Summer School: Geophysical Turbulence
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Turbulent transport:
Quantifying the role of coherent structures

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`Turbulence is the last great unsolved problem of classical physics'
- Richard Feynman (or maybe Einstein, or Heisenberg, or Sommerfeld)

`I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.'
- Horace Lamb (or maybe Einstein, or Heisenberg)

$\Rightarrow$ Turbulence is important and hard!

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What is “the turbulence problem”?

what are the characteristics of small-scale turbulent motions, how do these depend on the properties of the large-scale motions from which they derive, and knowing them, how can we model the transport of scalar and vector quantities, such as concentration, energy, or momentum?

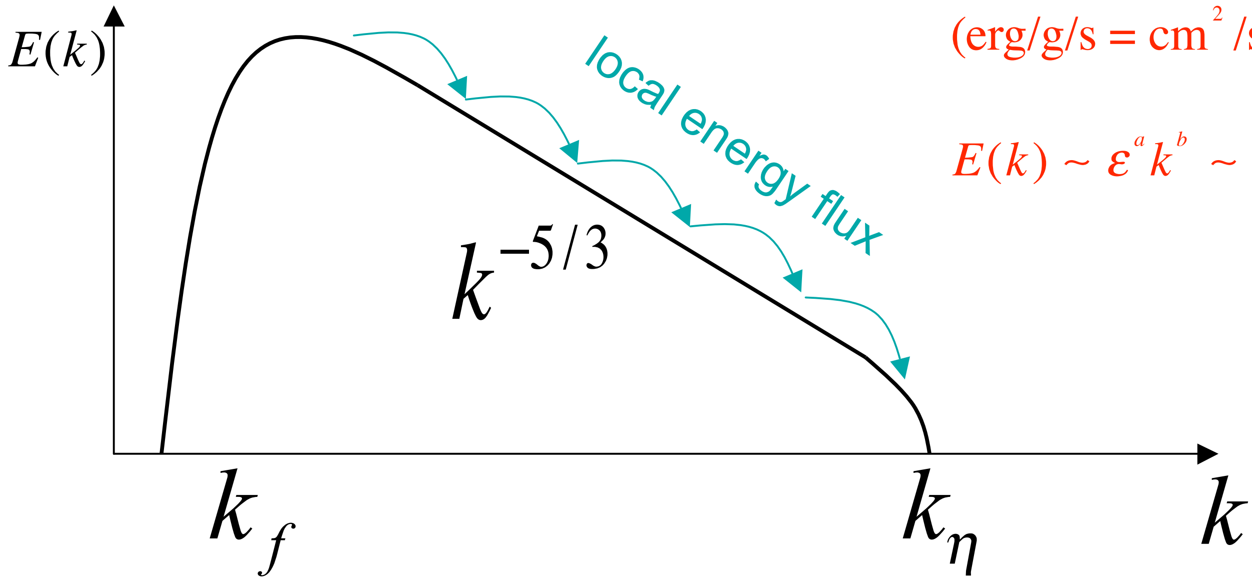
Two components to this problem:

Transport – fluid parcels, “mixing” on continuum scale (forward and inverse cascade)

Dissipation – homogenization within/between parcels, mixing on molecular scale

Spectra: all phase information lost
$\rightarrow$ model of transport in spectral space

Kolmogorov (1941):



$E(k)\,\mathrm{d}k \equiv$ energy/mass in interval dk $(\mathrm{cm}^2/\mathrm{s}^2)$

$\varepsilon \equiv$ energy/mass dissipation/injection $(\mathrm{erg/g/s} = \mathrm{cm}^2/\mathrm{s}^3)$

$E(k) \sim \varepsilon^a k^b \sim \varepsilon^{2/3} k^{-5/3}$

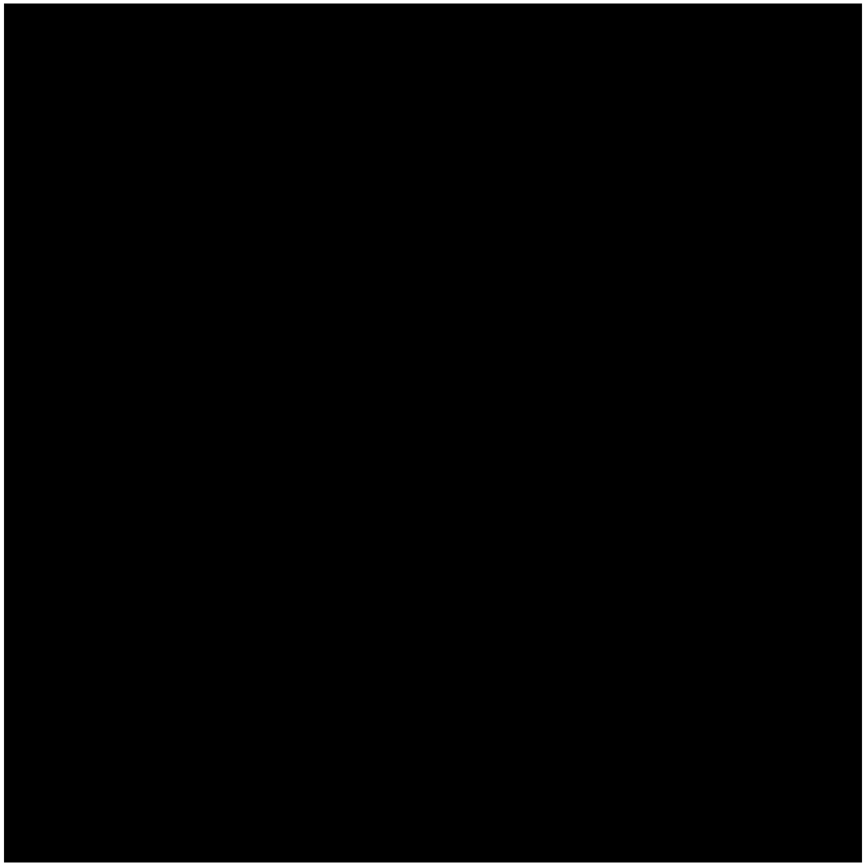
- Fluid instabilities produces ever smaller scales from large scale motions

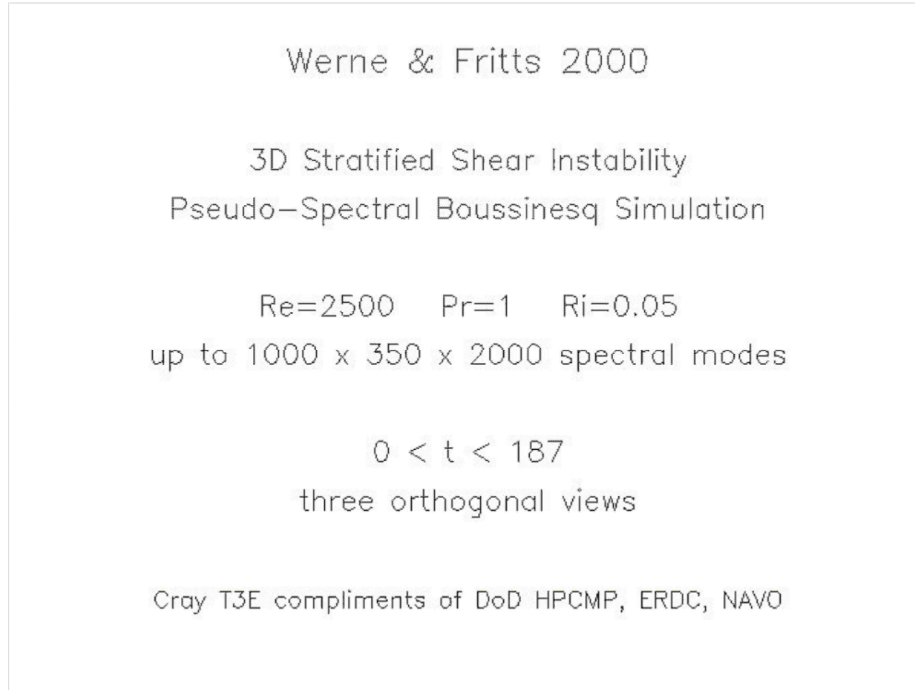
  Big whirls have little whirls, which feed on velocity, and little whirls have lesser whirls, and so on to viscosity (Richardson 1922 after Jonathan Swift)

- In steady state, energy at any size scale depends only on injection/dissipation rate and size scale -- spectral slope by dimensional analysis

Coherent Structures: all about phasing

$\rightarrow$ transport in physical space?





How to objectively define coherent structures?
How to use them in a transport model?

Molecular transport (Maxwell 1866):

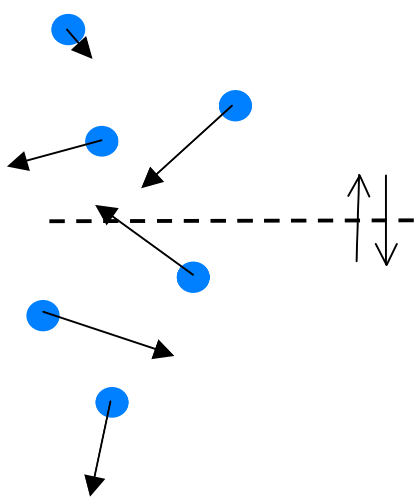
Random molecular motions yield

- viscosity by the transport of net momentum
- conductivity by the transport of net energy
- diffusion by the transport of molecular identity

Chapman – Enskog method:

properly takes into account nonequilibrium particle distribution functions due to the presence of the background variations – transport occurs because $f \neq f_0$ (distribution is nonMaxwellian)

Assume (lowest order): $f = f_0$



$$\Phi(u_z) = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(\frac{-mu_z^2}{2kT}\right)$$

$$N_+ = \int_0^{\infty} \Phi(u_z)\, u_z du_z$$

$$N_- = \int_{-\infty}^{0} \Phi(u_z)\, u_z du_z$$

$$\Downarrow$$

$$N_+ = N_- = \frac{1}{4} N \langle u \rangle$$

Transport of x-momentum:

$$z_0 + \frac{2}{3}\lambda$$

$$\rho_+ \approx \frac{1}{4} N \langle u \rangle \left[m \left(u_x(z_0) - \frac{2}{3}\lambda \frac{\partial u_x}{\partial z}\bigg|_0 \right) \right]$$

$$z_0$$

$$\rho_- \approx \frac{1}{4} N \langle u \rangle \left[m \left(u_x(z_0) + \frac{2}{3}\lambda \frac{\partial u_x}{\partial z}\bigg|_0 \right) \right]$$

$$z_0 - \frac{2}{3}\lambda$$

$$\Downarrow$$

$$F_\rho = \frac{1}{3} N m \langle u \rangle \lambda \frac{\partial u_x}{\partial z}\bigg|_0 \Leftarrow$$

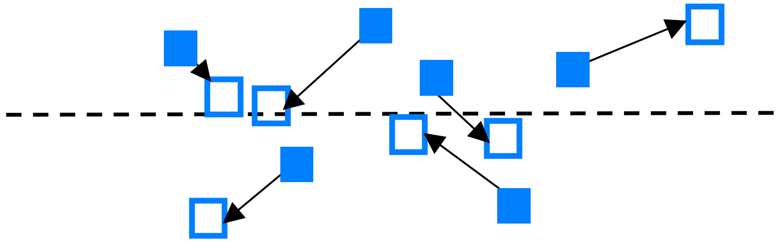
$$\Leftarrow$$

Dynamic Viscosity

$$\mu = \frac{1}{3} N m \langle u \rangle \lambda$$

Mixing length theory of turbulent transport (Prandtl 1925):

$$\mu = \frac{1}{3} Nm \langle u \rangle \lambda$$



The turbulence problem:

How to determine the velocity fluctuations from knowledge of the large scale flow (the closure problem).

