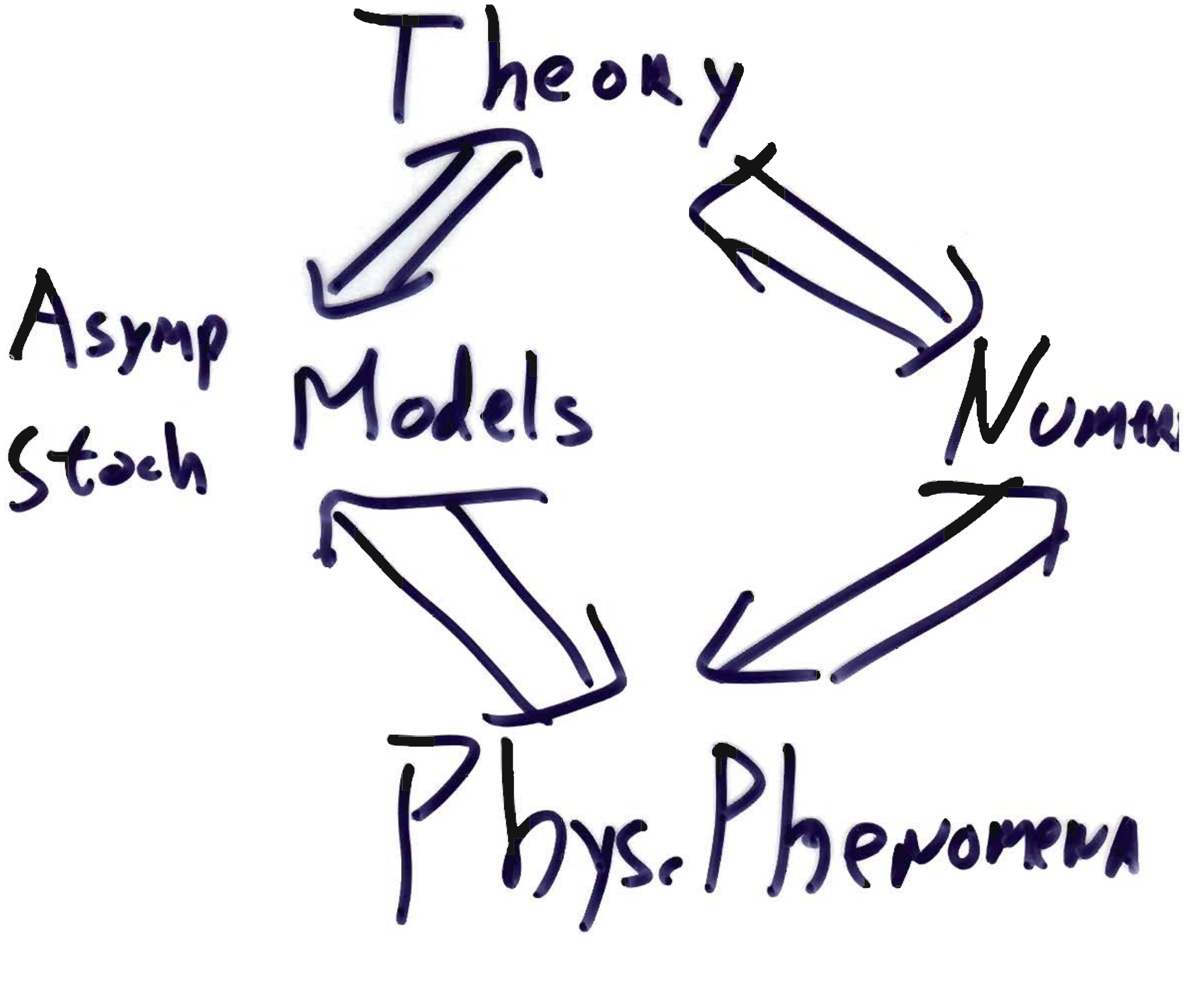
Introductory Lecture for Class

From "Waves, PDE's & Coarse-Grained Stochastic Models for Tropics"

Modern Appl. Math.



Applied and Theoretical Challenges for Multi-scale Hyperbolic PDE's in the Tropics

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Hierarchy of Organized Deep Convection over the Tropical Western Pacific

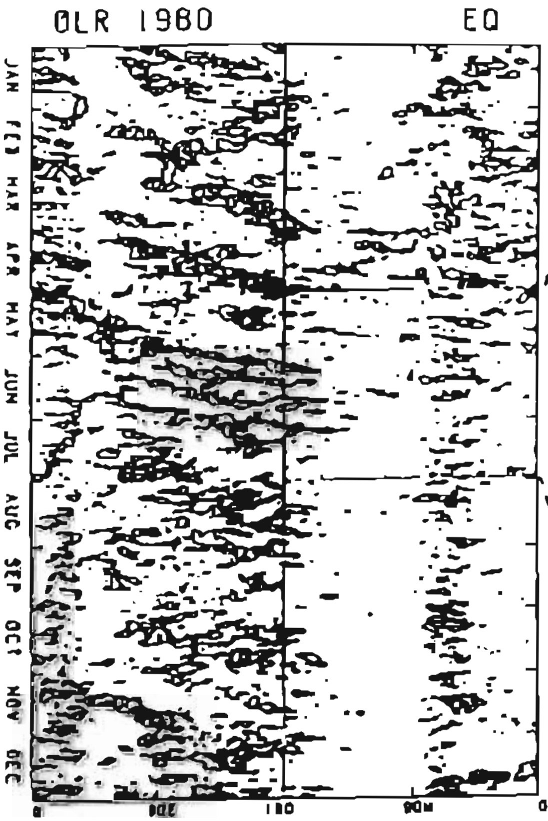
Mesoscale

Precipitating Clouds	1-10 km	<1hr
Cloud Ensemble	10-100 km	>hrs
Mesoscale Cloud Systems	500 km	hrs-days
"Super" Cloud Clusters	1000-2000km	days-week

Large Scale

Tropical Waves (Kelvin, Rossby-gravity)	Synoptic	week
Mean Circulations (Hadley, Walker)	Planetary	interseasonal
ENSO	Planetary	interannual

Hierarchy à la Nakazawa
in the tropical intraseasonal variability



Madden - Julian waves
(Eastward)
40 - 60 days
10^4 km

N Nakazawa

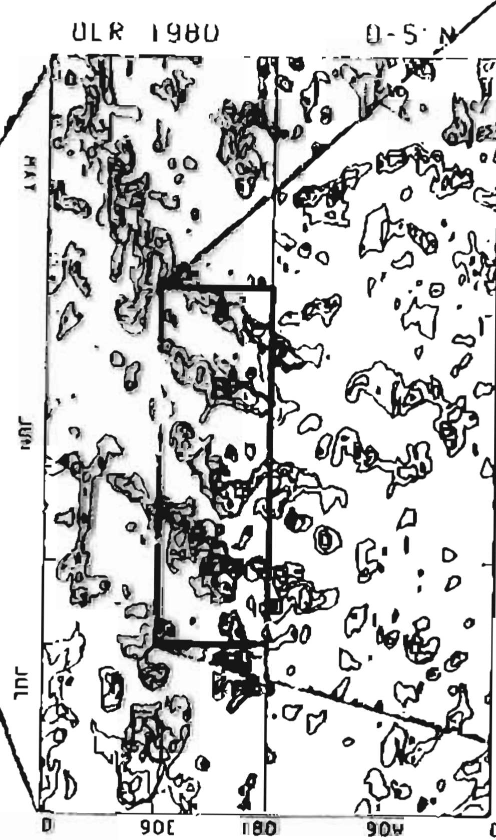


Fig. 1. Time-longitude section of transient (seasonal trend removed) OLR averaged between the equator and 5°N from May to July in 1980. Negative (active convective) regions are contoured. Contour interval decrements of 30 Wm^{-2} starting at $-15\ Wm^{-2}$. Symbols A to D indicate super clusters.

Super-Clusters
(Eastward)
2000 km, 10 days

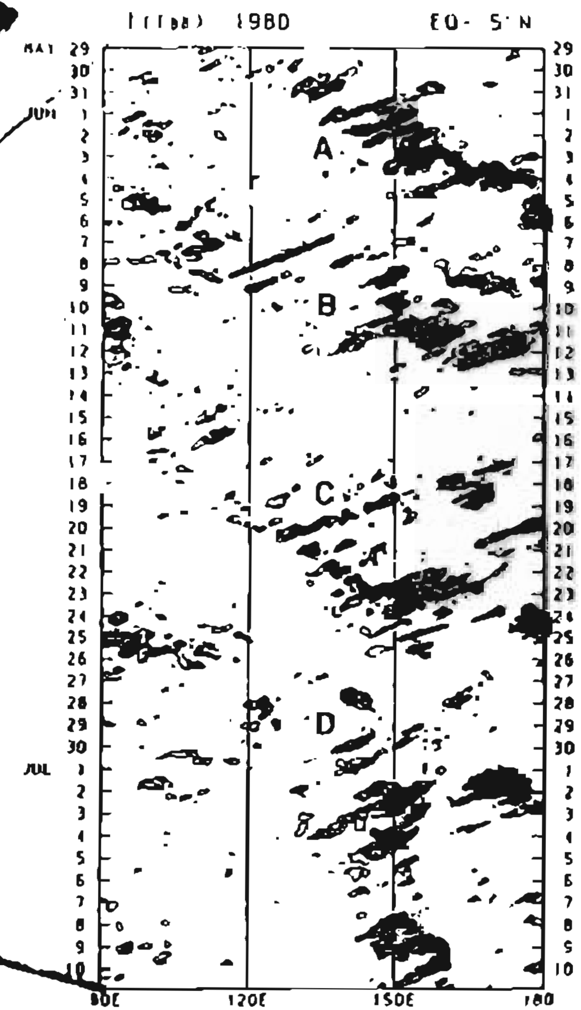
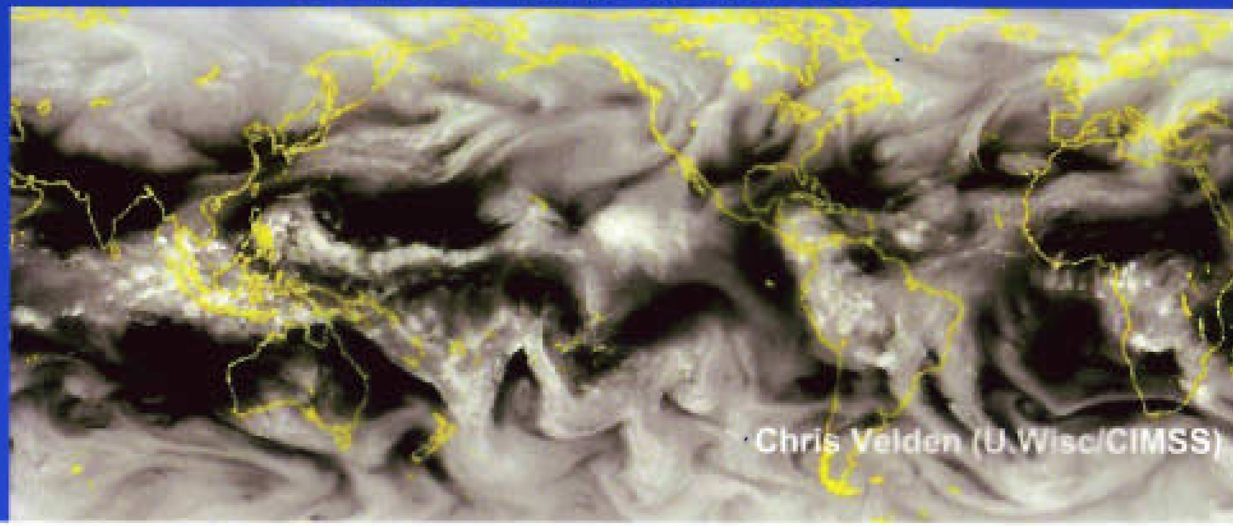


