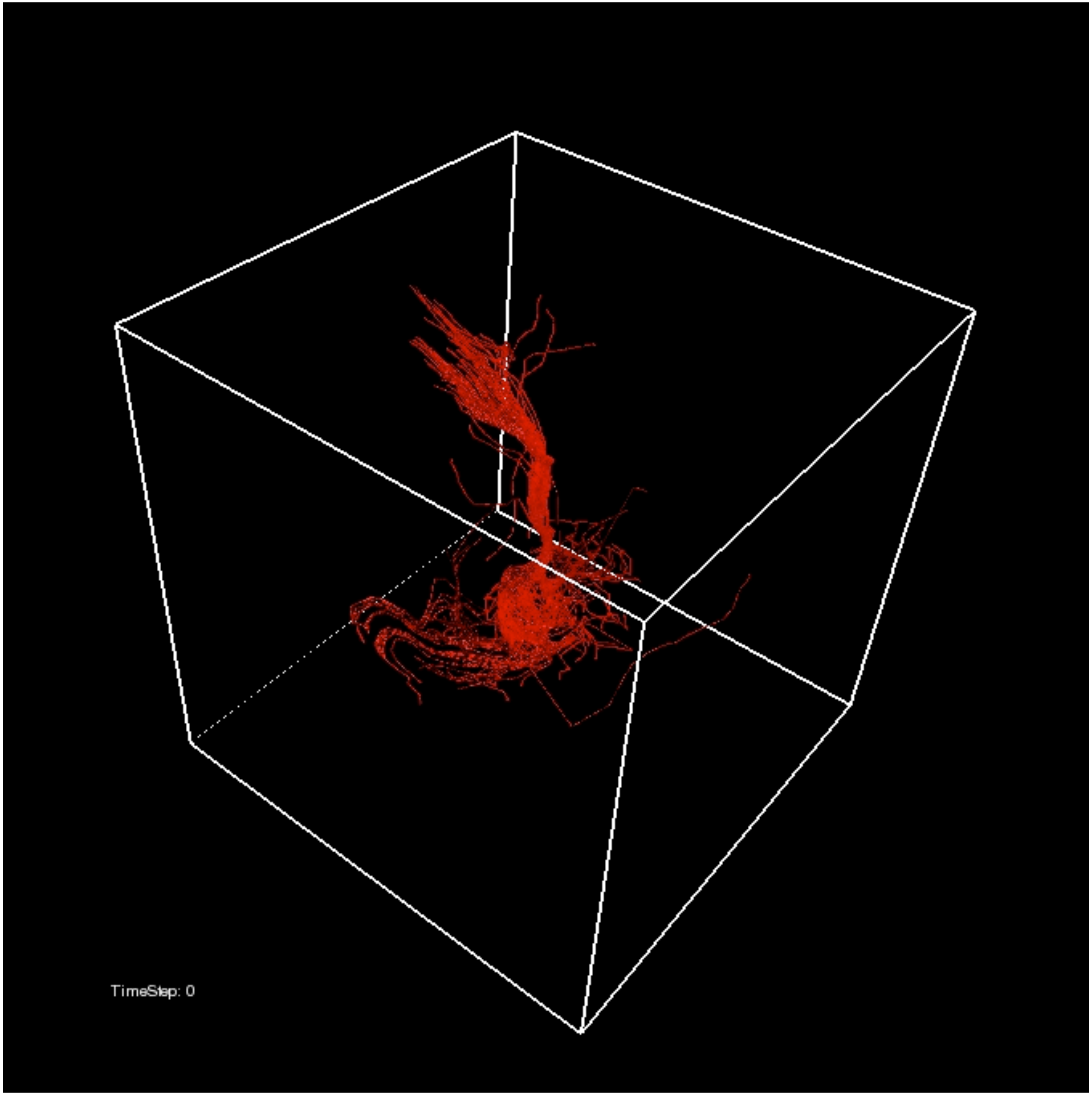


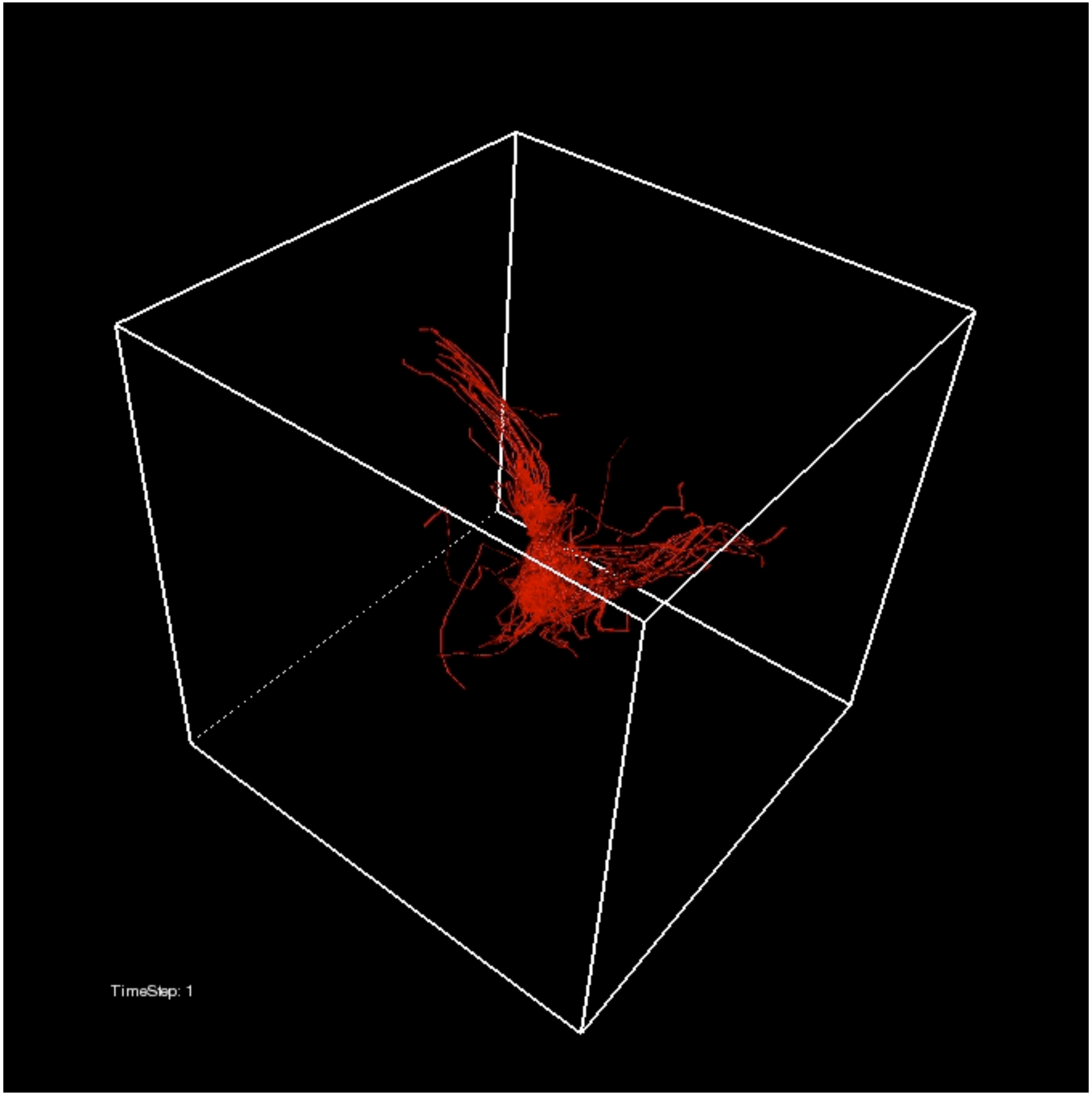
FIG. 2: Energy spectrum: The inertial scaling, say, the Kolmogorov $-5/3$ law, and the absolute equilibrium spectrum k^2 are also shown for reference; the vertical dashed line denotes the position of k_G.

$$\tilde{v}(r) = v_x(x + r, y, z) - v_x(x, y, z)$$

a direct numerical simulation of hyperviscous Navier-Stokes

a “cartoon” for thermalization




TimeStep: 1

conclusions and perspectives

- convergence to Galerkin truncation, partial thermalization;
- secondary bottleneck caused by eddy viscosity.

