Reduced Modeling of the Magnetorotational Instability

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Outline

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   - Hydrodynamics
   - MRI

2. Derivation of Reduced Model
   - Formulation
   - Shearing Sheet
   - Reduced Model

3. Modeling the Reduced Equations
   - Large Elsasser Regime
   - $O(1)$ Elsasser Regime
Accretion is a Fundamental Astrophysical Process

Accretion is the process by which a massive object collects surrounding matter by gravitation. Accretion disks are observed in many astrophysical processes (binary star systems, center of galaxies).

Keplerian balance in mom. eqn.

\[ \Omega^2 = \frac{GM}{r^3} \]

Angular momentum transport needed!
Geometry

- Astrophysical Disks: Differential Rotation
  - Keplerian balance: $ \Omega^2 = \frac{GM}{r^3}$
- Laboratory: Taylor-Couette Geometry
  - Imposed $\Omega(r_1), \Omega(r_2)$
- Shearing Sheet - Simplified Geometry
Keplerian Disks are Hydrodynamically Stable

- Linear stability for the astrophysical regime
  - Rayleigh stable $\frac{d}{dr} (r^2 \Omega) < 0$
- Finite amplitude disturbance can cause instability
- Turbulent mixing - Does it produce the needed angular momentum transport?
  - $\nu_{turb}$ large ("eddy") viscosity
Magnetorotational Instability

Shear instability in the presence of magnetic fields

- First discovered by Velikhov (1959) and Chandrasekhar (1960)
- Rediscovered by Balbus and Hawley (1991)
  - MRI generates the level of angular momentum transfer needed
- Operates in the Rayleigh stable regime $\frac{d}{dr}(r^2 \Omega) < 0$
- Axisymmetric
- Linear instability
Magnetorotational Instability

- Balbus & Hawley’s original simulation of the MRI
  - Contour plot of angular momentum perturbations \((r, z)\)
- Transfer of angular momentum
- Large vertical gradient of the “fingering instability” motivates scalings
Previous Work

- Balbus and Hawley (1991)
  - Efficient transfer of angular momentum
  - Didn’t include dissipative processes - viscous or ohmic
  - No Saturation

- Sano et. al. (1998)
  - First to show saturation of this instability
  - Compressible flow
  - Resistivity and viscosity included
  - Critical parameter, Elsasser number $\Lambda = v_A^2/\Omega\eta$
    - $\Lambda > 1$ Saturation
    - $\Lambda < 1$ No saturation

- Goodman and Xu (1994)
  - Nonaxisymmetric perturbations can saturate

- Lathrop group
  - First experimental observation of MRI (Sisan et. al. 2004)
Full Equations

Conservation of Momentum

\[
\frac{Du}{Dt} + 2\Omega \times u = -\frac{1}{\rho} \nabla \rho - \frac{1}{2\mu_0\rho} \nabla |B|^2 + \frac{1}{\mu_0\rho} B \cdot \nabla B + \nu \nabla^2 u,
\]

Induction Equation

\[
\frac{DB}{Dt} = B \cdot \nabla u + \eta \nabla^2 B
\]

Incompressibility, Solenoidal Condition

\[
\nabla \cdot u = 0, \nabla \cdot B = 0
\]
Reduced Modeling

Astrophysical Regime - Large Parameters

- $\nu \ll 1, \eta \ll 1$
  - $R_e = \frac{UL}{\nu}, \ R_m = \frac{UL}{\eta} \gg 1$

- Several different time scales:
  - Rotational time scale $\Omega^{-1}$
  - $L/\nu_A^2$
  - Diffusive time scale $L^2/\nu, L^2/\eta$
  - $\Omega^{-1} \ll L/\nu_A \ll L^2/\nu, L^2/\eta$

- To reach saturation we must integrate far in time
Shearing Sheet

- Shearing sheet approximation at $r^*$ with local angular velocity $\Omega^*(r^*)\hat{z}$
- Local Cartesian coordinates $(r, \phi, z) \sim (x, y, z)$
- Straight channel:
  $-L^*/2 \leq x^* \leq L^*/2, -\infty < y^* < \infty, -\infty < z^* < \infty$
- Linear Shear: $U_0^* = (0, \sigma^* x^*, 0)$
- Constant Background Magnetic field: $B_0^* = (0, B_{tor}^*, B_{pol}^*)$
Formulation of the Model Problem

- Non-dimensionalize
- \( \nu_A = \frac{B_{pol}^*}{\sqrt{\mu_0 \rho^*}} U^* \), \( \Omega, \nu, \eta \) are the dimensionless Alfven speed, rotation rate, kinematic viscosity and ohmic diffusivity
- \( \nabla \cdot u = 0 \), \( \nabla \cdot b = 0 \), axisymmetry allow the use of a stream function and flux function
  - \( u = (-\psi_z, \nu, \psi_x), \ b = (-\phi_z, b, \phi_x) \)
Nondimensionalized Equations