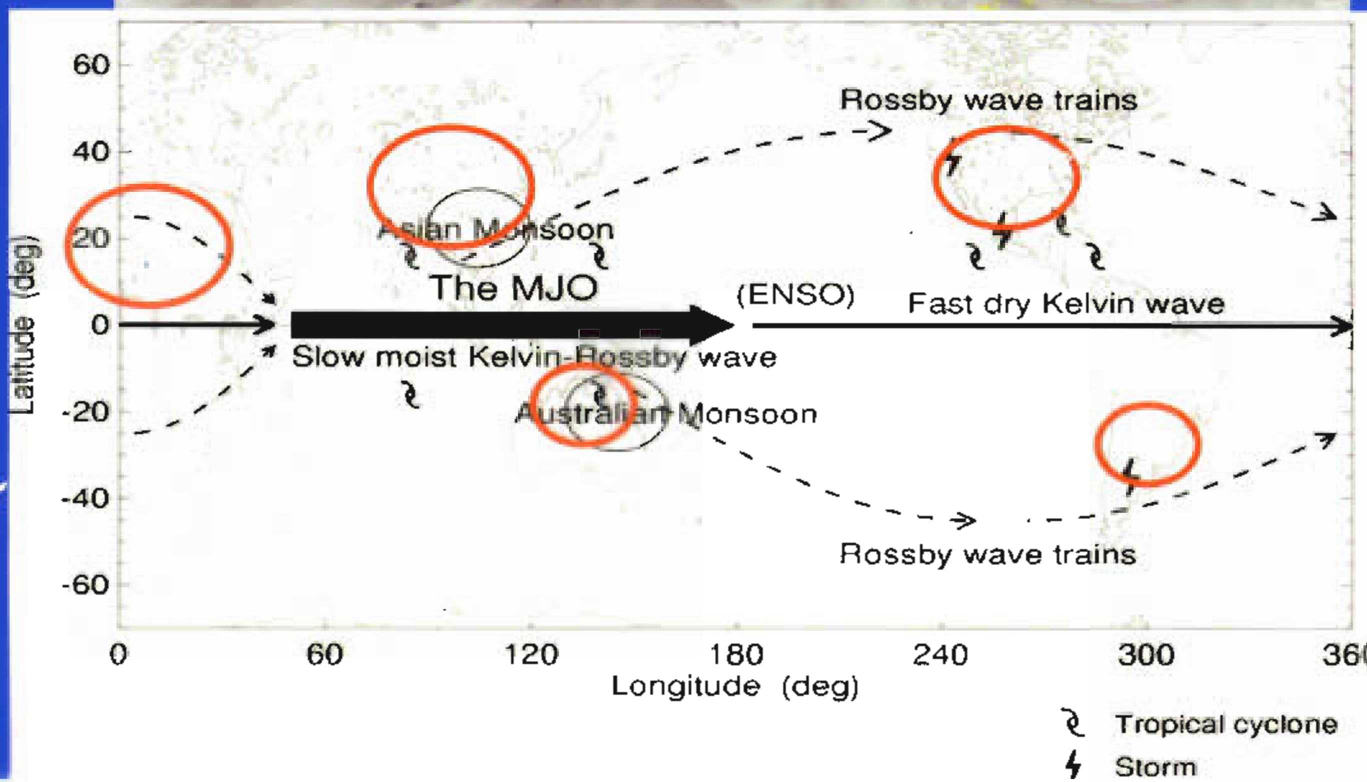
Fig. 2. Time-longitude section of I_{BB} index (I_{BB}) integrated between the equator and 5°N obtained from the 3-hourly GMS IR data from 29 May 03Z to 10 July 21Z, 1980. Symbols A to D denote the same super clusters as in Fig. 1. Contour interval is 10, and shading denotes the region whose values are greater than 20.

Cloud Clusters
(Westward)
10^2 km, 1 day

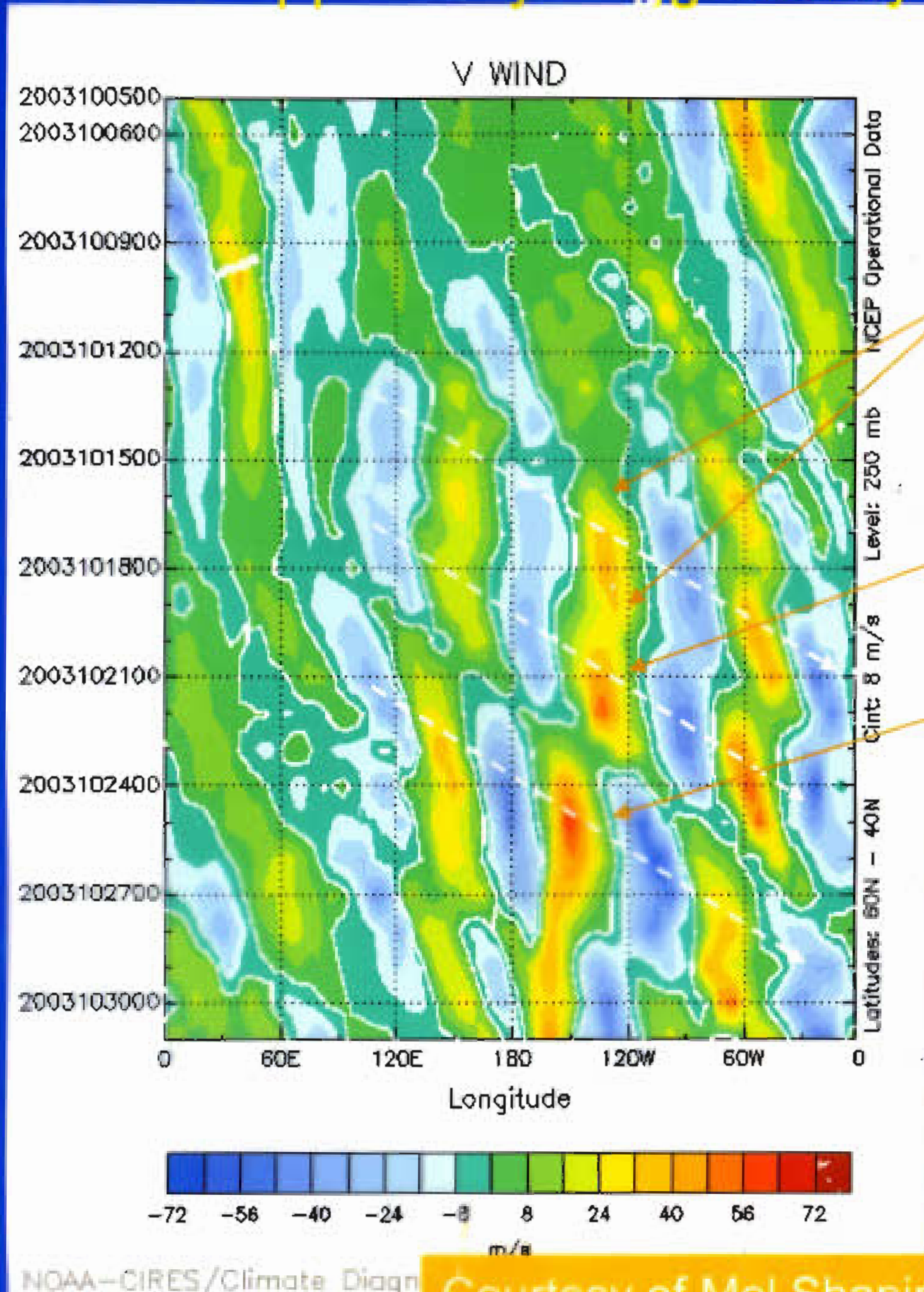
Tropical – Extratropical Interaction and weather

Water Vapor Channel





Rossby Wave Trains Apparently Triggered by Tropical Convection



BC's flood of the Century (18.5")

Western WA Flood (Seattle 1-day record)

CA Wild Fires (downslope winds)

Courtesy of Mel Shapiro and David Parsons

I. Equatorial Shallow Water with Topography

$$\begin{aligned}
\eta_t + [(1+\eta-h)u]_x + [(1+\eta-h)v]_y &= 0 \\
u_t + uu_x + vu_y + \eta_x - yv &= 0 \\
v_t + uv_x + vv_y + \eta_y + yu &= 0.
\end{aligned}$$

Length and Time Scales

$$L = (\frac{c}{\beta})^{\frac{1}{2}}, \quad T = \frac{L}{c}, \quad c = \sqrt{gH},$$

H equivalent height

I. Linear Equatorial Waves

Kelvin Wave

$$
\begin{aligned}
\eta &= K(x-t)e^{-\frac{y^2}{2}} \\
u &= K(x-t)e^{-\frac{y^2}{2}} \\
v &= 0,
\end{aligned}
$$

K an arbitrary function

Dispersive Equatorial Waves

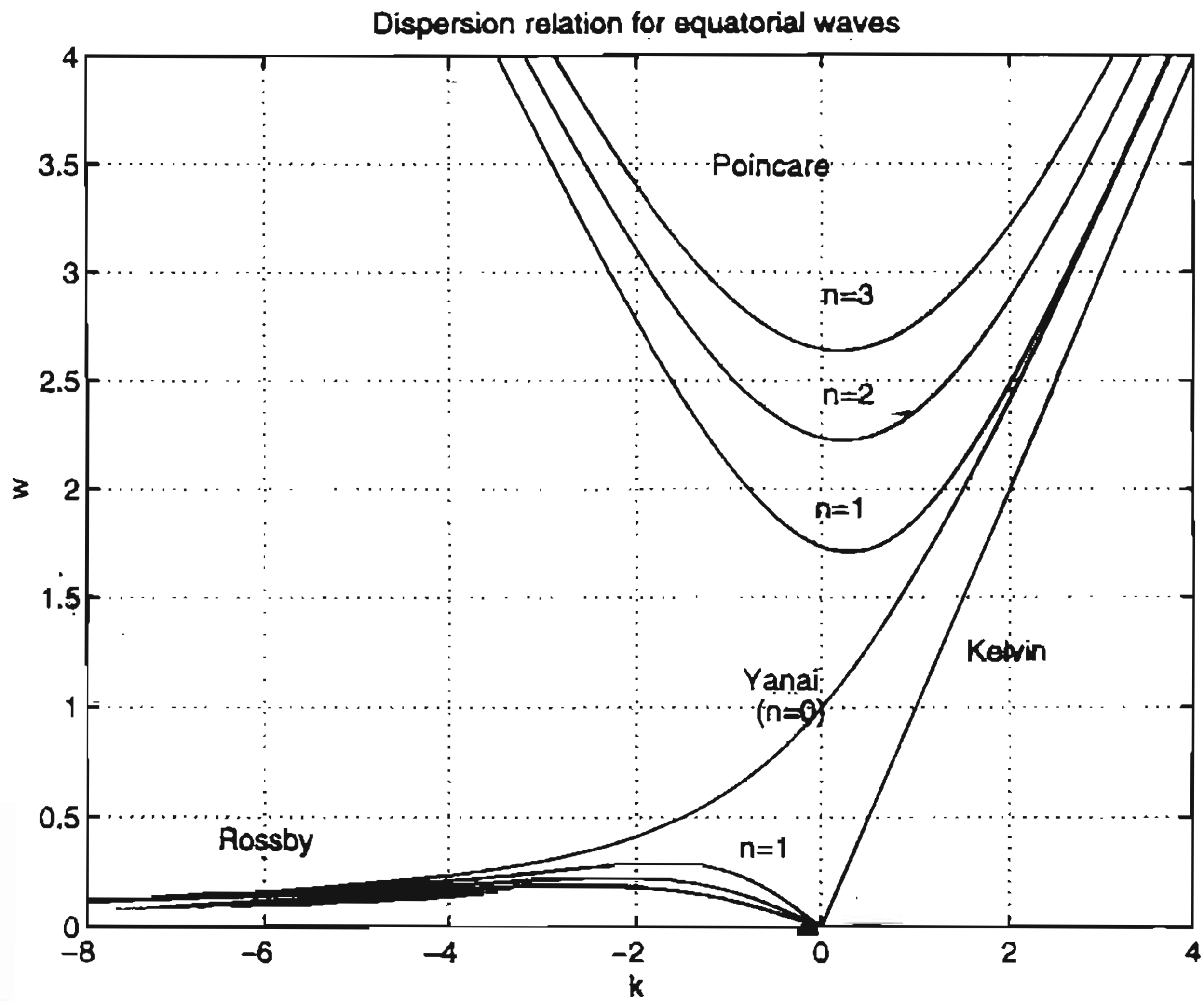
$$
\begin{aligned}
\eta &= \left[\frac{y}{k-\omega}H_n(y) + \frac{\omega}{\omega^2-k^2}H_n'(y)\right] e^{i(kx-\omega t)}e^{-\frac{y^2}{2}} \\
u &= \left[\frac{y}{k-\omega}H_n(y) + \frac{k}{\omega^2-k^2}H_n'(y)\right] e^{i(kx-\omega t)}e^{-\frac{y^2}{2}} \\
v &= iH_n(y)e^{i(kx-\omega t)}e^{-\frac{y^2}{2}},
\end{aligned}
$$