- dynamics/mechanics, flow structures ......
- more “rugged” invariants, depression of thermalization ......

conclusions and perspectives

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p^* + \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \boldsymbol{B}_0 \cdot \nabla \boldsymbol{b} + \nu \nabla^2 \boldsymbol{v},$$

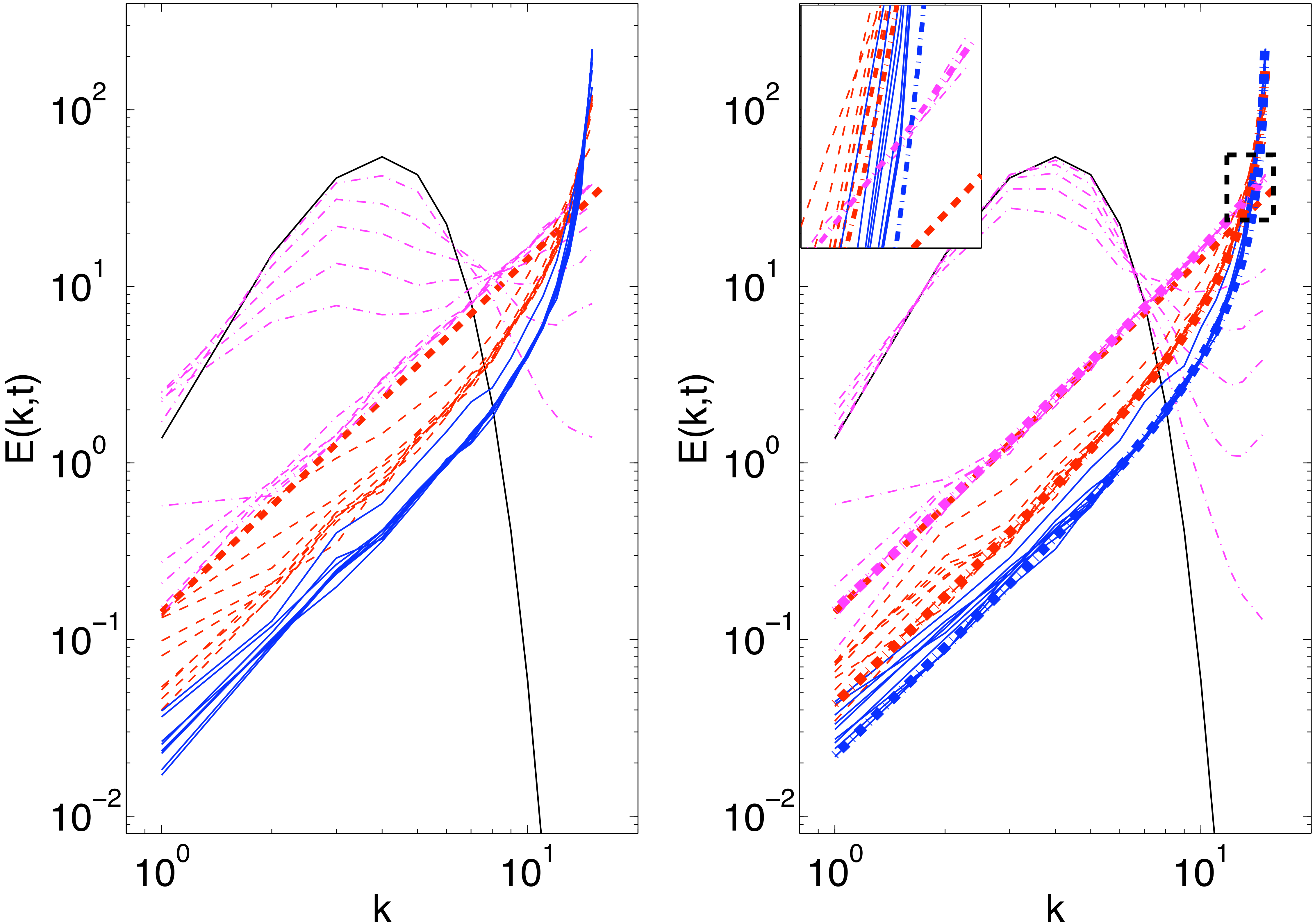
$$\frac{\partial \boldsymbol{b}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{b} = \boldsymbol{b} \cdot \nabla \boldsymbol{v} + \boldsymbol{B}_0 \cdot \nabla \boldsymbol{v} + \eta \nabla^2 \boldsymbol{b},$$

$$\nabla \cdot \boldsymbol{v} = 0,$$

$$\nabla \cdot \boldsymbol{b} = 0,$$

Thank You !

Comparison of Truncated 3D Euler Energy Spectra



absolute equilibria with helicity