How to mix – provided by elastic collisions in molecular transport (understanding the interface between continuum and molecular dynamics)

Prandtl's answer (as quoted by Bradshaw 1974)

- typical values of the fluctuating velocity components are each proportional to $l\,\partial U/\partial z$ where l is the mixing length (*Mischungsweg*)
- l "may be considered as the diameter of the masses of fluid moving as a whole in each individual case; or again, as the distance traversed by a mass of this type before it becomes blended in with neighboring masses"
- l is "somewhat similar, as regards effect, to the mean free path in the kinetic theory of gases"

–

Turbulent viscosity:

$$\mu_T = \rho l^2 \frac{\partial U}{\partial z}$$

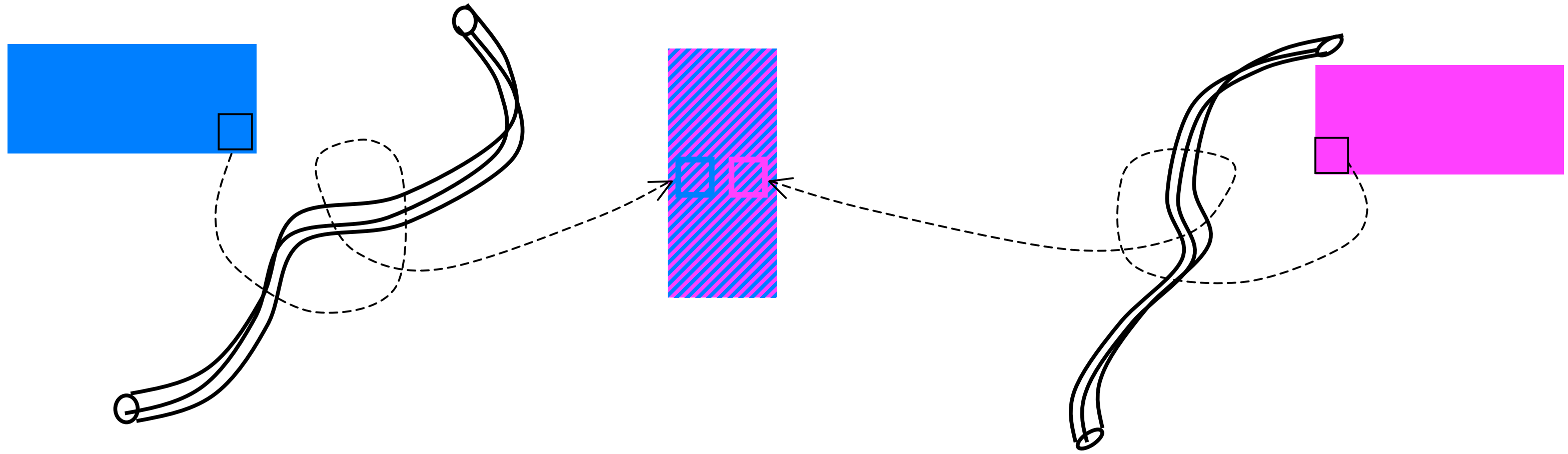
Shear stress:

$$\tau = \rho \left(l \frac{\partial U}{\partial z} \right)^2$$

1. If turbulent motions were strictly random, turbulent viscosity would be proportional the average magnitude of the velocity perturbations
    – It is the coherent motions that are key, phase relations are critical

2. Turbulent transport requires a "mixing length" – turbulent mean free path
    – Must understand the interface between continuum and molecular dynamics to understand where in the flow the transported quantity is deposited

→ Formulate a statistical description of turbulent coherent structures, Lagrangian dynamics in their presence, and mixing between parcels



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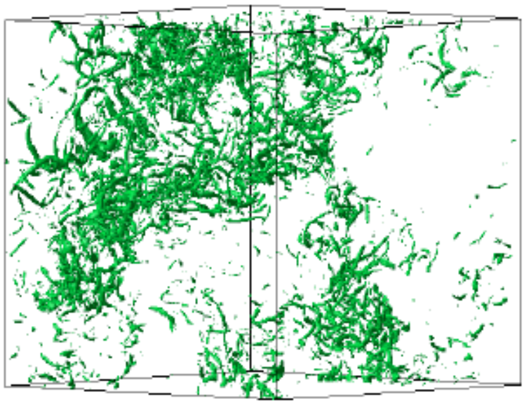
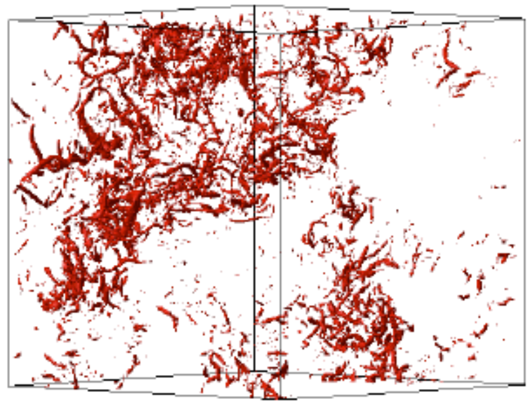
Why the emphasis on coherent structures?

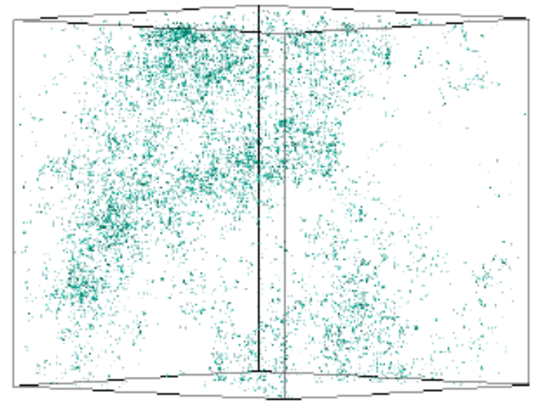
Example: Farge, M., Pellegrino, G., & Schneidr, K. 2001, Phys Rev Lettt 87, 054501
Yoshimatsu, K., et al. 2007, Comp Phys and New Perspectives in Turb, p. 235



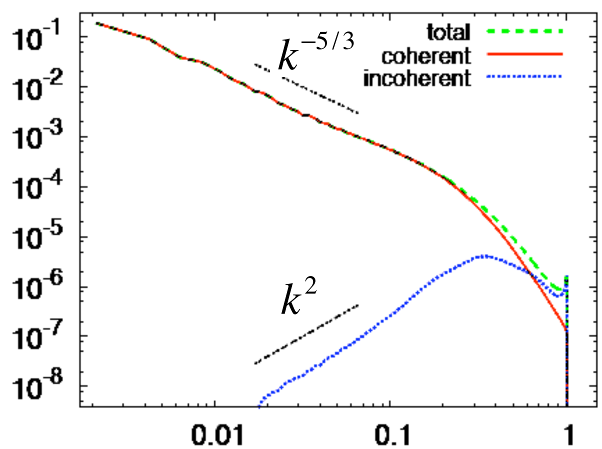
Isosurfaces: Total vorticity



Coherent vorticity



Incoherent vorticity



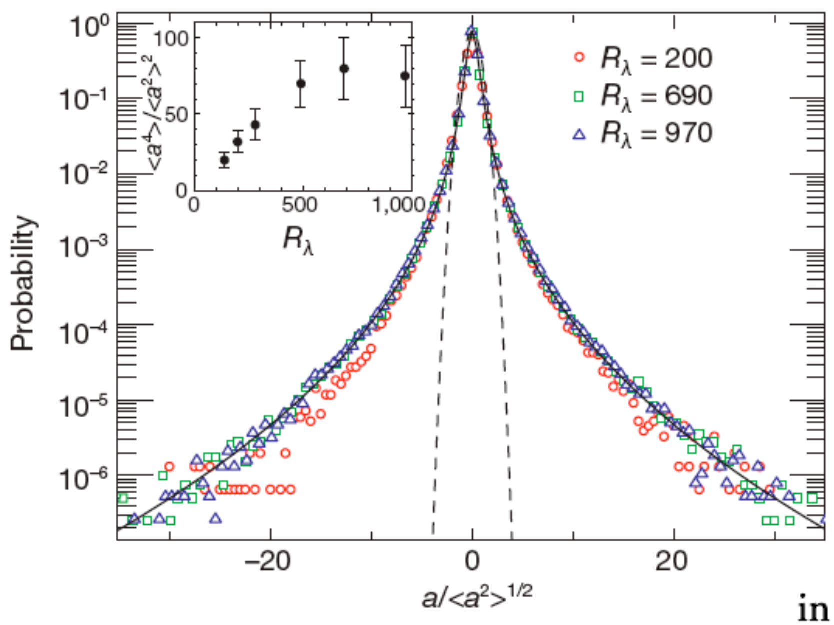
- 1024^3 incompressible homogeneous isotropic turbulence (Earth Simulator)
- Wavelet filtered (largest 2.9% coefficients retained)
- Filtered flow contains 99.7% of the energy and 81.0% of the enstrophy
- Strong scale-by-scale correlation between velocity field induced by coherent vortices and the total velocity

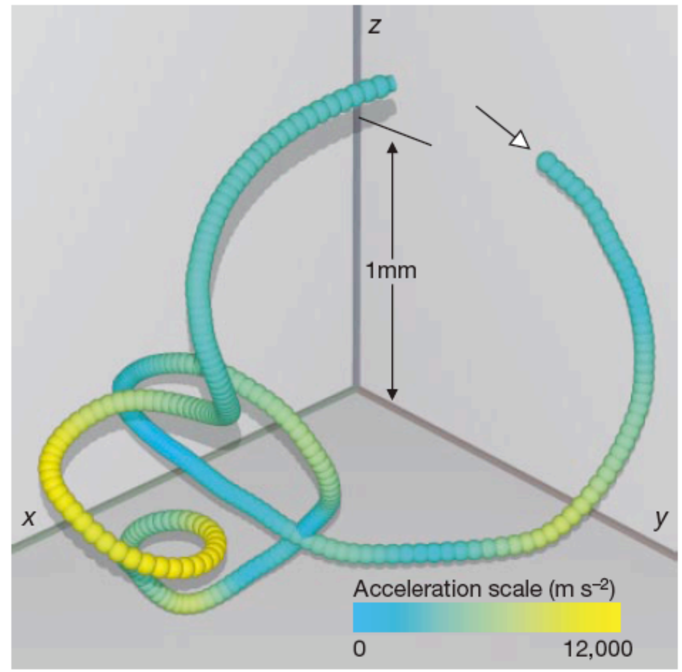
Lagrangian statistics in turbulent flows:

$\Delta_\tau v(t) = v(t+\tau) - v(t)$

- Velocity measurements follow a Gaussian distribution
- For small temporal increments velocity-difference PDFs are highly nonGaussian

La Porta et al. 2001, Nature, 409, 1017





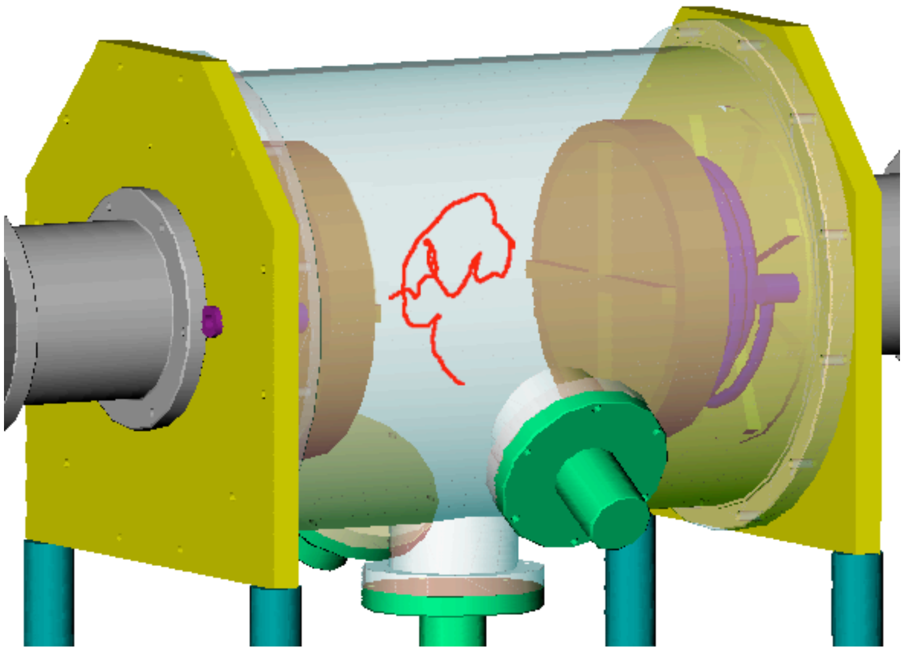
Laboratory water flow with Reynolds number of up to 63000

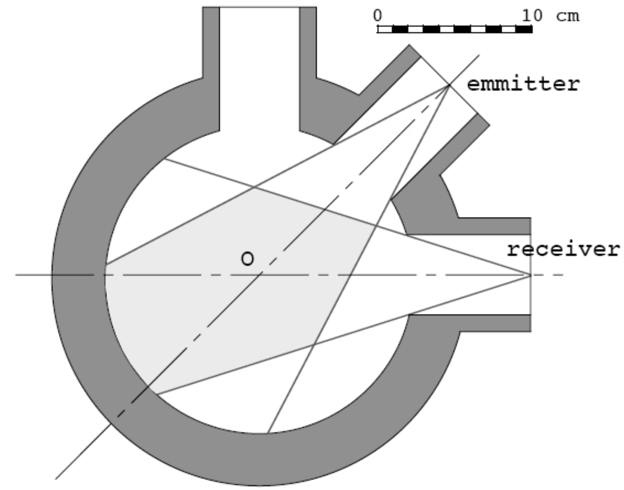
Acceleration up to 1500g measured

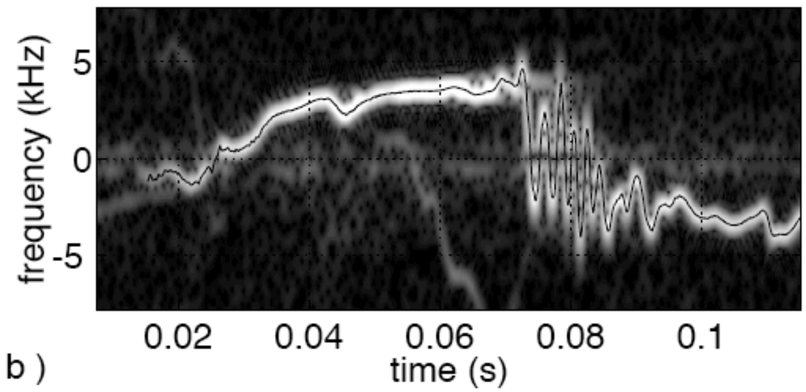
in practical flows with varying Reynolds numbers[3,21,22]. Our research also has surprising implications for everyday phenomena. For instance, a mosquito flying on a windy day (wind speed 18 km h^{-1} and an altitude of 1 m) would experience an r.m.s. acceleration of 15 m s^{-2}. But given the extremely intermittent nature of the acceleration, the mosquito could expect to experience accelerations of 150 m s^{-2} (15 times the acceleration of gravity) every 15 seconds. This may explain why, under windy conditions, a mosquito would prefer to cling to a blade of grass rather than take part in the rollercoaster ride through the Earth's turbulent boundary layer[23]. □

LASP

Mordant, N., Lévèque, E., & Pinton, J.-F.
Phys Fluids 2006



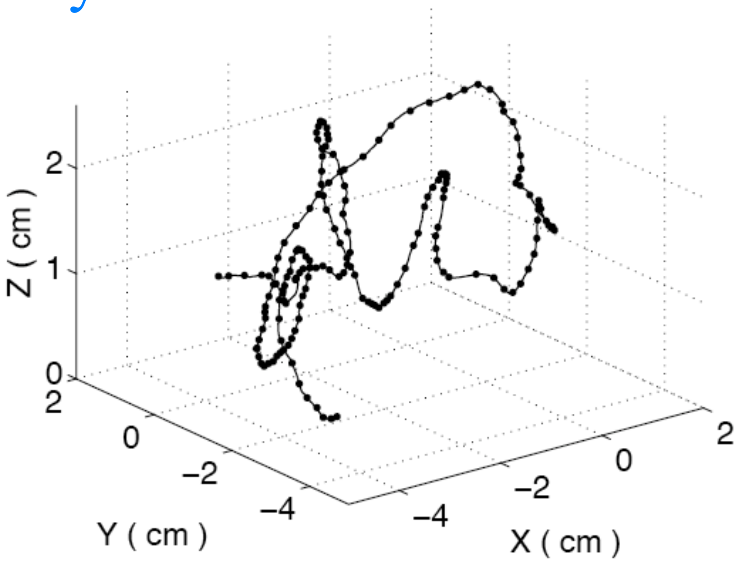


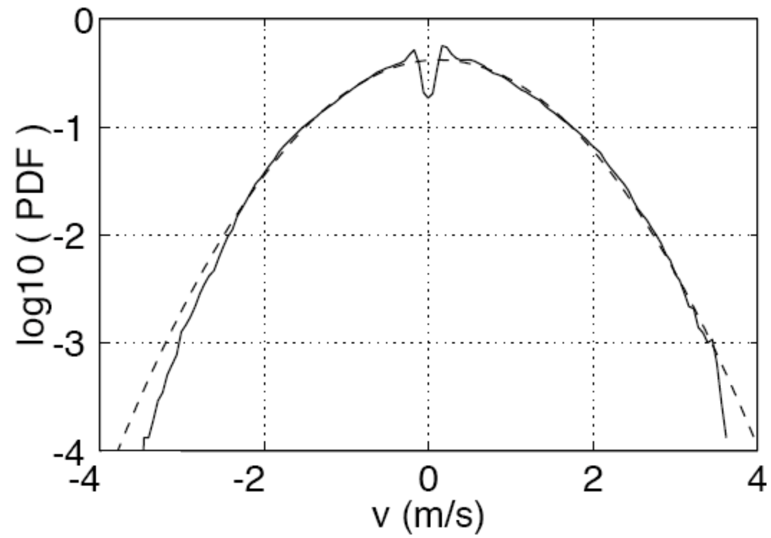


b)

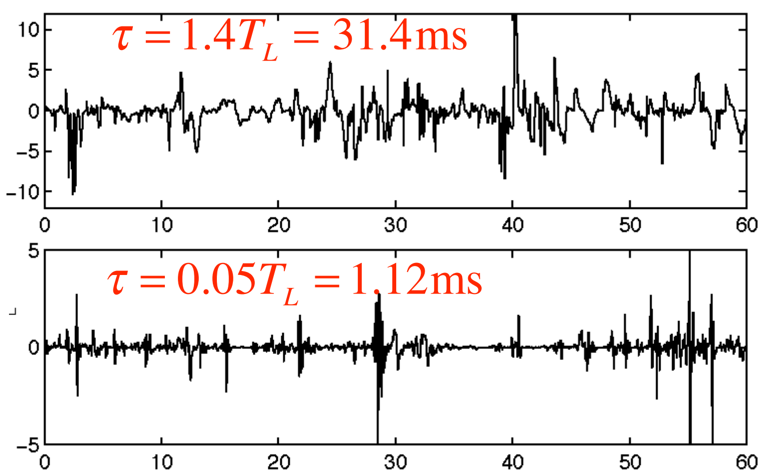
- Tank radius 10cm (9 liter volume) filled with water
- Counter rotating disks (9.5cm diameter, 18cm separation)
- 250μm diameter 1.06 g/cm^3 tracer particles (smaller than Taylor microscale)
- Beam width at center of volume (no mean flow) 10 cm
  (larger than the integral scale -- sample full range of Lagrangian motions)
- Total number of tracers small (less than two in sample volume at one time)

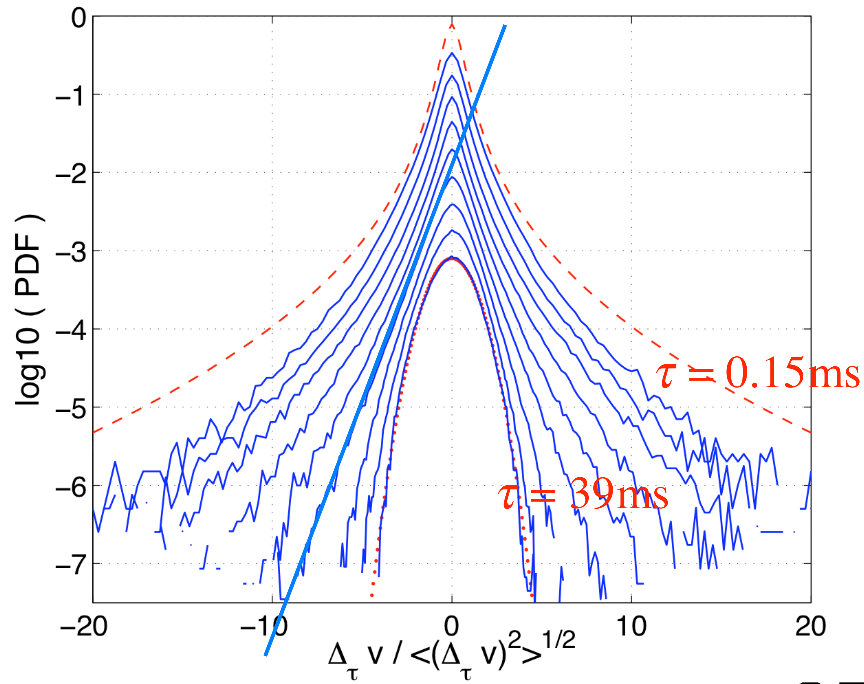
Mordant, N., Lévèque, E., & Pinton, J.-F.
Phys Fluids 2006