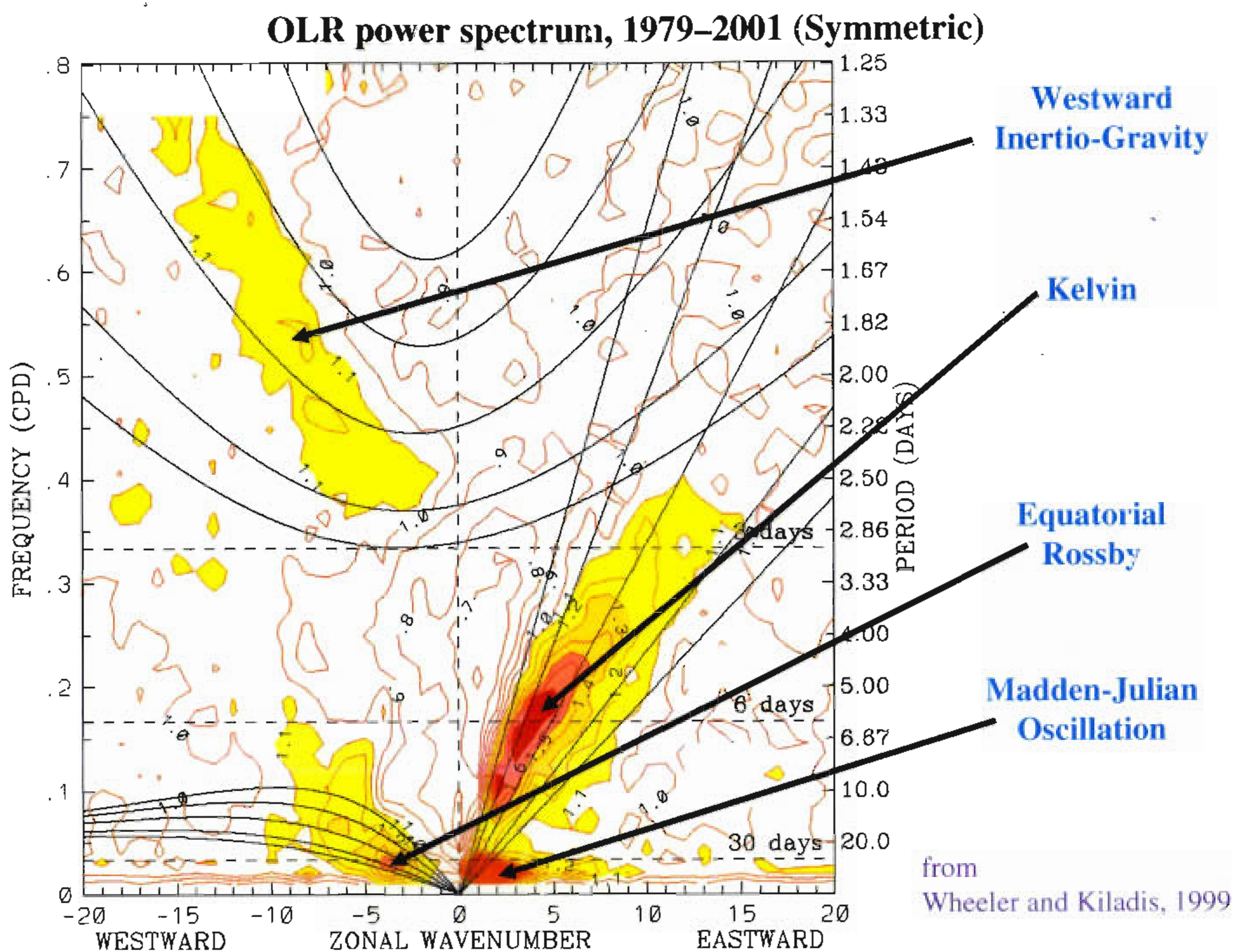
$H_n(y)$ Hermite polynomial of order n

$\omega = W(k)$ satisfies the dispersion relation

$$
-2n + \frac{(\omega+k)\left(\omega^2 - k\omega - 1\right)}{\omega} = 0.
$$

Dispersion Relation for Equatorial Waves

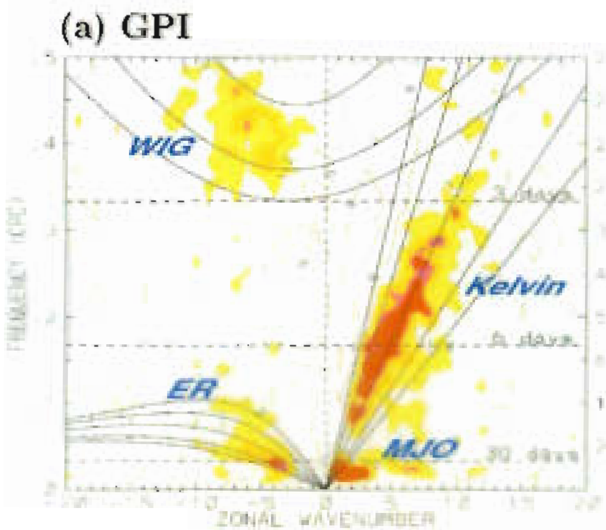


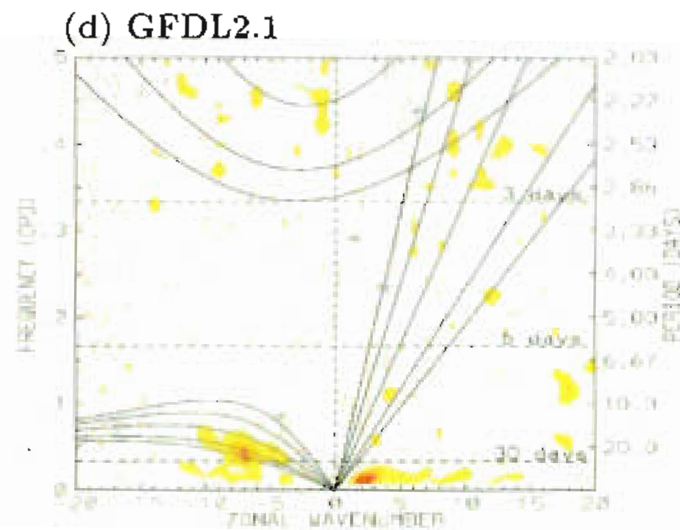

OLR power spectrum, 1979–2001 (Symmetric)
FREQUENCY (CPD)
PERIOD (DAYS)
ZONAL WAVENUMBER
WESTWARD
EASTWARD
3 days
6 days
30 days
Westward Inertio-Gravity
Kelvin
Equatorial Rossby
Madden-Julian Oscillation
from Wheeler and Kiladis, 1999

From Lin et al. (2006)

Left: Observations

Right: GCM





Basic References

A. J. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, (Chapter 9) 2005, Courant Lecture Series #8 Publ. Amer. Math. Soc.

Special Issues

#1) Theoretical Developments in Tropical Meteorology
Vol. 20, #5-6, November 2006
Theoretical & Computational Fluid Dynamics

#2) Atmospheric Convection and Wave Interactions: Convective Life Cycles and Scale Interactions in Tropical Waves
Dynamics of Atmospheres & Oceans
Vol. 42 2006

Current Research Directions: Applied Math and Tropical Dynamics

1) Hierarchical Multi-Scale Modeling:

-Majda, Klein JAS 2003; Moisture, Majda JAS 2006 A, B; Klein, Majda TCFD 2006

-MJO Theory - Biello, Majda PNAS 2004, JAS 2005; DAO, TCFD 2006

-Superrotation in MJO – Biello, Majda, Moncrieff JAS 2006

2) Midlatitude Troposphere Connections with Tropics: What Are Routes?

-Weakly Nonlinear Theory – Majda, Biello JAS 2003

-New Coupled Dispersive Waves - GAFD, Stud. Appl. Math. 2004

3) Convectively Coupled Superclusters: Theory and Observations:

-Majda, Shefter JAS 2001; Majda, Khouider, Kiladis JAS 2004

-Majda, Khouider New Multicloud Models JAS 2005, 2006; TCFD, DAO 2006

4) Stochastic Parameterization of Convection:

- Majda, Khouider PNAS 2002
- Khouider, Majda, Katsoulakis PNAS 2003

5) Moisture Dynamics in Tropics from Large Scale Perspective

-Novel Waves and Relaxation Limits – Frierson, Majda, Pauluis Comm. Math. Sci. 2004

-High Resolution Balanced Numerics – Khouider, Majda TCFD 2005

-Nonlinear Waves – Stechmann, Majda TCFD 2006

6) Novel Singular Limits with Fast Variable Coefficients

-Fast Wave Averaging for the Equatorial Shallow Water Equations - Dutrifoy, Majda Comm. PDE 2006