Axisymmetric perturbations of the form
\( u = (u, v, w) = (-\psi_z, v, \psi_x), \quad b = (-\phi_z, b, \phi_x) \) give the following equations

\[
\begin{align*}
\nabla^2 \psi_t + 2\Omega v_z + J(\psi, \nabla^2 \psi) &= v_A^2 \nabla^2 \phi_z + v_A^2 J(\phi, \nabla^2 \phi) + \nu \nabla^4 \psi, \\
v_t - (2\Omega + \sigma) \psi_z + J(\psi, v) &= v_A^2 b_z + v_A^2 J(\phi, b) + \nu \nabla^2 v, \\
\phi_t + J(\psi, \phi) &= \psi_z + \eta \nabla^2 \phi, \\
b_t + J(\psi, b) &= v_z - \sigma \phi_z + J(\phi, v) + \eta \nabla^2 b
\end{align*}
\]

Here, \( J(f, g) = f_x g_z - f_z g_x \).
Scaling Assumptions

- Traditional approach to nonlinear saturation: weakly nonlinear theory with \((\Lambda - \Lambda_c)/\Lambda_c \ll 1\).
  
  - Our approach: strongly nonlinear theory

- Shear is the dominant source of the energy for the MRI
  
  - rapid rotation, strong shear: \((\Omega, \sigma) = \delta^{-1}(\hat{\Omega}, \hat{\sigma})\)

- MRI itself requires the presence of a (weaker) vertical magnetic field
  
  - magnetic field: \(v_A \sim 1, \, v_A^* \equiv B_{pol}^*/\sqrt{\mu_0 \rho^* U^*}\)

- Dissipative effects are weaker still but cannot be ignored since they are ultimately responsible for the saturation of the instability
  
  - weak dissipative processes: \((\nu, \eta) = \epsilon(\hat{\nu}, \hat{\eta})\)

- Take \(\epsilon \sim \delta \ll 1, \, \Lambda = \mathcal{O}(1)\) (Case A) or \(\epsilon \ll \delta \ll 1, \Lambda \gg 1\) (Case B)
  
  - \(Rm = |\sigma^*|L^2*/\eta^*, \, Pm = \nu^*/\eta^*, \, S = v_A^*L^*/\eta^*\)
  
  - so \(Rm \gg S \gg \max(1, Pm)\), while \(\Lambda = \mathcal{O}(1)\)
Multiple Scales Expansion

- Motivated by Balbus & Hawley
- Large wavenumber in $z$ - large variation
  - $\partial_z = \epsilon^{-1} \partial_z$
- Have large gradients in $x$ direction
- Fast dynamic time scale, slow evolution to a statistically steady state
  - $\partial_t = \epsilon^{-1} \partial_t + \partial_T$
- Set $\epsilon$ by considering the size of our domain
  - $L_z$ wavelength of fastest growing mode - Linear Dispersion
  - $L_X \sim \epsilon^{-1} L_z$
Asymptotic Expansion

- To solve the scaled equations we expand every variable

\[ \psi(x, X, z, t, T) = \sum_{i,j} \epsilon^{i/2} \delta^{j/2} \psi_{ij}(x, X, z, t, T) + \ldots, \]

- Deduction: Leading order azimuthal fields \( \nu_{00}, b_{00} \) represent large-scale adjustment to background shear and toroidal field due to MRI

- Separate all variables into their mean and fluctuating components
  
  \[ \psi_{ij}(x, X, z, t, T) = \overline{\psi}_{ij}(X, T) + \psi'_{ij}(x, X, z, t, T) \]
  
  \[ \overline{\psi}_{ij}(X, T) \equiv \lim_{\tau, \nu \to \infty} \frac{1}{\tau \nu} \int_{\tau, \nu} \psi_{ij}(x, X, z, t, T) dx dz dt \]

- We now collect terms at each order in the evolution equations
Reduced Fluctuating Equations