$\Delta_\tau v(t) = v(t+\tau) - v(t)$





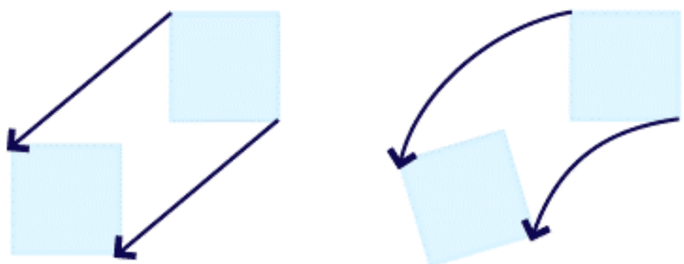
$T_L \simeq 22.4$ms $\quad T_L \equiv$ Lagrangian velocity autocorrelation time

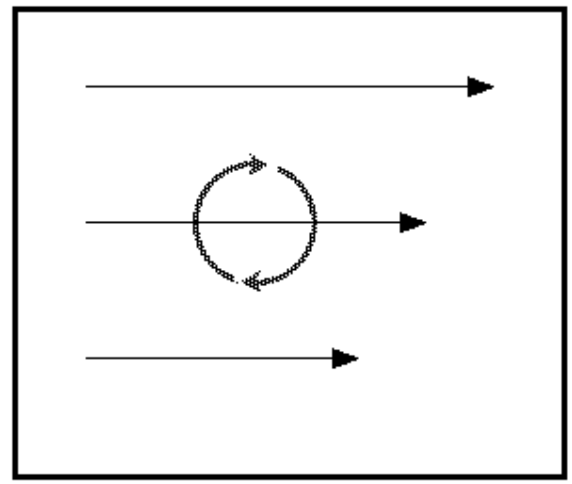
Vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{P}{\rho}\right) + \nu \nabla^2 \mathbf{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$$





Point vortex flow:
(two-dimensional flow)

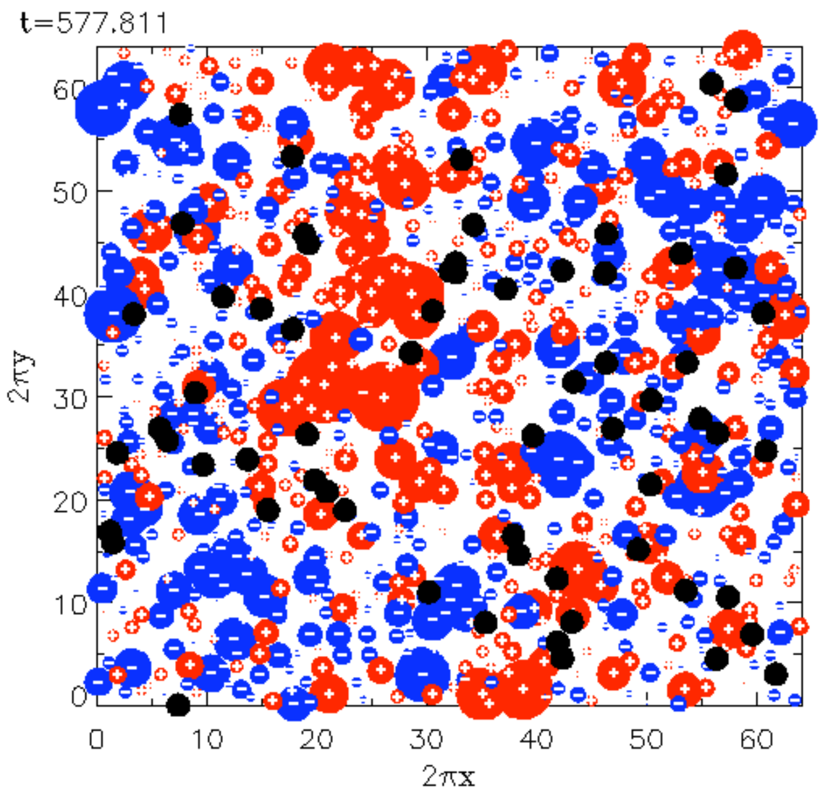
$$\mathbf{u}(\mathbf{r}) = \frac{\Gamma}{2\pi r}(\hat{z} \times \hat{r})$$

$$u_\theta \sim 1/r$$
$$u_r \sim 0$$

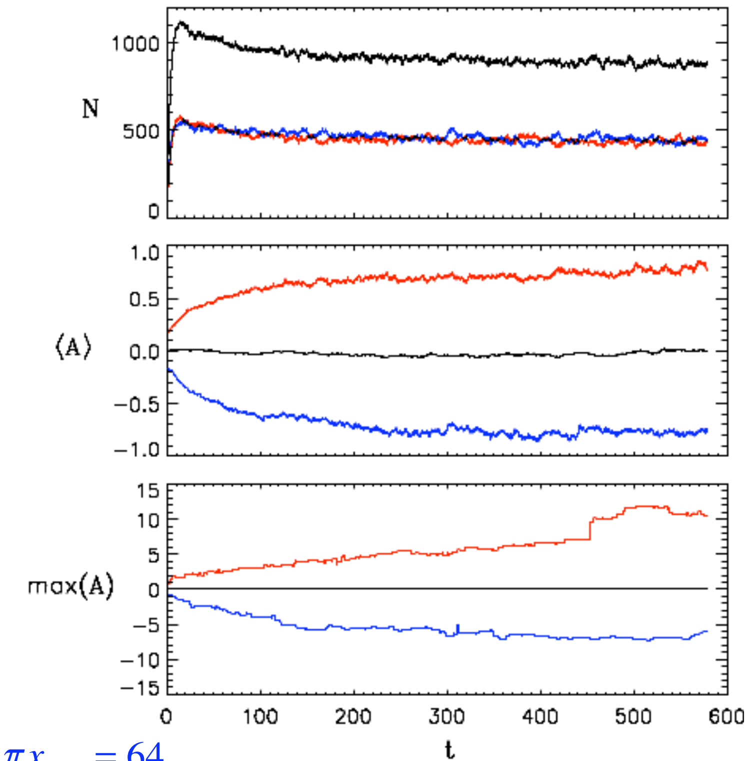
Point vortex simulations:

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi |\mathbf{x} - \mathbf{x}_k|} \left(\hat{z} \times \left(\widehat{x - x_k} \right) \right)$$

Each vortex advected in the flow field of all the others.
Reduction of continuum equations to n-body interactions.
Merger of close vortices
Stirring by vortex creation