-The Dynamics of Equatorial Long Waves; A singular Limit with Fast Variable Coefficients - Dutrifoy, Majda Comm. Math. Sci. 2006

http://www.math.nyu.edu/faculty/majda/

Self-similarity in tropical convection

Squall lines

Zipser 1969

Zipser et al 1981

Two-day waves

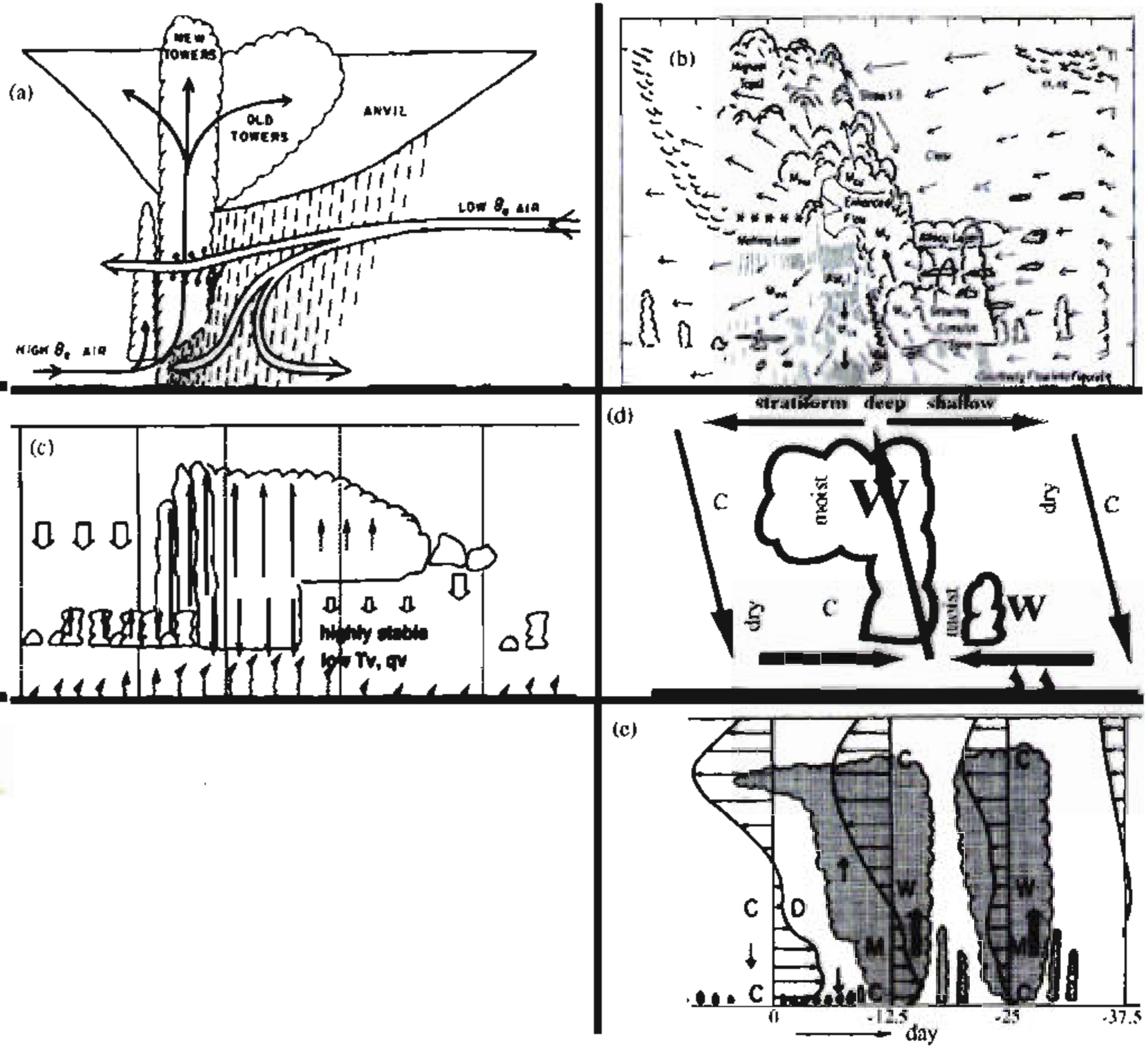
Takayabu et al 1996

C. C. Kelvin waves

Straub and Kiladis 2003

Madden–Julian Osc.

Lin and Johnson 1996



Source: Mapes et al. 2006 DAO

New Multiscale Models
and
Self-Similarity in Tropical Convection

Majda 2007, *J. Atmos. Sci.*

Multiscale Models with Moisture
and
Systematic Strategies for Superparameterization

Majda 2007, *J. Atmos. Sci.* (in press)

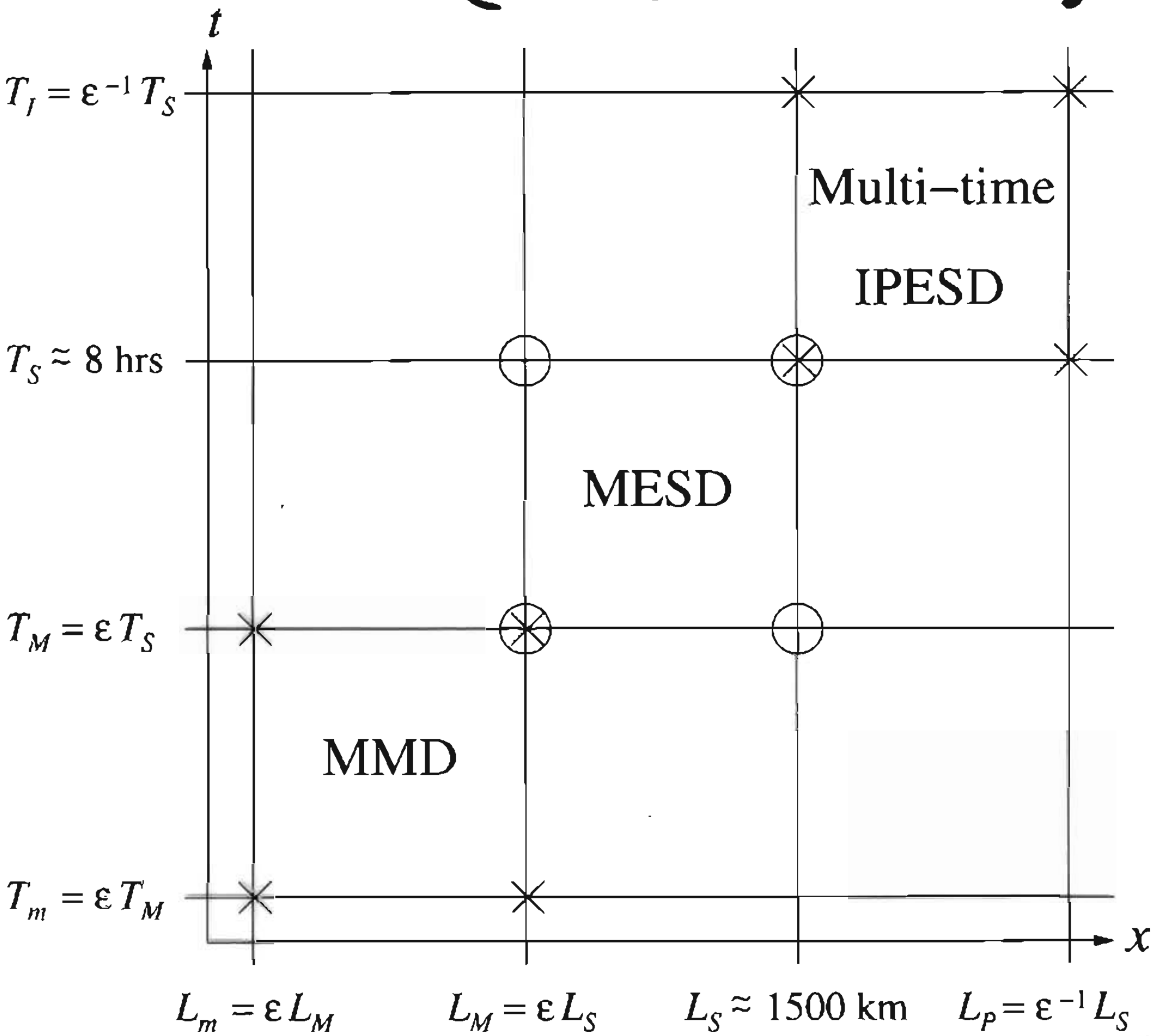
Assumptions:

1. Low Froude Number, $\mathbf{u}_h = \epsilon \mathbf{u}_{h,1}$
2. Weak Temperature Gradient, $\theta = \epsilon\theta_1, p = \epsilon p_1$

Hierarchical Multi-Scale Models
for Theory, Observations, and Numerical Strategies

Majda & Klein (2003); Klein & Majda (2006); Majda (2006a,b)

(JAS) (TCFD) (JAS)



IPESD – MJO – Appl.: Biello, Majda PNAS04 JAS05 DAO06

IPESD – Intraseasonal Planetary Equatorial Synoptic Scale Dynamics

MESD – Mesoscale Equatorial Synoptic Dynamics

MMD – Microscale Mesoscale Dynamics (B, M, Moncrieff) ↔ JAS07

(superotation)

The Interlocking
IPESD
MESD Multi-scale Models
derived under Two universal Assumptions:

#1) Low Froude #

$$\varepsilon = \frac{|u|}{|c_g|}, \quad c_g = 50\,ms^{-1}$$

#2) Weak Temp. Gradient

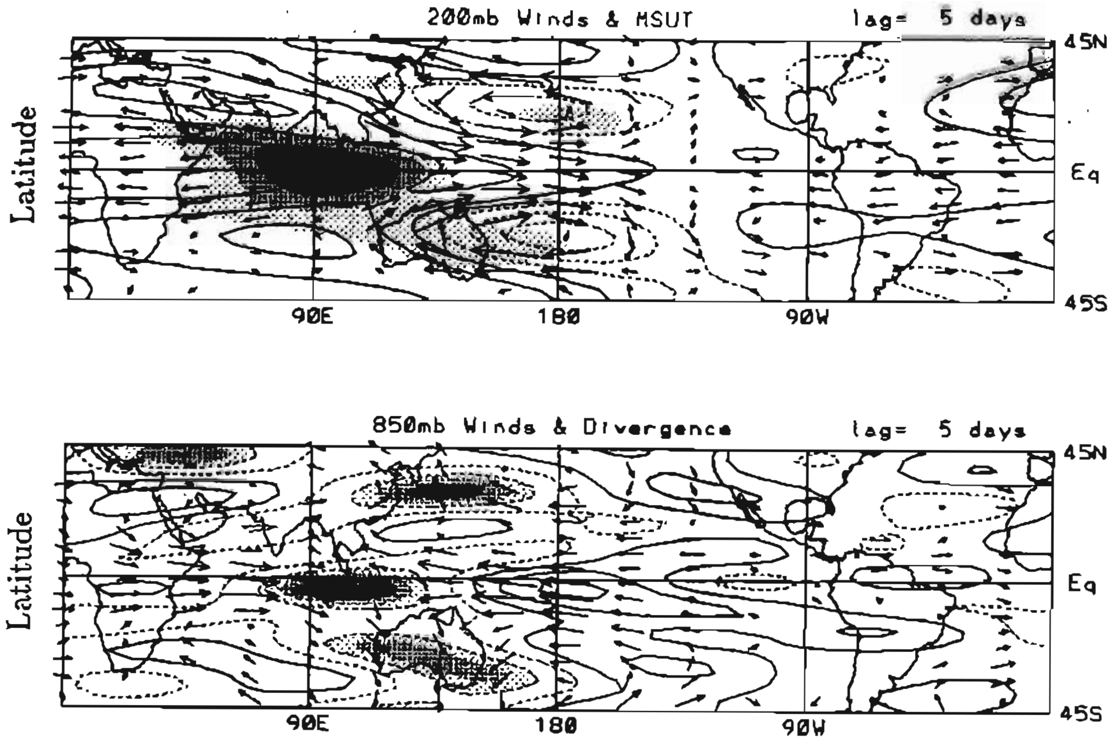
$$\Theta = \bar{\Theta} + \varepsilon \Theta_1$$

$$[\Theta_1] \cong 3°K$$

See M, JAS 2003

#1), #2) Obs. consistent!!