At $O(\epsilon^{-\frac{1}{2}} \delta^{-\frac{1}{2}})$

$$
\nabla^2 \psi_{00t} + 2\hat{\Omega} \nu_{11z} + \left( \frac{\epsilon}{\delta} \right)^{\frac{1}{2}} J_x(\psi'_{00}, \nabla^2 \psi_{00}) = v_A^2 \nabla^2 \phi_{00z} + \nabla^4 \psi_{00},$$

$$
\nu_{11t} - \left( 2\hat{\Omega} + \hat{\sigma} + \partial_x \bar{v}_{00} \right) \partial_z \psi_{00} + \left( \frac{\epsilon}{\delta} \right)^{\frac{1}{2}} J_x(\psi'_{00}, \nu'_{00}) = \nabla^2 \nu_{11},$$

$$
\phi_{00t} + \left( \frac{\epsilon}{\delta} \right)^{\frac{1}{2}} J_x(\psi'_{00}, \phi'_{00}) = \psi_{00z} + \left( \frac{\epsilon}{\delta} \right) \nabla^2 \phi_{00},$$

$$
b_{11t} - \partial_x \bar{b}_{00} \psi_{00z} + \left( \frac{\epsilon}{\delta} \right)^{\frac{1}{2}} J_x(\psi'_{00}, b'_{11}) = v_{11} - (\hat{\sigma} + \partial_x \bar{v}_{00}) \phi_{00z} + \nabla^2 b_{11}.$$

Ben Jamroz  Reduced Modeling of the MRI
Feedback onto the Shear

- For both sets of scalings ($\epsilon \ll \delta$ & $\epsilon \sim \delta$) there is feedback onto the imposed shear
- Readjustment of the local background state
  - $(\hat{\sigma} + \partial_X \mathbf{v}_{00})$ terms
  - Tends towards solid body rotation, $|\hat{\sigma} + \partial_X \mathbf{v}_{00}| < |\hat{\sigma}|$
  - Angular momentum must have been transported
- At $O(\epsilon \delta)$ we close this system and get relations for $\partial_X \mathbf{v}_{00}$ and $\partial_X \mathbf{b}_{00}$

\[
\hat{\nu} \partial_X \mathbf{v}_{00} = -\psi_{00z} \nu_{11} + \nu_A^2 \phi_{00z} b_{11}
\]
\[
\hat{\eta} \partial_X \mathbf{b}_{00} = -\psi_{00z} b_{11} + \phi_{00z} \nu_{11}
\]

- $\partial_X \mathbf{b}_{00}$ is the $O(1)$ correction to the background toroidal field $B_{tor}$
Single Mode Theory

  - Solutions
  
$$\psi'_{00} = e^{\lambda t} \cos(nz)\hat{\psi}'_{00}, \quad \nu'_{11} = e^{\lambda t} \sin(nz)\hat{\nu}'_{11}, \quad \phi_{00} = e^{\lambda t} \sin(nz)\hat{\phi}'_{00}, \quad b_{11} = e^{\lambda t} \cos(nz)\hat{b}_{11}\n$$

- No variation in $x$
- Nonlinear ODE → evolves to saturation
- Does not work for a combination of such modes
Modeling the Reduced Equations

Numerical Method

- **Time integration**: Runge-Kutta scheme (Spalart et. al. 1991)
  - diffusion terms treated implicitly
  - all other explicitly

- **Spectral**
  - Periodic boundary conditions in $z$
  - Rigid (no slip), Stress Free, or Periodic in $x$
Case B: $\epsilon \ll \delta \ll 1, \Lambda \gg 1$

- Nonlinear and dissipative terms are subdominant
- Unbounded algebraic growth at leading order
- Saturation of $\bar{v}_{00}$
  - Decaying and growing terms $\rightarrow$ steady product
- No quadratic nonlinearities - energy transferred through modification of the dispersion relation
Case A: $\epsilon \sim \delta$, $\Lambda = \mathcal{O}(1)$

Single Mode results

- Good agreement with linear theory for a robust set of boundary conditions
- In particular the saturated value of $\partial_X \mathbf{v}_{00}$ matches the theory
- Can be of use in a parameterization model
Case A: Channel Initial Condition

- $\phi'_{00}$ Gaussian in $x$, vertical magnetic field lines
- Small random perturbation in $\psi'_{00}$
- $\nu'_{11} = b'_{11} = 0$
- Critical wavenumber grows
- Saturated state takes up the largest allowed scale in $z$ (coarsening)
Coarsening

- Begin from a random state with a fixed box length in $z$
- Fastest growing linear mode dominates early evolution
- Flow coarsens to fill computational domain
- The saturated value of $\partial_x \bar{v}_{00}$ matches the single mode theory for a mode with corresponding vertical wavenumber
  - Suggest using a parameterization model for large scale simulations
- Comparison with Case B
Summary & Future Work

- Derived reduced asymptotic models for the saturation of the MRI
- Found a back reaction on the imposed shear which allows for saturation
- Numerical results

Future Work

- Parameterization model
- Non-axisymmetric
  - Non-axisymmetric saturation
  - Dynamo → saturation