Every 250 time steps shown



$2\pi x_{max} = 64$

$2\pi r_{min} = 1$

$\Gamma_{rms} = 0.5$

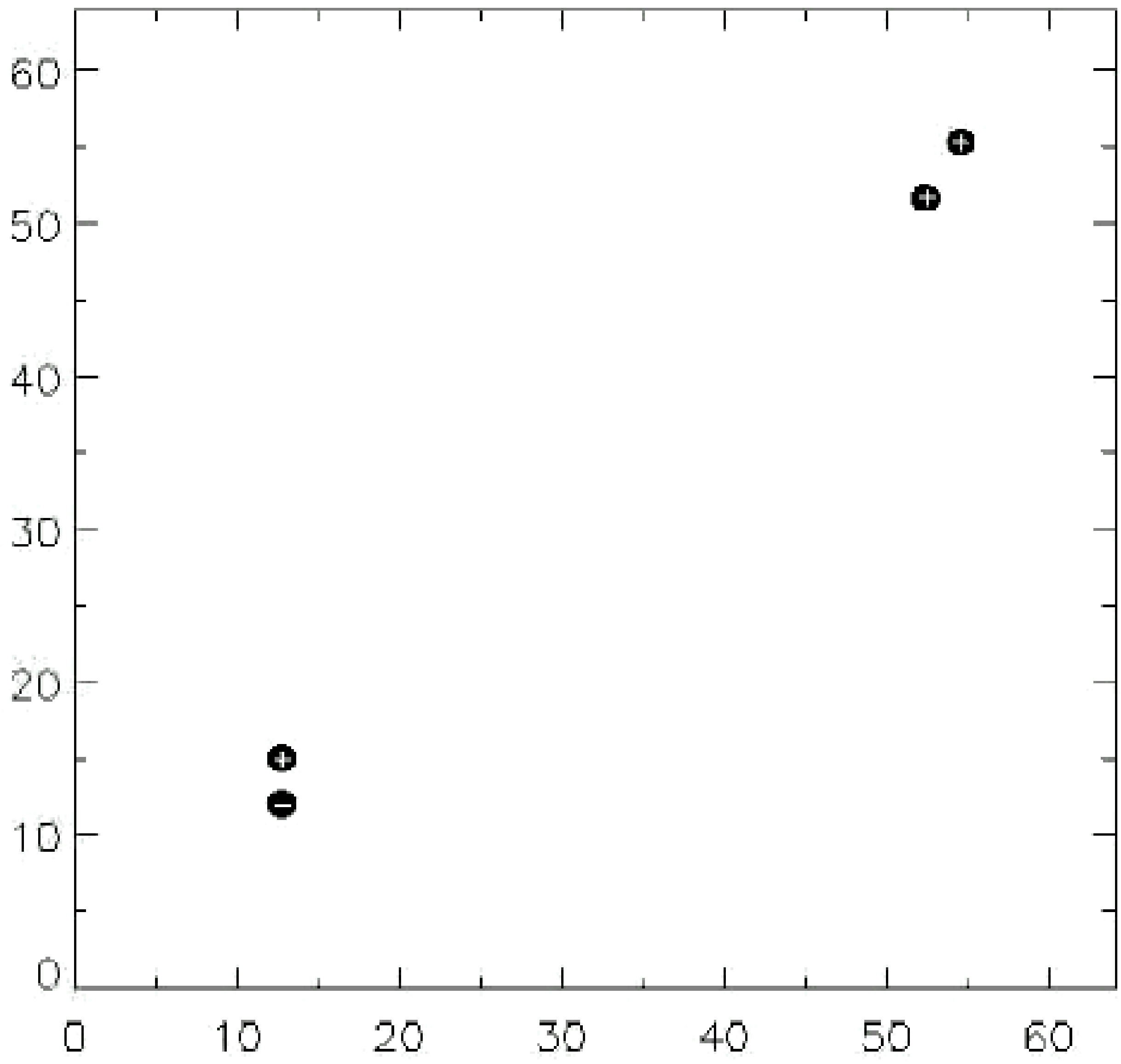
256 new vortices per unit time interval

$dt = 10^{-3}\tau$

$\Downarrow$

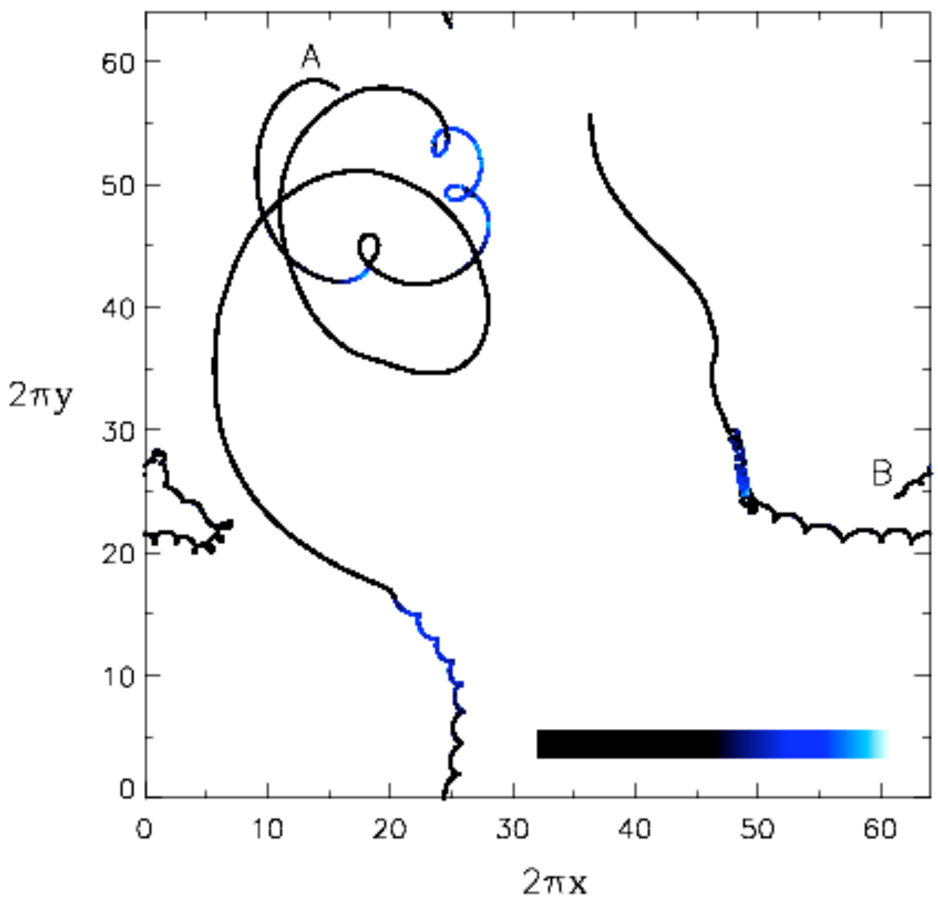
$u_\theta = \Gamma$ at $r = r_{min}$

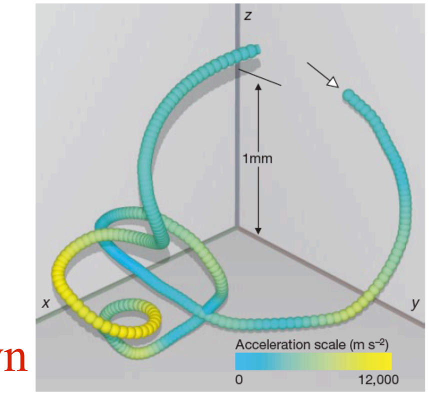
For $\Gamma = 1$ one orbit takes 2π time units at $r = r_{min}$



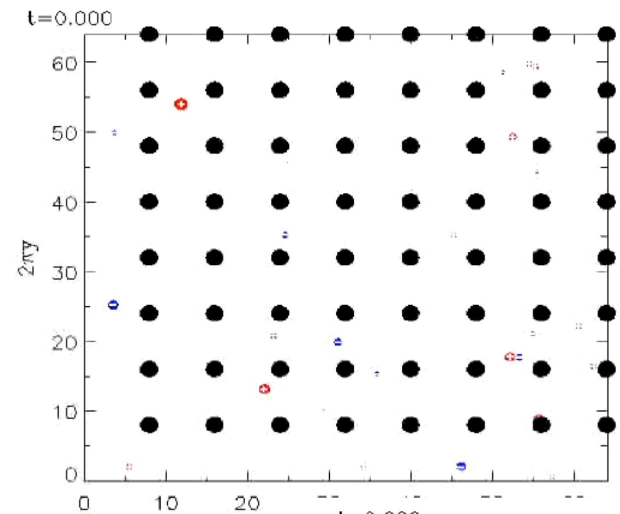
- Like sign vortices orbit
- Oppositely signed vortices translate
- Scattering leads to preferential merger of oppositely signed pairs

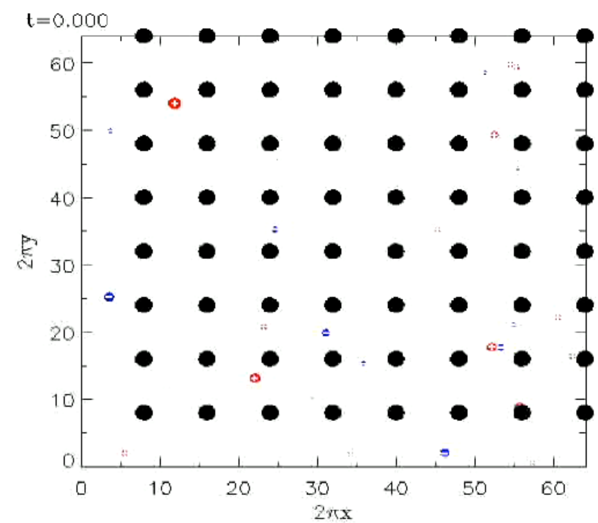
Particle trajectories in point – vortex model:





Every 125 time steps shown



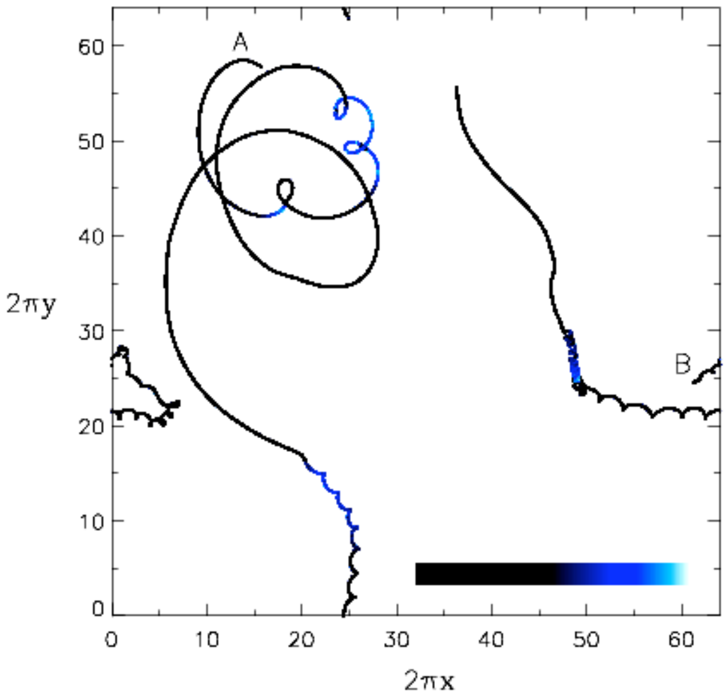


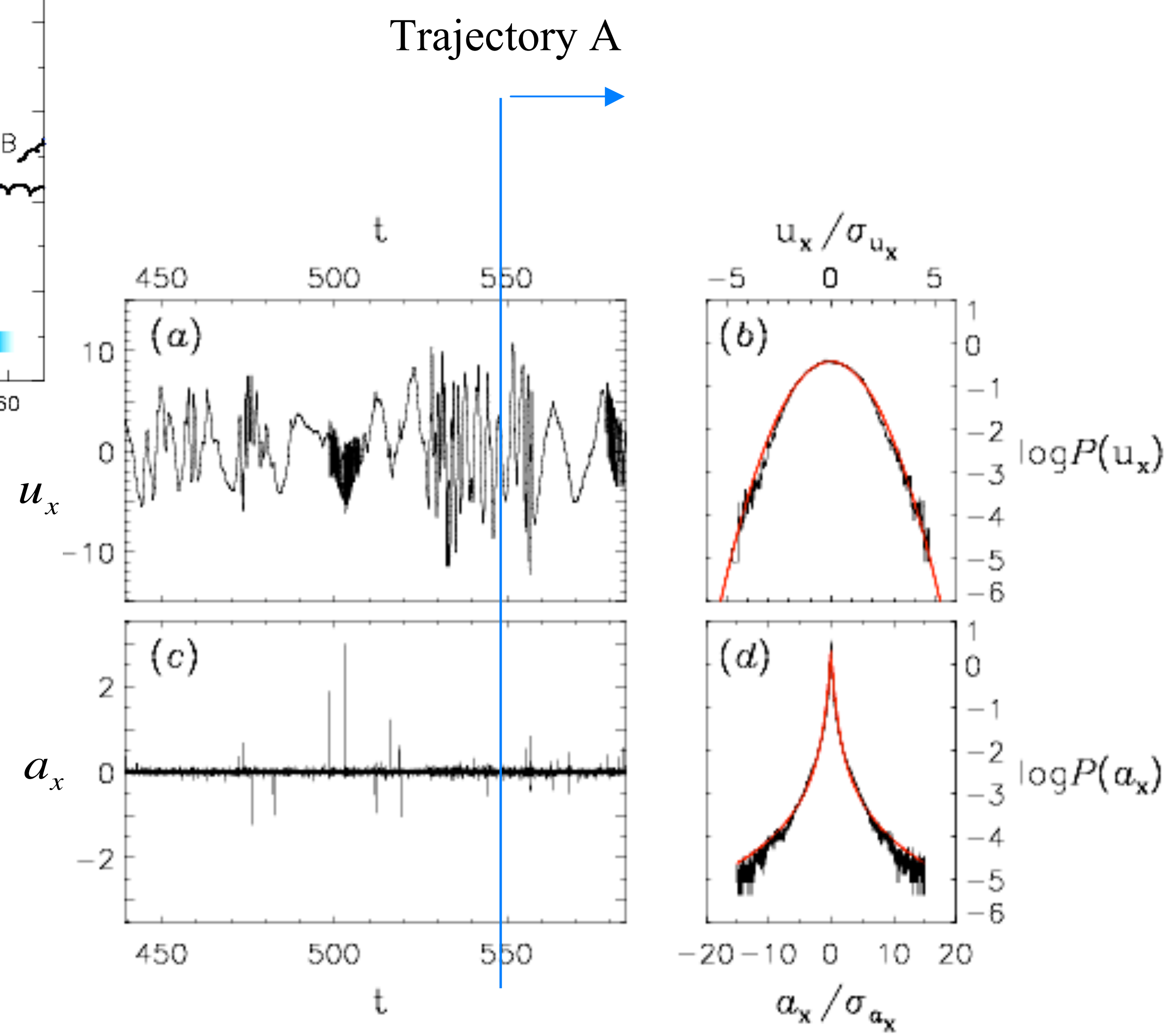
'Trapping events' (Biferale et al. 2005) are common, have range of frequencies, and are best seen at high temporal resolution (small temporal increment)