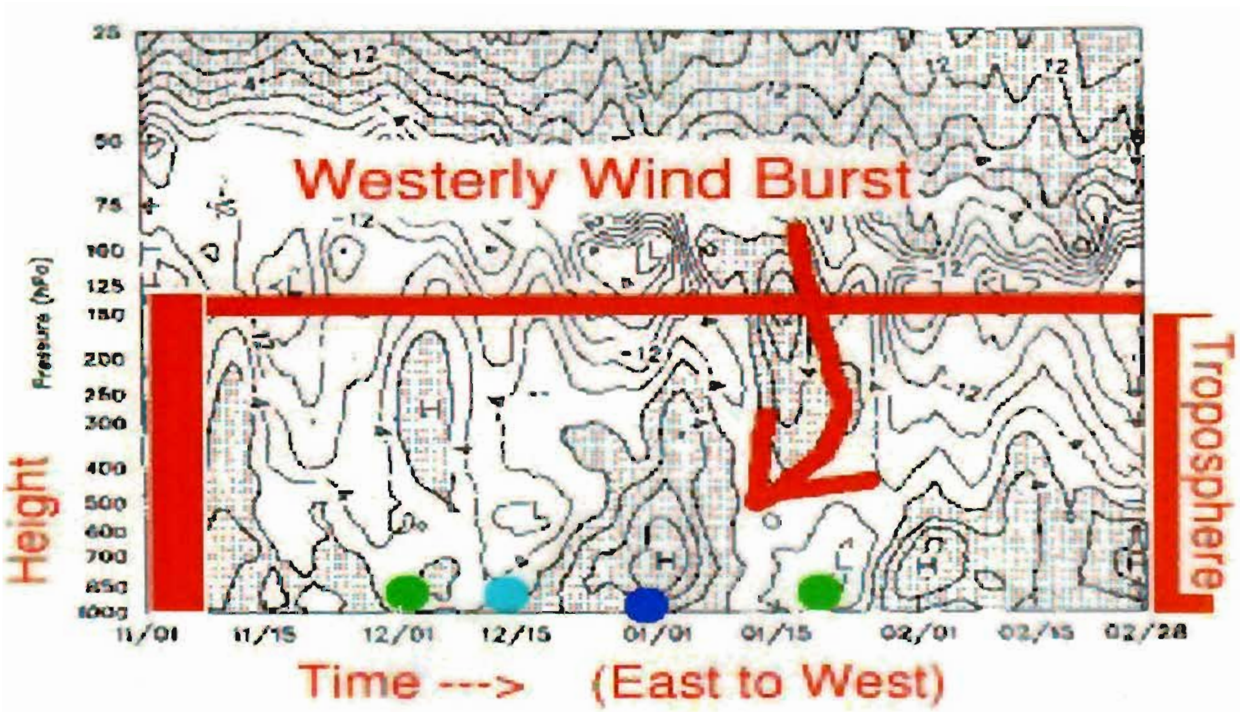
MJO: Large scale wind pattern.



- Top: 200 mb winds and precipitation.
- Bottom: 850 mb winds and divergence.
- Filtered at large scales.

From Hendon & Salby *J. Atmos. Sci.*, **51**, p 2230, fig. 3.

MJO: Vertical Shear



Lin & Johnson *J. Atmos. Sci.*, **53**, p 701, fig. 3 (a).

- Westerly wind burst zonal/vertical profile over a fixed position near the equator.
- Time goes from left to right and can be interpreted as left = east, right = west.
- Background easterlies.
- Westerly onset region.
- Strong westerly region.

MJO: Central Issue

Which aspects of the planetary scale dynamics are due to

1. Planetary scale mean heating as in traditional Matsuno-Gill models?

2. Upscale transport of potential temperature from synoptic to planetary scales through eddy fluxes.

3. Upscale transport of zonal momentum from synoptic to planetary scales through eddy fluxes.

IPESD framework can isolate these causes and their effects on planetary scale organized flows.

The IPESD Multiscale Model for the MJO

Joseph A. Biello - University of California, Davis
Andrew Majda - Courant Institute, New York University

- FRAMEWORK: IPESD multiscale models (Majda/Klein 2003).
- STRUCTURE: MJO structure given by specified heating profiles (PNAS 2004, JAS 2005, DAO 2006, BM+ Moncrieff JAS 2007)
  - planetary scale direct heating
  - upscale fluxes of momentum and heat from synoptic scales
- DYNAMICS: Khouider/Majda multi-cloud model. (Khouider/Majda JAS 2005, 2006, 2007)
  - active moisture through cloud model
  - nonlinear feedback from planetary to synoptic scales
  - organized embedded structures in a traveling envelope (Majda, Stechmann, Khouider PNAS)

FRAMEWORK: IPESD Theory (Majda/Klein 2003)

Synoptic Scale (Balanced) Dynamics:

$$
\begin{aligned}
u'_\tau - y\,v' + p'_x &= S'_u \\
v'_\tau + y\,u' + p'_y &= S'_v \\
\theta'_\tau + w' &= S'_\theta \\
p'_z &= \theta' \\
u'_x + v'_y + w'_z &= 0 \\
\overline{S'_\theta} &= 0
\end{aligned}
$$

Planetary Scale Quasi-Linear Dynamics:

$$
\begin{aligned}
\overline{U}_t - y\overline{V} + \overline{P}_X &= F^U \; - d_0 \overline{U} \\
y\overline{U} + \overline{P}_y &= 0 \\
\overline{\Theta}_t + \overline{W} &= F^\theta \; - d_\theta \overline{\Theta} + \overline{S}_\theta \\
\overline{P}_z &= \overline{\Theta} \\
\overline{U}_X + \overline{V}_y + \overline{W}_z &= 0
\end{aligned}
$$

The fluxes from the synoptic scales are given by

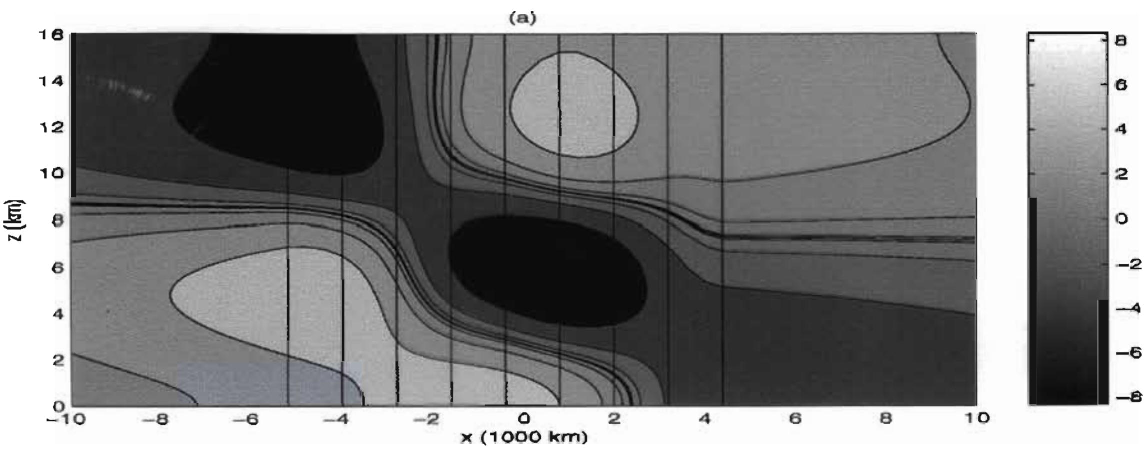
$$
\begin{aligned}
F^U &= -\overline{(v'\,u')_y} - \overline{(w'\,u')_z} \\
F^\theta &= -\overline{(v'\,\theta')_y} - \overline{(w'\,\theta')_z}
\end{aligned}
$$

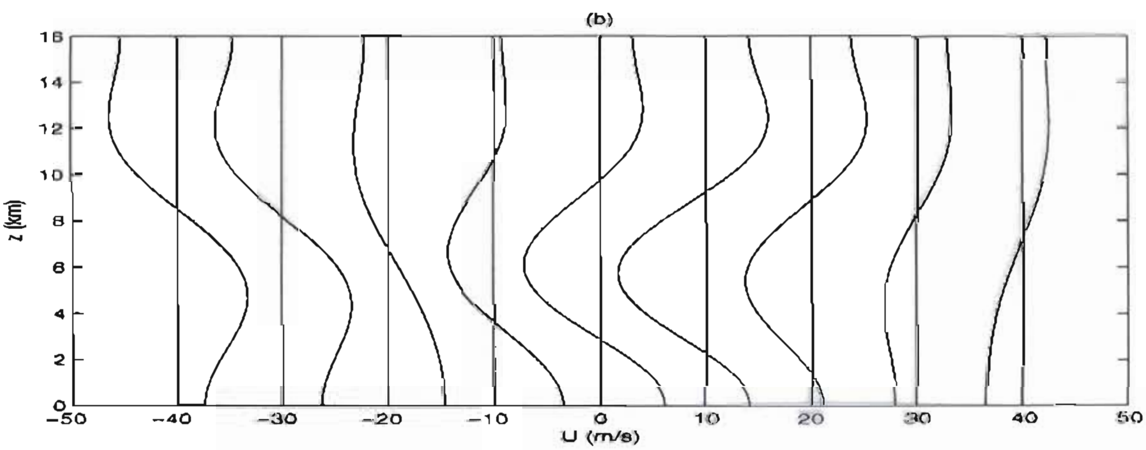
Each forcing effect, i.e. upscale vertical and meridional momentum and temperature transport and planetary scale mean heating can be considered separately and superposed

Equatorial MJO model:

- Congestus heating in the east and westward tilted superclusters in the west of a moving warm pool.

- Planetary mean heating is weaker, but has same structure of synoptic scale fluctuations.

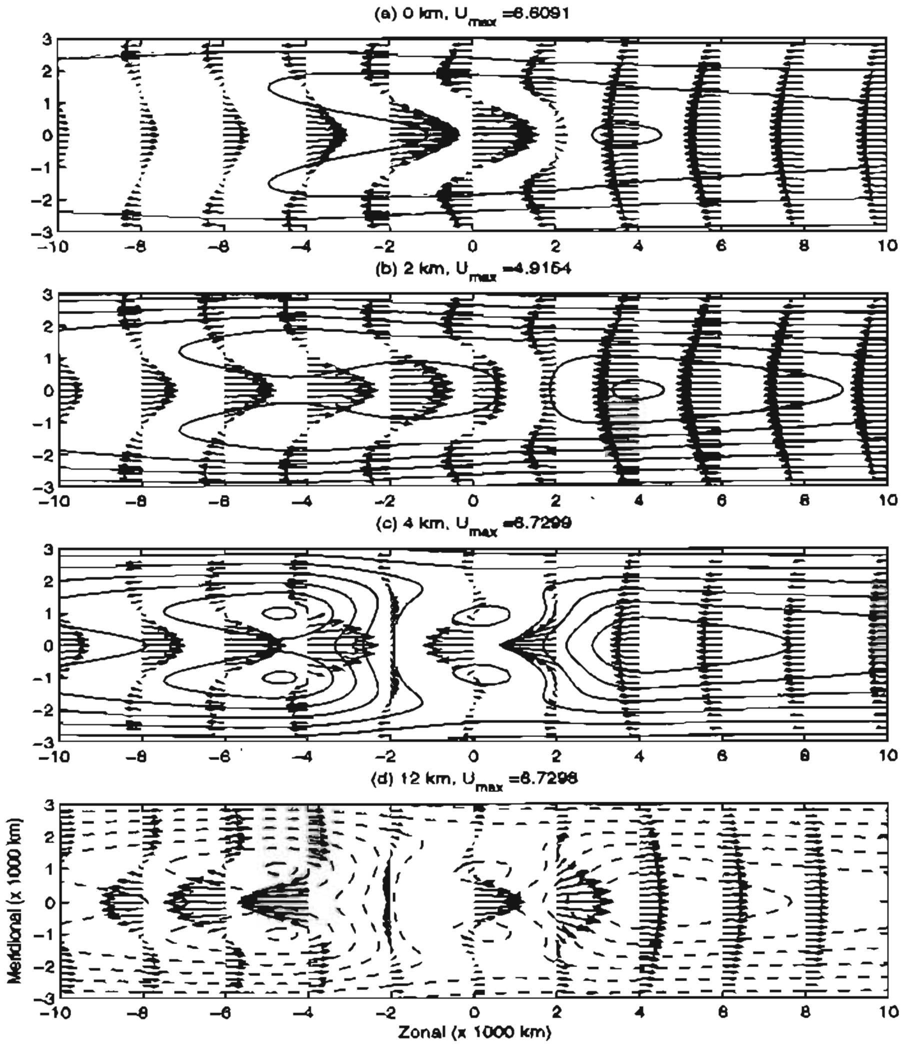




- (a) Zonal velocity (westerly = light, easterly = dark) as a function of height and longitude above equator and (b) as a function of height above the cuts from (a).

Equatorial MJO model:

- Congestus heating in the east and westward tilted superclusters in the west of a moving warm pool.
- Planetary mean heating is weaker, but has same structure of synoptic scale fluctuations.



- Pressure and flow at $z = 0, 2, 4, 12$ km.

Summary: Which aspects of the MJO arise from upscale transport versus direct heating?

Lower/Upper Troposphere Direct Heating Alone

- Easterlies Leading Westerlies
- Maximum westerlies at base of troposphere

Addition of Congestus/Supercluster Upscale Transport

- Quadrupole structure
- Upward/Westward tilt of westerly wind burst
- Maximum westerlies at 4-6 km height
- Intensification of midlevel easterly jet leading the westerly onset region
- Cyclone pair trailing convective activity
- Upper troposphere outflow from convective region

Vortical Hot Towers

Hot towers :
intense deep convection cores with small horizontal scales (of order 10 km) and short convective lifetimes (of order 1 hour).

Next:

- Build elementary models which exhibit basic characteristics of hot towers to study the evolution of radial eddies (which represent "vortical hot towers") in various radial preconditionings.

- How heat (mass) sources can generate vortices?

- Explore the role of heat sources in cyclogenesis through a reduced form of the asymptotic system (1.3).

Although the terminology of hot towers (cloud scales) is used here, the model is also good for larger (meso) scale systems e.g. mesovortices under synoptic-scale preconditionings.

System in Axisymmetric Case

$$\mathbf{u} = u^r \mathbf{e}_r + u^\theta \mathbf{e}_\theta + w\mathbf{e}_z,$$

The system (1.3) is reduced to

$$\begin{aligned}
\frac{\partial u^\theta}{\partial t} + u^r \frac{\partial u^\theta}{\partial r} + u^r \frac{u^\theta}{r} + w\frac{\partial u^\theta}{\partial z} + f u^r &= 0, && (4.2a)\\
\frac{\partial (r u^r)}{\partial r} + \frac{\partial (rw)}{\partial z} &= 0, && (4.2b)\\
wN^2(z) &= S_\theta. && (4.2c)
\end{aligned}$$

Hence the radial velocity u^r is directly specified by the heat source

$$u^r = -\frac{1}{r}\int_0^r s\frac{\partial w}{\partial z} ds.$$

Note: no vertical shear due to axisymmetry

System in Axisymmetric Case

We decompose the heat source into a large scale mean and small scale perturbation as

$$S_\theta(t,r,z) = \overline{S}_\theta(t,z) + S'_\theta(t,r,z)$$

Hence, the flow quantities are decomposed as

$$\begin{aligned}
w(t,r,z) &= \overline{w}(t,z) + w'(t,r,z). \\
\omega(t,r,z) &= \overline{\omega}(t,z) + \omega'(t,r,z), \\
\overline{u}^\theta(t,r,z) &= \overline{u}^\theta(t,r,z) + \left(u^\theta\right)'(t,r,z), \\
\overline{u}^r(t,r,z) &= \overline{u}^r(t,r,z) + \left(u^r\right)'(t,r,z),
\end{aligned}$$

where $\overline{u}^r$ and $\overline{u}^\theta$ are respectively obtained from (4.3) and (4.4)

$$\begin{aligned}
\overline{u}^r(t,r,z) &= -\frac{1}{2}\frac{\partial \overline{w}(t,z)}{\partial z} r, \\
\overline{u}^\theta(t,r,z) &= \frac{1}{2}\overline{\omega}(t,z) r.
\end{aligned}$$

System in Axisymmetric Case

The equation for the evolution of vorticity (4.6) is simplified for $\overline{\omega}(t,z)$ as

$$\frac{\partial \overline{\omega}(t,z)}{\partial t} + \overline{w}(t,z)\frac{\partial \overline{\omega}(t,z)}{\partial z} = \frac{\partial \overline{w}(t,z)}{\partial z}\left(\overline{\omega}(t,z) + f\right).$$

Preconditioned Large Scale Flows

The mean flow satisfies (4.5) by

$$\frac{\partial \overline{u}^{\theta}}{\partial t} + \overline{u}^{r}\overline{\omega} + \overline{w}\frac{\partial \overline{u}^{\theta}}{\partial z} + f\overline{u}^{r} = 0.$$

Equation for the evolution of small scale perturbation $(u^{\theta})'$ in a large-scale preconditioning:

$$\frac{\partial (u^{\theta})'}{\partial t} + \left(\overline{u}^{r} + (u^{r})'\right)\left(\frac{\partial (u^{\theta})'}{\partial r} + \frac{(u^{\theta})'}{r}\right) + \left(\overline{w} + w'\right)\frac{\partial (u^{\theta})'}{\partial z}$$

$$= \; -f\,(u^{r})' - (u^{r})'\,\overline{\omega} - w'\frac{\partial \overline{u}^{\theta}}{\partial z},$$

stretching

vertical advection of mean flow by perturbation

Mean Flow

Physical problems of interest for the mean flow

$f \neq 0 \Leftrightarrow$
preconditioned
Barotropic Mean Flow

- Barotropic mean vorticity

  $\overline{w} = 0$ and hence, $\overline{u}^r = 0$

  $\overline{\omega} = \overline{\omega}_0(z)$

- Deep convective mean flow

  $\overline{w} = A(t)\sin(\pi z), \quad 0 \leq z \leq 1$

  $\frac{d\bar{\omega}}{dt} = (\bar{\omega} + f)\overline{w}_z$ in the characteristic coordinates.

  This yields *cyclones in the lower troposphere and anti-cyclones in the upper troposphere.*

- Stratiform mean flow

  $\overline{w} = -A(t)\sin(2\pi z), \quad 0 \leq z \leq 1$

  This leads to *a mid-level cyclone and high and low level anti cyclone generation.*

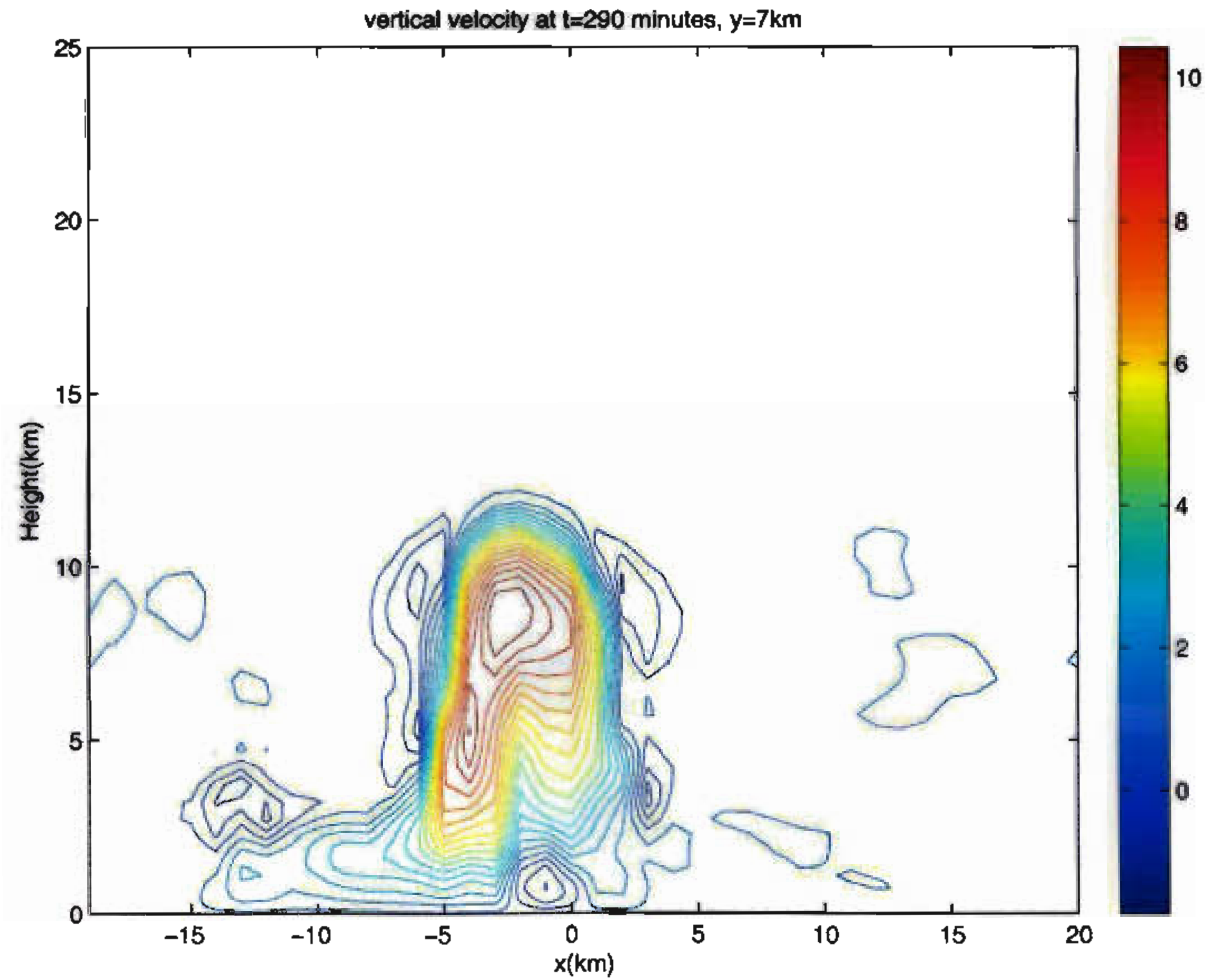
Elementary model for small-scale hot towers

Introduce a perturbation flow with a compact support which represents basic characteristics of hot towers to study the role of hot towers in the hurricane embryo.

Hot Towers exhibit the following basic features:

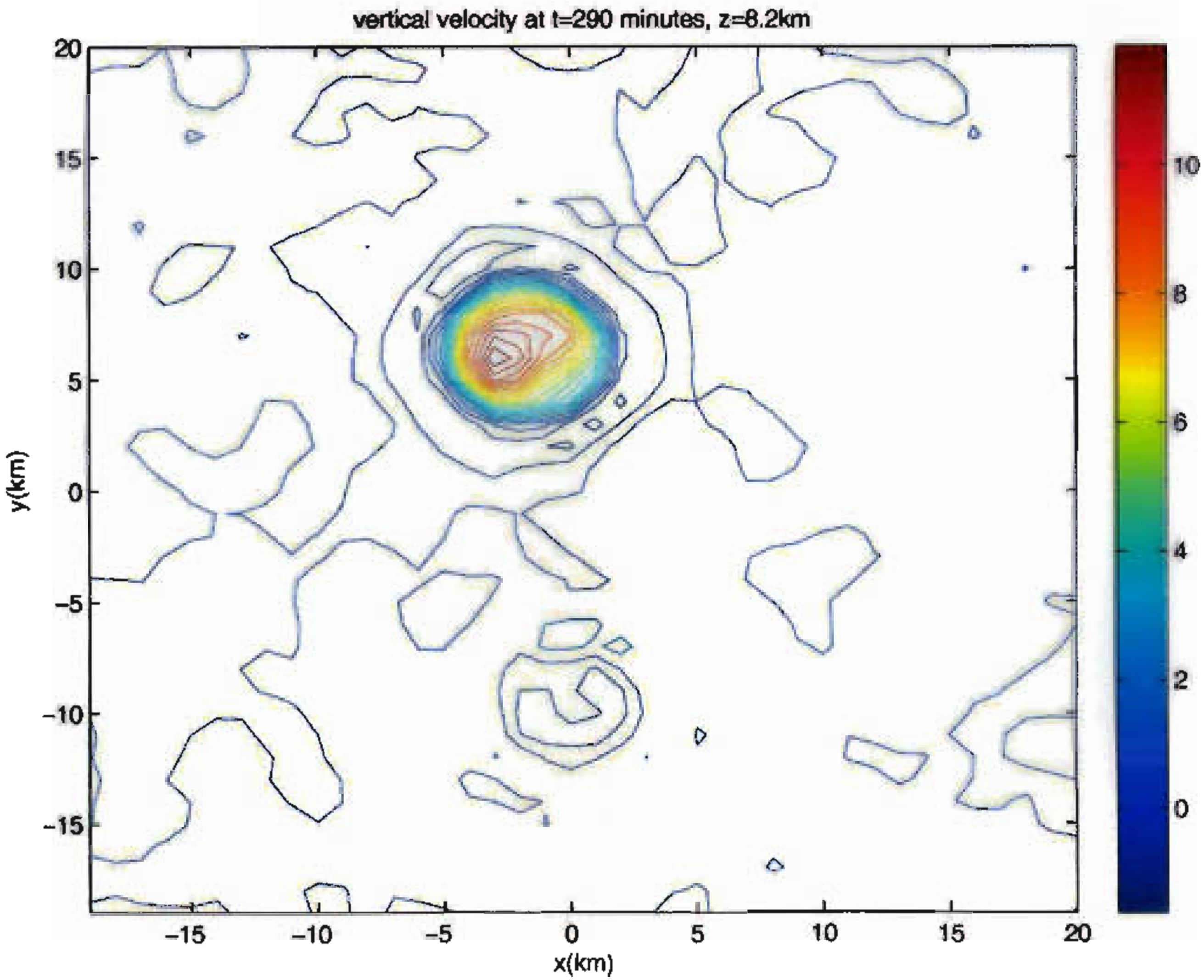
- they have a horizontally small-scale compact support ;
- their vertical structure resembles deep convective rising plumes;
- they consist of an intense updraft in their center and milder downdrafts around;
- they exhibit short convective lifetimes including generation, mature, and decaying stages.