Every 500 time steps shown

LASP

Velocity and acceleration distributions in point – vortex model:



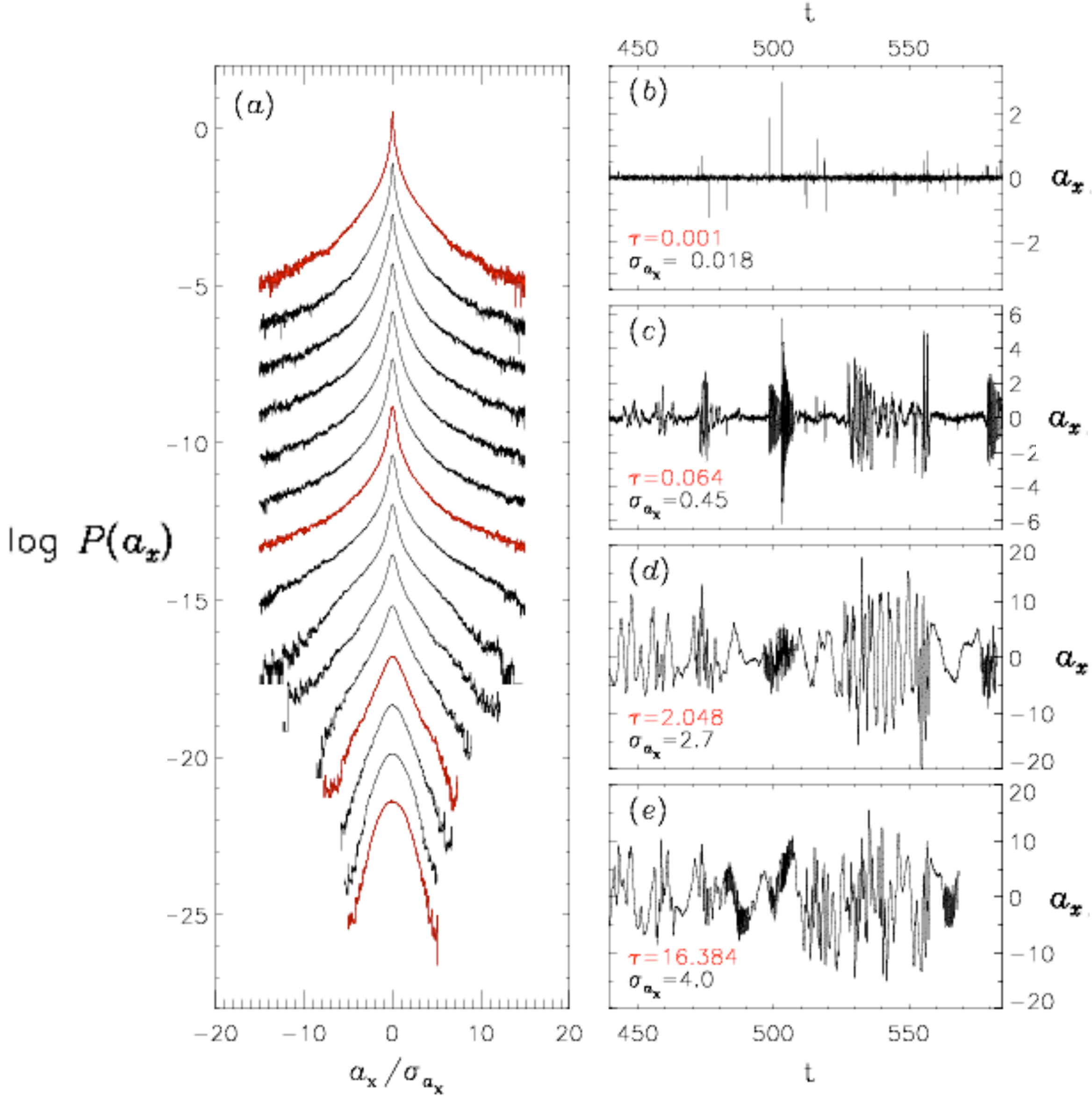


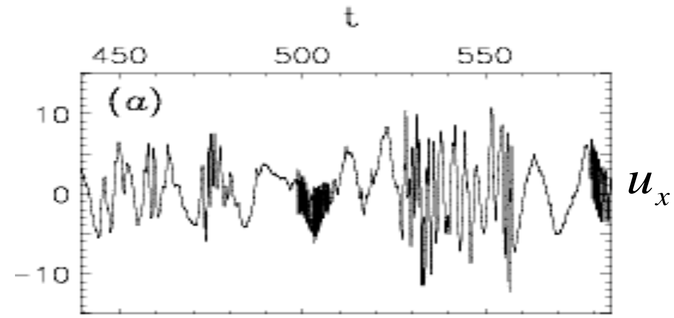
$$\sigma_{u_x} = 2.8$$

$$\sigma_{a_x} = 0.018$$

Dependence on temporal increment τ:

$$\Delta_\tau v(t) = v(t+\tau) - v(t)$$





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Bivariate transformation of random variables:

Let x and y be independent random variables with probability densities $P(x)$ and $P(y)$ and joint probability density $P_{xy}(x,y) = P(x)P(y)$

Let $u = f(x,y)$ and $v = g(x,y)$ be functions of the random variables with inverse functions $x = h_1(u,v)$ and $y = h_2(u,v)$

Then the joint probability density of u and v is $P_{uv}(u,v) = P_{xy}(h_1,h_2)\begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{vmatrix}$

and $P(u) = \int P_{uv}(u,v)\, dv$ and $P(v) = \int P_{uv}(u,v)\, du$

Example:

Consider two Gaussianly distributed independent random variables each with a Mean value of zero and variance equal to one:

$$P_{xy}(x,y) = P(x)P(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

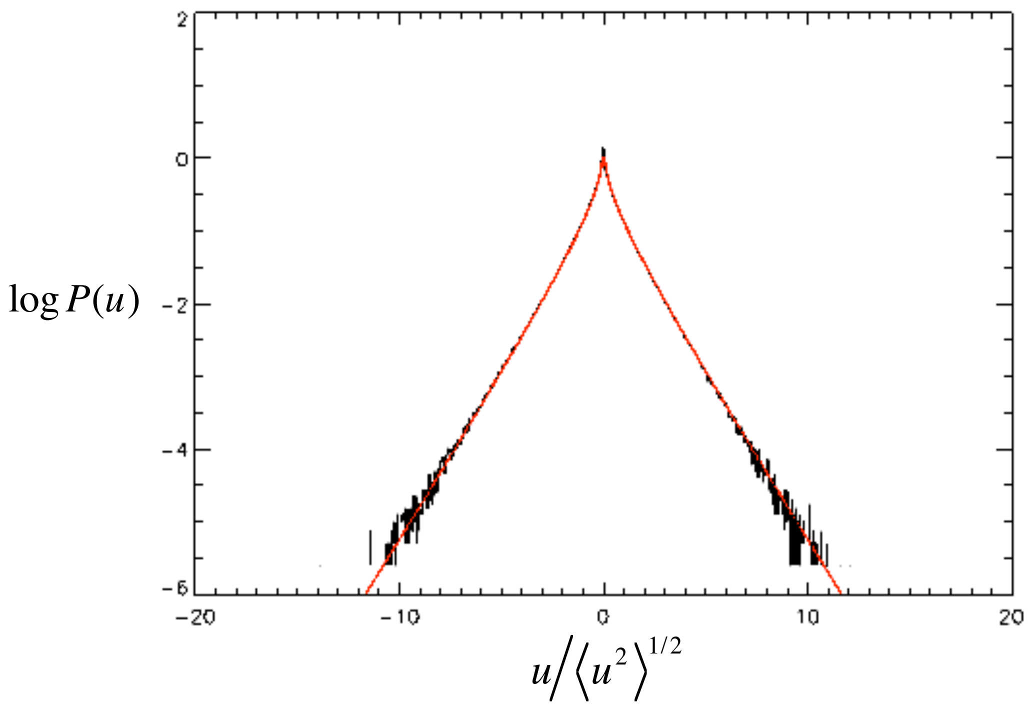
To derive the probability density of their product, let $u = xy$ and $v = y$ with inverses $x = u/v$ and $y = v$

The joint probability density of u and v is then

$$P_{uv}(u,v) = \frac{1}{2\pi} e^{-(u^2/v^2+v^2)/2} \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2\pi v} e^{-(u^2/v^2+v^2)/2}$$

and integrating over "dummy" function v yields $P(u) = \frac{1}{\pi} K_0\left(\sqrt{u^2}\right)$ $u = xy$

K_0 is the lowest order modified Bessel function of the second kind



Monte Carlo vs. analytic probability density for the Gaussian product $N_1 N_2$

Evidence for vortical motion:

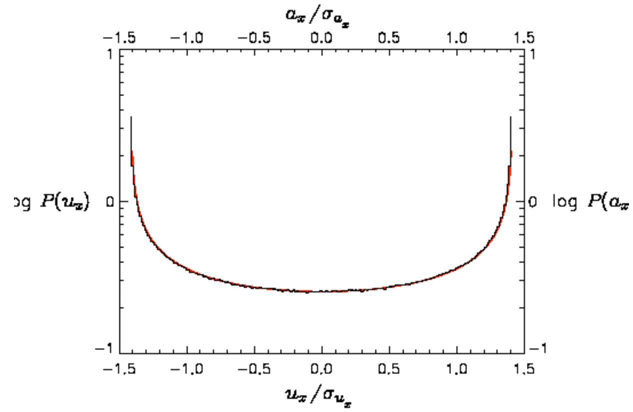
$$\mathbf{u} = U_0 / r \ \hat{\theta}$$

$$u_x = \frac{U_0}{r} \sin\theta$$

$P(\theta)$ is uniform with θ

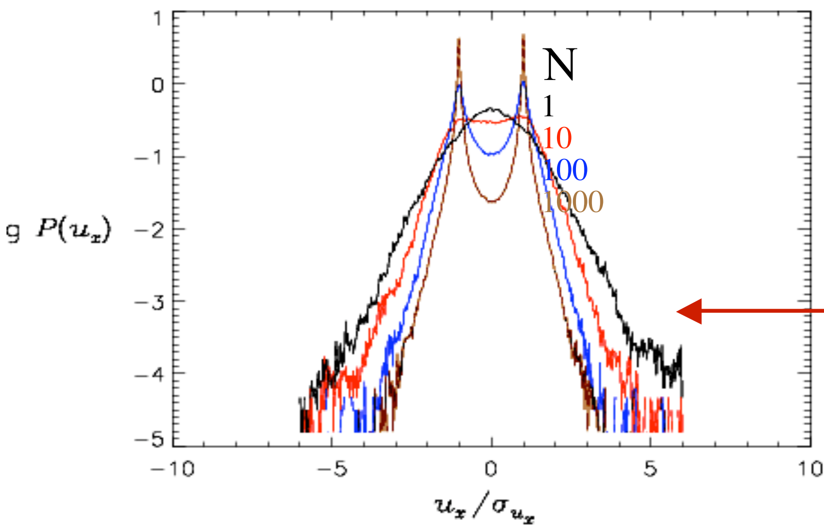
so at fixed radial distance is

$$P(u_x) \propto \frac{1}{\cos\theta}$$



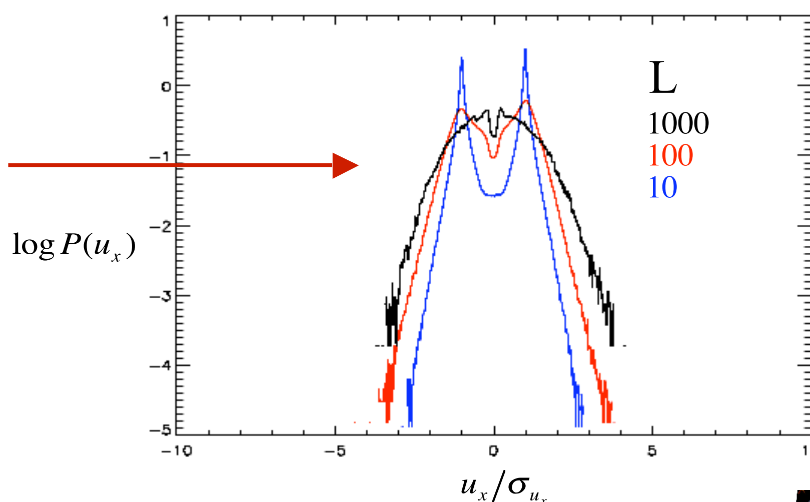
Note: Probability density diverges but probability of observing any value is finite

$$\int P(u_x)\ du_x = \int U_0\ d\theta$$



Point vortex model

Laboratory data



Evidence of vortical motion
OR
The projection of any randomly oriented vector uniformly distributed in direction

Velocity around a single point – vortex:

When radial distance to vortex is sampled randomly in the plane:

$$\mathbf{u} = U_0 / r \ \hat{\theta}$$

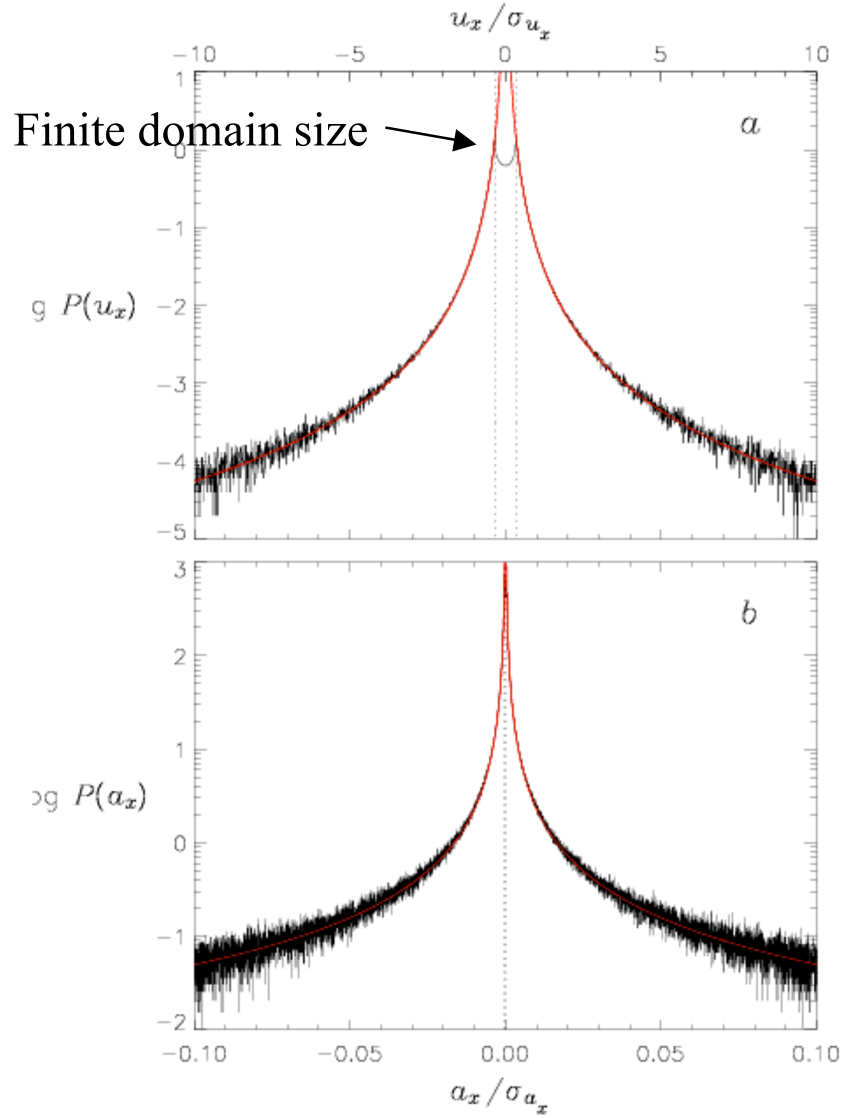
$$u_x = \frac{U_0}{r}\sin\theta \qquad a_x = \frac{U_0^2}{r^3}\tau\cos\theta \ \text{ for small } \frac{U_0}{r^2}\tau$$

$U_0 = \text{constant}$

$$P(r) \propto r \qquad P(\theta) \text{ uniformly distributed}$$

Bivariate transformation of random variables:

$$\Longrightarrow \quad P(u_x) \propto \frac{1}{u_x^3} \qquad P(a_x) \propto \frac{1}{a_x^{5/3}}$$



Velocity in field of randomly placed point – vortices:

Keeping only nearest neighbor contribution:

$$u_x = \frac{U_0}{r}\sin\theta$$

$$P(r) = 2\pi n r e^{-\pi n r^2}$$

U_0 = constant

$P(\theta)$ uniformly distributed

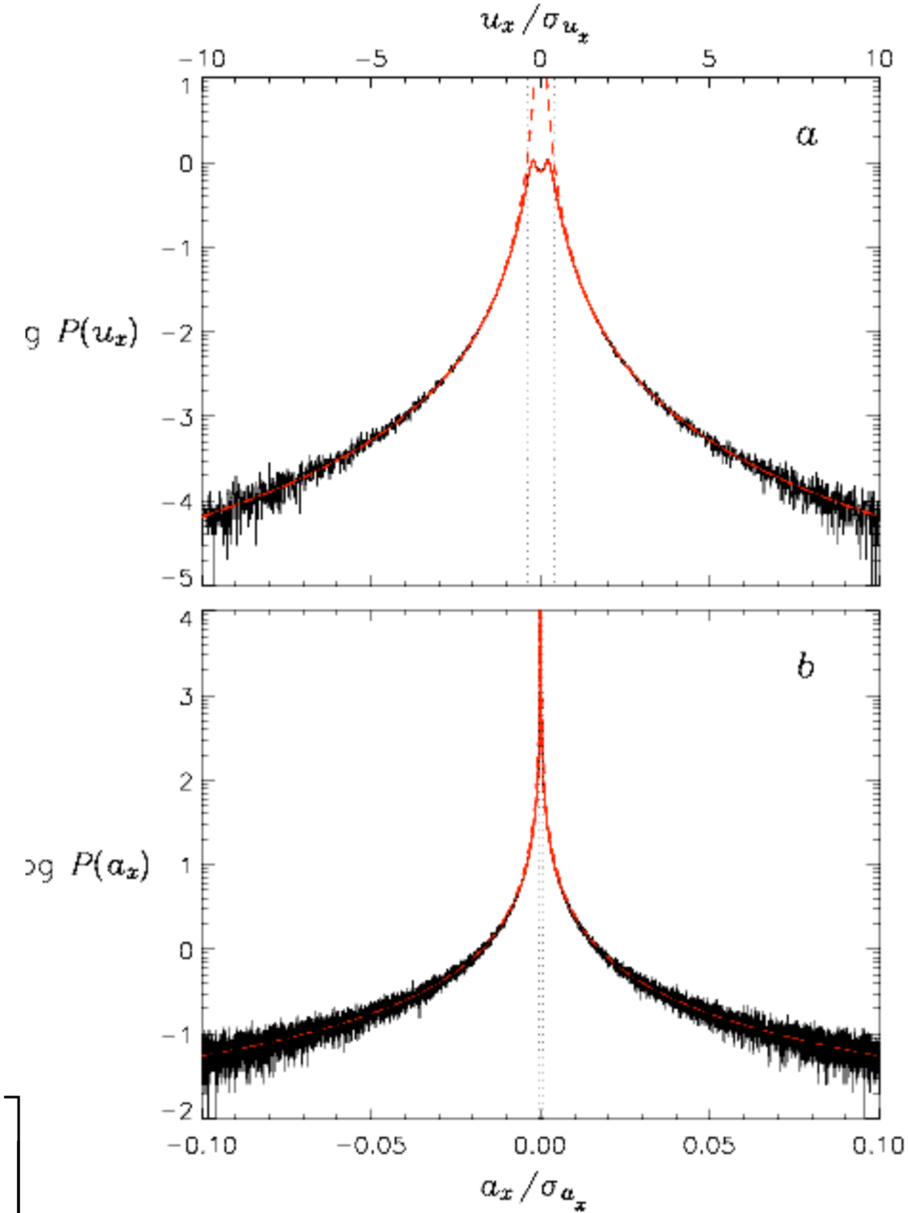
- nearest neighbor distance r
- n is vortex field number density

$$P(u_x) = \frac{n\pi}{u_x^3} e^{-\frac{\pi n}{2u_x^2}} \left[I_0\left(\frac{\pi n}{2u_x^2}\right) - I_1\left(\frac{\pi n}{2u_x^2}\right) \right]$$

modified Bessel functions of integer order

$$P(u_x) \propto \frac{1}{u_x^3}\left[u_0 + \frac{u_1 n}{u_x^2} + \frac{u_2 n^2}{u_x^4} + \cdots \right]$$

$$P(a_x) \propto \frac{\delta t^{2/3}}{a_x^{5/3}}\left[a_0 + a_1 n \left(\frac{\delta t}{a_x}\right)^{2/3} + a_2 n^2 \left(\frac{\delta t}{a_x}\right)^{4/3} + \cdots \right]$$