Elementary model for small-scale hot towers



horizontal CROSS-section

Elementary model for small-scale hot towers



Vortical
View
from
top

Elementary model for small-scale hot towers

Motivated by the above typical features of hot towers, we consider the following profile of vertical velocity as an elementary hot tower model

$$\widehat{w} = \begin{cases} z^4(z-1)^4[850r(r-1)^6 + \frac{255}{2}(r-1)^6 + (1700r(r-1)^6 \\ \quad +5100r(r-1)^5 + \frac{255}{2r}(r-1)^6 + 765(r-1)^5] & 0 \le r, z \le 1, \\ 0 & otherwise. \end{cases} \tag{4.28}$$

Using the continuity equation, we obtain

$$\widehat{u^r} = \begin{cases} [-4z^3(z-1)^4 - 4z^4(z-1)^3]\left(850r^2(r-1)^6 + \frac{255}{2}r(r-1)^6\right), & 0 \le r, z \le 1, \\ 0 & otherwise. \end{cases} \tag{4.29}$$

The life cycle of a hot tower may be modeled by the $\sin^+$ function

$$\sin^+(\theta) = \begin{cases} \sin(\theta), & if \ \sin(\theta) > 0, \\ 0 & otherwise. \end{cases}$$

, T_{max} – lifetime

Hence,

$$w' = \widehat{w}\sin^+(\pi t/T_{max}),$$

$$(u^r)' = \widehat{u^r}\sin^+(\pi t/T_{max}).$$

What's needed in general to make a hot tower of compact support
w = "Heat Source"

(1) Start with $w_+(r) \geq 0$, comp. supp.

& $\int_0^\infty r\, w_+(r)\, dr > 0$

(2) Let

$$w = A\left(\lambda_+^{-2} w_+\left(\frac{r}{\lambda_+}\right) - \lambda_-^{-2} w_+\left(\frac{r}{\lambda_-}\right)\right) / \text{[illegible]}$$

$$\lambda_+ >> \lambda_-$$

First term is updraft - stronger
Second term is downdraft

(3) The term in (2) is convective heating

Elementary model for small-scale hot towers

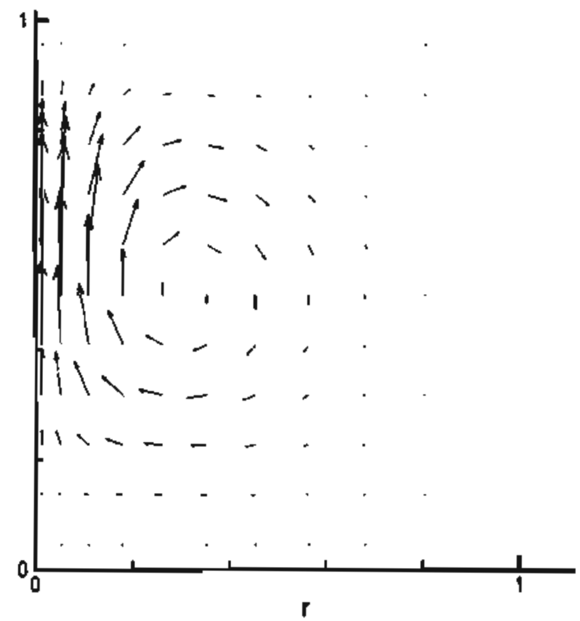


Figure 4.14: The perturbation flow field generated by the hot tower given by (4.28) and (4.29).

a deep convective rising plume, with an intense updraft in its center and a mild downdraft around.

Elementary model for small-scale hot towers

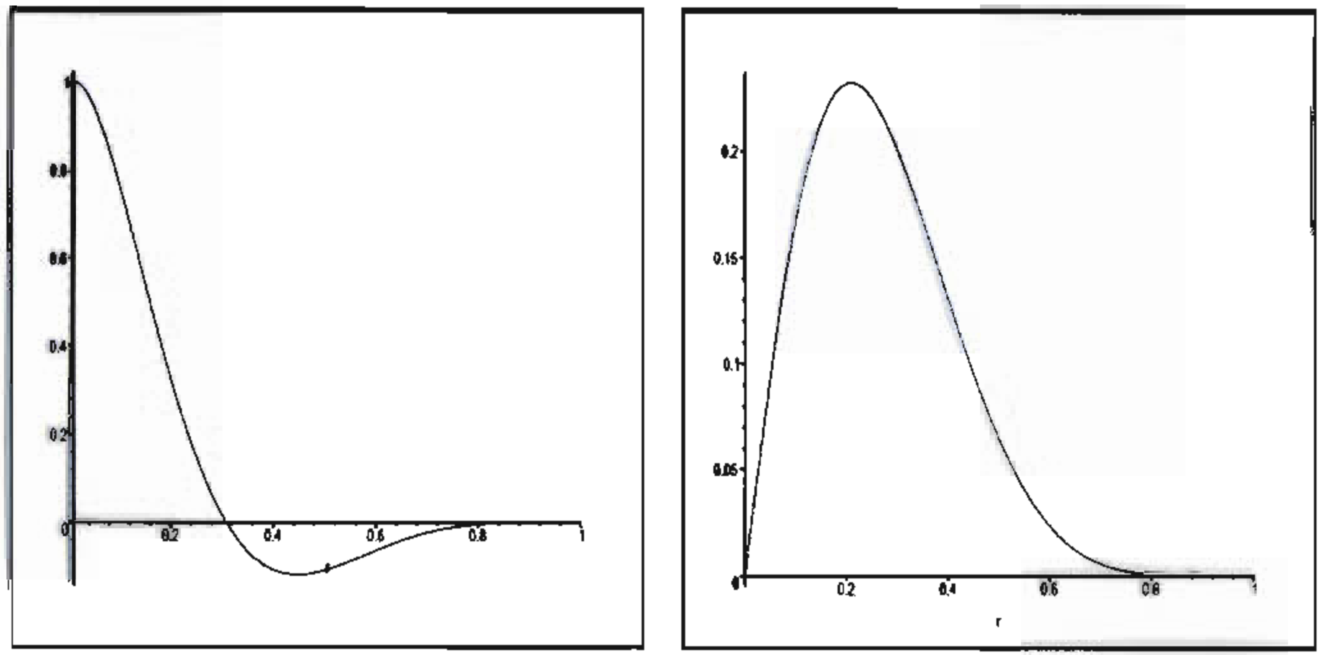


Figure 4.13: $\widehat{w}$ at $z = \frac{1}{2}$ (left) and $\widehat{u}^r$ at $z = \frac{1}{2} - \frac{\sqrt{7}}{14}$ where it is maximum (right) given by (4.28) and (4.29).

Evolution of hot towers in the absence of mean flows

Here we assume no background rotation ($\overline{\omega} = 0$ and $\overline{w} = 0$).
Show how initial conditions and the Coriolis parameter can affect the evolution of a hot tower in the absence of mean flow.

$f \neq 0 \Leftarrow$ Barotropic Mean Flow

The equation (4.16) for the evolution of $(u^\theta)'$

$$\frac{\partial (u^\theta)'}{\partial t} + (u^r)' \left(\frac{\partial (u^\theta)'}{\partial r} + \frac{(u^\theta)'}{r} \right) + w' \frac{\partial (u^\theta)'}{\partial z} = -f\,(u^r)'.$$

Due to the fact that $(u^r)' < 0$ for $z < ½$, and $(u^r)' > 0$ for $z > ½$, we know:

the source term $-f\,(u^r)'$ generates cyclones in the lower troposphere ($z < ½$) and anti-cyclones in the upper troposphere.

the term $(u^r)'\,(u^\theta)'/r$ is an amplification term when $z < ½$ and a dissipation term when $z > ½$. In the regions close to center, where r is very small, this term is dominant.

Evolution of hot towers in the absence of mean flows ⇔

No small scale initial vorticity $(u^\theta)'$ Barotropic preconditioned Mean Flow

The initial vorticity is produced in the domain according to the term $\frac{\partial w'}{\partial z} f$ in the vorticity equation (4.6). As the time increases, a huge vorticity is produced close to the center. This is because the vorticity is given by $\frac{\partial (u^\theta)'}{\partial r} + \frac{(u^\theta)'}{r}$; in the regions close to the center, the term $\frac{(u^\theta)'}{r}$ is dominant and this leads to a huge vorticity in the inner core.

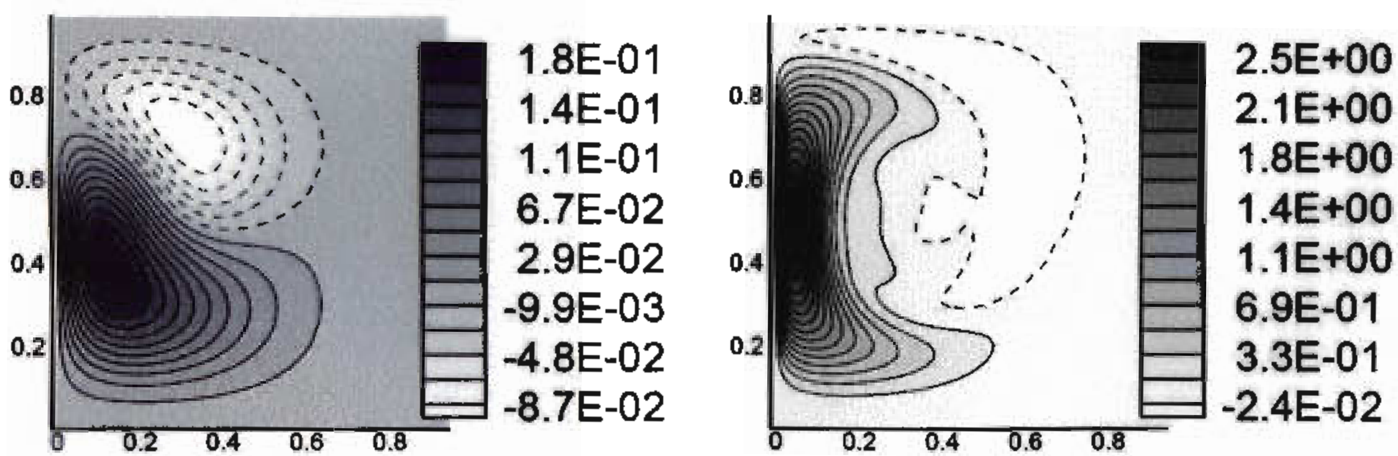


Figure 4.15: Case A1, contour plots of $(u^\theta)'$ for $(u^\theta)'_0 = 0$, $f = 1$, results at $t = T_{max}$ for $T_{max} = 1$ (left) and $T_{max} = 10$ (right). Dash lines show negative values (anti-cyclonic flow). Horizontal axis is r and vertical axis is z.

Effect of steady mean flows on hot towers

For simplicity, we assume zero initial small scale vorticity.

The equation (4.16) for the evolution of $(u^\theta)'$

$$\frac{\partial (u^\theta)'}{\partial t} + (u^r)'\left(\frac{\partial (u^\theta)'}{\partial r} + \frac{(u^\theta)'}{r}\right) + w'\frac{\partial (u^\theta)'}{\partial z} = -f\,(u^r)' - (u^r)'\overline{\omega} - w'\frac{\partial \overline{u}^\theta}{\partial z}$$

Effect of steady mean flows on hot towers

Low level cyclonic and high level anti-cyclonic mean flow

The mean vorticity here corresponds to a large-scale deep convective flow and it is defined by $\overline{\omega} = \sin(2\pi z)$.

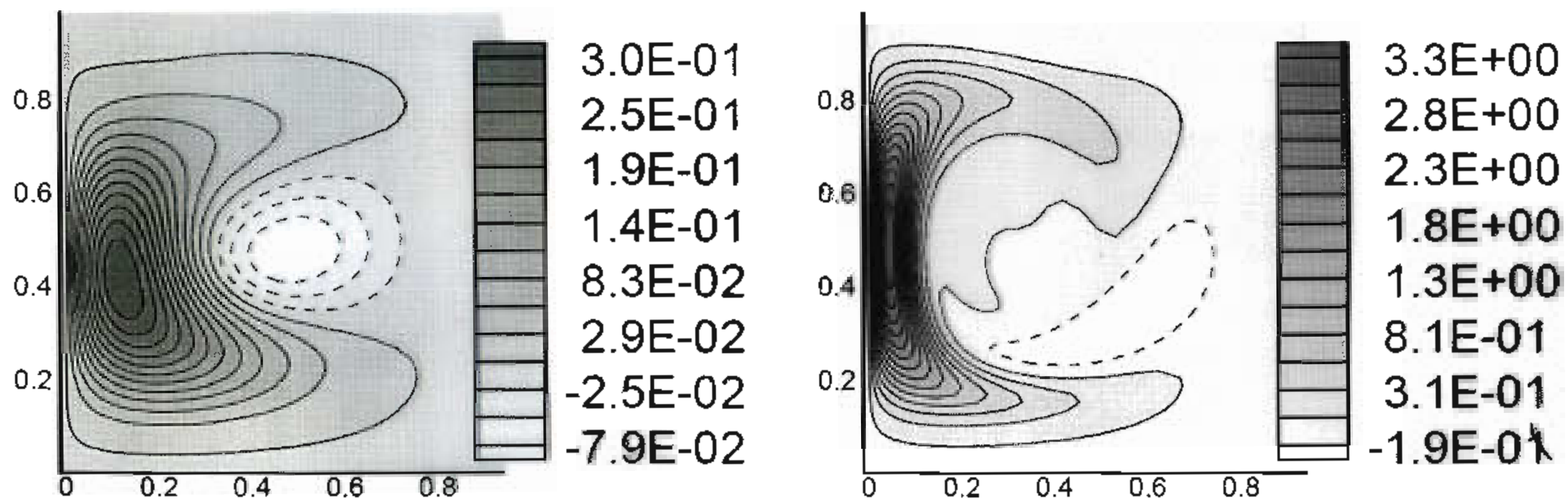


Figure 4.19: Same as Fig. 4.15 for Case B1. $(u^{\theta})'_0 = 0$, $f = 0.5$.

Summary

We have shown how a heat/mass source can generate large vorticity in a suitable preconditioning and useful elementary insight into the role of hot towers in cyclogenesis has been obtained through combination of exact solutions and simple numerics.

Although the terminology of hot towers and cloud scales are used here, the canonical model studied in this paper, is also relevant for larger scales, and the elementary model study also gives insight into how mesovortices may be generated due to the heat sources by mesoscale convective systems under various synoptic scale preconditionings.

Future work:

The insights obtained in this study, are useful as elementary structures in multi-scale models for the hurricane embryo.

Vertically Sheared Horizontal Flow with Mass Sources: A Canonical Balanced Model

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1: Vertically Sheared Horizontal Flow with Mass Sources: A Canonical Balanced Model, to appear in Geophysical and Astrophysical Fluid Dynamics (GAFD).
2: Multiscale equations for the hurricane embryo in a WTG environment, in preparation.

Introduction

Two common assumptions in the development of multi-scale models:

- horizontal weak temperature gradient (WTG) approximation for potential temperature

$$\Theta = \overline{\Theta}(z) + \epsilon\theta(\mathbf{x}_h, z, t), \qquad \epsilon \ll 1,$$

- low Froude number approximation for the horizontal flow U_h

$$\mathbf{U}_h = \epsilon \mathbf{u}_h \ , \qquad \epsilon \ll 1.$$

$\epsilon \cong \frac{1}{10}$ to $\frac{1}{7}$ are typical observed values for the lower/middle troposphere

Introduction

With the above background, the goal here is to study the following canonical balanced model.

Vertically sheared horizontal flow with mass (heat) sources (VSHFS):

$$
\begin{aligned}
\frac{D\mathbf{u}_h}{Dt} + f u_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \qquad (1.3)\\
div\mathbf{u}_h + w_z &= 0,\\
wN^2(z) &= S_\theta,
\end{aligned}
$$

Introduction

The equations in (1.3) arise in a variety of multiple spatial scale balanced dynamics for the tropics:

- on horizontal scales of order 1500 km and time scales of order 8 hours (see the BMESD model in Majda 2007b);
- on horizontal scales of order 10 km and time scales of order 15 minutes (Klein 2000; Klein and Majda 2006);
- with the beta plane approximation, $f = \beta y$, on horizontal scales of order 800 km and time scales on the order of 1 day (Sobel et al. 2001; Majda and Klein 2003);
- on seasonal planetary scales (see SPEWTG model in Majda and Klein 2006).

The Canonical Balanced Model

The forced Boussinesq equations takes the non-dimensional form:

$$
\begin{aligned}
\frac{D\mathbf{u}_h}{Dt} + (\mathrm{Ro})^{-1}\,\mathbf{u}_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \\
\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \epsilon^{-1}\theta + S_w, \\
\frac{D\theta}{Dt} &= \epsilon^{-1}(-w + S_\theta), \\
div_h \mathbf{u}_h + w_z &= 0.
\end{aligned}
\tag{2.2}
$$

under the conditions:

1: WTG approximation,
2: low Froude number,
3: comparable horizontal and vertical velocity magnitudes,
4: large Rossby number $\mathrm{Ro}= \frac{LV}{f} \geq O(1)$,

Units: [**x**] = [z] = 10 *km*, [t] = 15 *minutes*, [❑] = 3 *K*,
[**u**] = [w] = 10 *m/s*, strong heating: 120 *K/hr*.

Key Unsolved Questions
for Hurricane Embryo:

What preconditioning background enviroments (Shear, Vorticity, Temp, Moisture) Lead to Tropical Cyclogenesis?

H E - Stage

O(10 m/s) winds

Hot Towers O(10 km), 120 K/hr

(*) Mesovortices of T.C. O(100 km)

What is involved in creating (*)?

Hot Towers, Montgomery JP, 2004 -

Moist Thermodynamics, Bister-Emanuel

The Canonical Balanced Model

The derivation of the canonical model (1.3) is straightforward. They are

- the leading order ϵ^0 equations for horizontal momentum and mass conservation
- the leading order ϵ^{-1} equations for the potential temperature.

If the temperature perturbation θ is expanded as $\theta = \epsilon\theta_1$, then θ_1 is determined from the solution of (1.3) as given by

$$\theta_1 = \frac{Dw}{Dt} + \frac{\partial p}{\partial z} - S_w.$$

The Canonical Balanced Model

I) The canonical models in (1.3) has direct relevance for the troposphere with

horizontal scales of order 10 km and time scales of order 15 minutes;

Hot Tower Scales

II) It also apply on

horizontal scales of order 100 km and time scales of order 2.5 hours;

These time scales are relevant for the formation of mesovortices in the hurricane embryo.

The Canonical Balanced Model

To establish this fact, introduce the aspect ratio A = H/L , A ≤ 1 and the new rescaled variables

$$
\begin{aligned}
T &= At, \quad X = Ax_h, \\
w &= Aw_A, \\
(\mathrm{Ro})_A &= A\mathrm{Ro}, \\
AS_{\theta,A} &= S_\theta, \\
A\mathbf{S}_{\mathbf{u},A} &= \mathbf{S}_{\mathbf{u}}.
\end{aligned}
$$

<u>Note:</u> Horizontal Velocity,

$u = u_A(X, T)$ still has units

$[u] \cong 10\,m/s$

The Canonical Balanced Model

With these rescaling, the equations in (2.2) become

$$\begin{aligned}
\frac{D\mathbf{u}_h}{DT} + (\mathrm{Ro})_A^{-1}\mathbf{u}_h^{\perp} &= -\nabla_h p + S_{\mathbf{u},A}, \qquad (2.6)\\
A^2\frac{Dw_A}{Dt} &= -\frac{\partial p}{\partial z} + \epsilon^{-1}\theta + S_w,\\
\frac{D\theta}{DT} &= \epsilon^{-1}(-w_A + S_{\theta,A}),\\
div_X\mathbf{u}_h + (w_A)_z &= 0.
\end{aligned}$$

The same derivation can be repeated now for any A with A $\ll$ 1 to yield (1.3) as a canonical balanced model provided that $(\mathrm{Ro})^{-1}_{A}$ remains finite.

Choose $A = \varepsilon$ to have rotation important for 100 km spatial scales & 2.5 hrs.

Vertical Vorticity Dynamics

Vertically sheared horizontal flow with mass (heat) sources (VSHFS):

$$
\begin{aligned}
\frac{D\mathbf{u}_h}{Dt} + f u_h^{\perp} &= -\nabla_h p + \mathbf{S}_{\mathbf{u}}, \\
div\mathbf{u}_h + w_z &= 0, \\
wN^2(z) &= S_\theta,
\end{aligned}
\tag{1.3}
$$

Vertical Vorticity Dynamics

Use the horizontal Helmholtz decomposition

$$\mathbf{u}_h = \nabla_h \Phi + \nabla_h^{\perp} \Psi + \mathbf{b}(z,t),$$

where Ψ is the stream function, Φ is the velocity potential, and $\mathbf{b}(z,t)$ is the specified background shear.

Vertical Vorticity Dynamics

By taking curl$_h$ of (1.3), we have the **Vertical Vorticity Dynamic Equation:**

$$\frac{D\omega}{Dt} = \underbrace{(\omega + f)\,(S_\theta)_z}_{\text{stretching}} + \underbrace{(\frac{\partial}{\partial z}\mathbf{u}_h^\perp) \cdot \nabla_h S_\theta}_{\text{tilt}} + curl_h \mathbf{S}_\mathbf{u}.$$

It can be decomposed, using Helmholtz decomposition, as:

$$\frac{D\omega}{Dt} = (\omega + f)w_z - \nabla_h^\perp w \cdot \frac{\partial \mathbf{b}}{\partial z} - \nabla_h^\perp w \cdot \frac{\partial \nabla_h \Phi}{\partial z} - \nabla_h^\perp w \cdot \frac{\partial \nabla_h^\perp \Psi}{\partial z} + curl_h \mathbf{S}_\mathbf{u}.$$

We investigate this equation in this presentation.