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Velocity in field of randomly placed point – vortices of random amplitudes:

$$P(r) = 2\pi n r e^{-\pi n r^2}$$

nearest neighbor as before

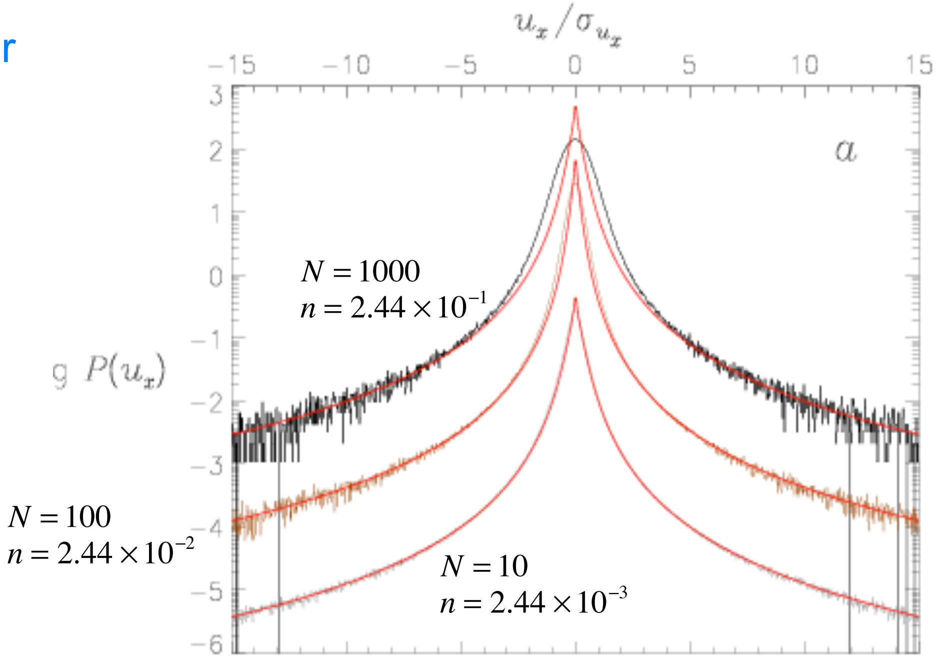
$P(\theta)$ uniformly distributed

$$P(U_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-U_0^2/2\sigma^2}$$

Gaussianly distributed amplitudes

$$u_x = \frac{U_0}{r}\sin\theta$$

$\Downarrow$



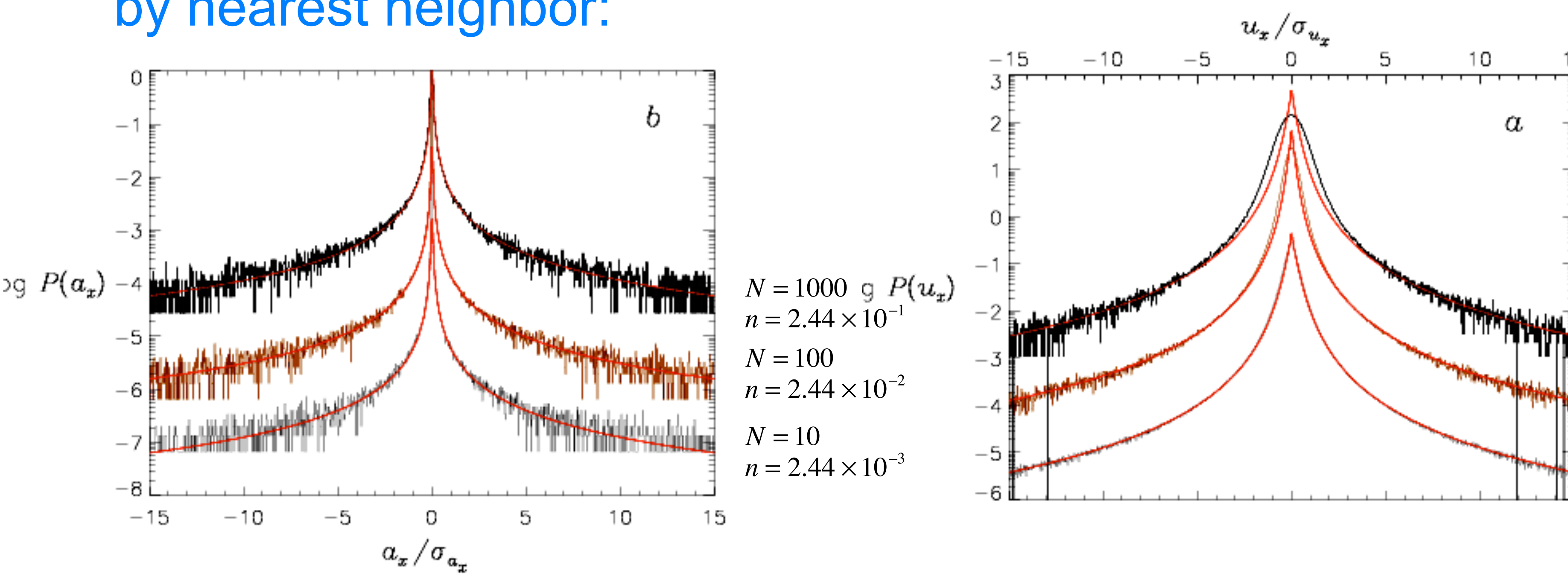
$$P(u_x) = \frac{2}{\pi^3}\frac{1}{\sqrt{u_x^2 + 2\pi n\sigma^2}}\left[K\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right) - E\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right)\right]$$

where K and E are the complete elliptic integrals of the first and second kind

Two important physical contributions to the velocity difference:

Advection over temporal increment τ by nearest neighbor:

Creation of new vortices in domain:

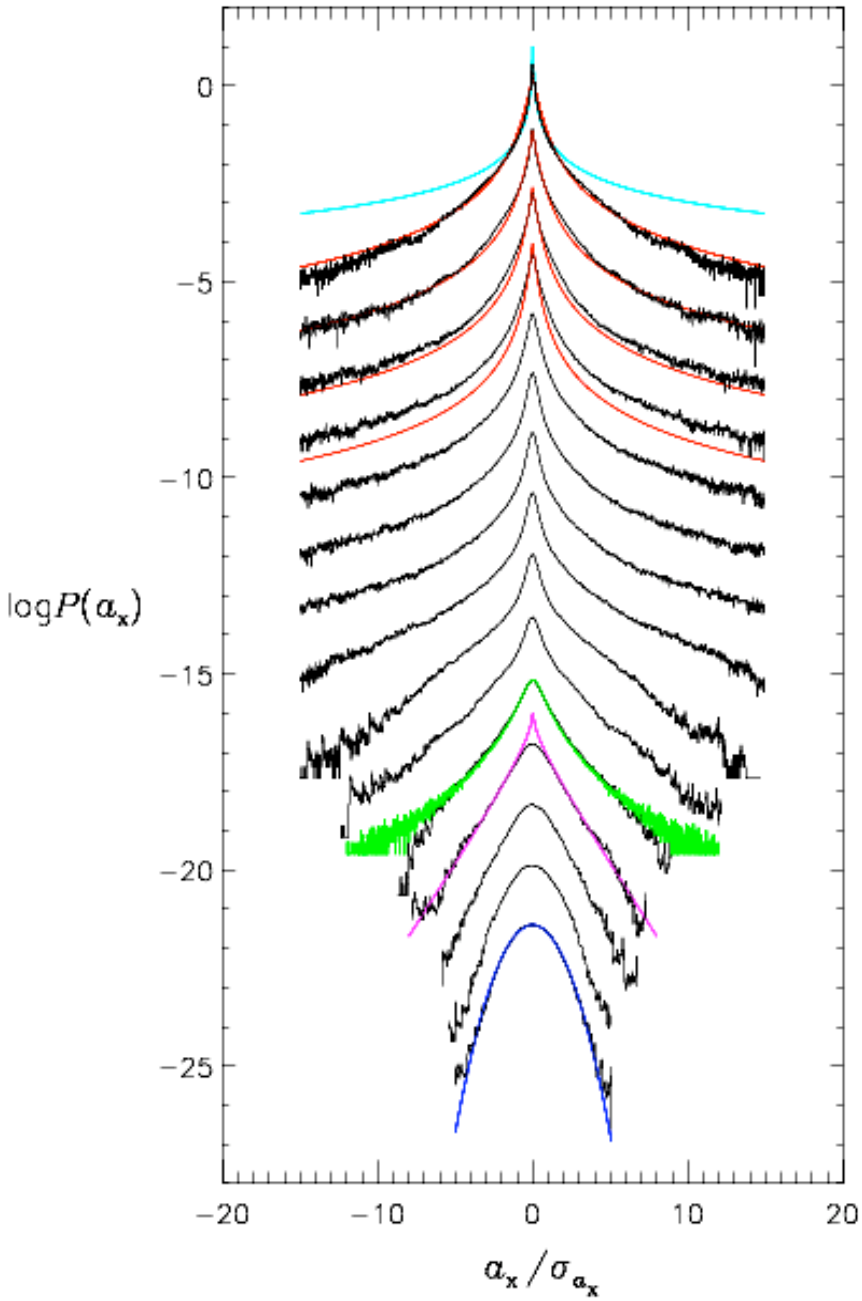


$$P(a_x) \propto \frac{\delta t^{2/3}}{a_x^{5/3}}\left[a_0 + a_1 n\left(\frac{\delta t}{a_x}\right)^{2/3} + a_2 n^2\left(\frac{\delta t}{a_x}\right)^{4/3} + \cdots\right]$$

Velocity difference in field of randomly placed point – vortices

$$P(u_x) = \frac{2}{\pi^3}\frac{1}{\sqrt{u_x^2 + 2\pi n\sigma^2}}\left[K\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right) - E\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right)\right]$$

Velocity in field of randomly placed point – vortices of random amplitudes



$$P(a_x) \propto \frac{\delta t^{2/3}}{a_x^{5/3}}\left[a_0 + a_1 n\left(\frac{\delta t}{a_x}\right)^{2/3} + a_2 n^2\left(\frac{\delta t}{a_x}\right)^{4/3} + \cdots\right]$$

$$P(u_x) = \frac{2}{\pi^3}\frac{1}{\sqrt{u_x^2 + 2\pi n\sigma^2}}\left[K\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right) - E\left(\frac{2\pi n\sigma^2}{u_x^2 + 2\pi n\sigma^2}\right)\right]$$

with NO fitting parameters!

Experimental data

$$u(t_0) = u_{nn} + N_1$$
$$u(t_0 + \tau) = u_{nn} + N_1(1 + N_2)$$
$$u(t_0 + \tau) - u(t_0) = N_1 N_2$$

$$P(a_x) = \frac{1}{\pi\sigma_1\sigma_2} K_0\left(\frac{\sqrt{a_x^2}}{\sigma_1\sigma_2}\right)$$

Gaussian (correlated)

Gaussian (uncorrelated)

Implications:

- Lagrangian statistics are dominated by noise in the core and nearest neighbor contributions in the wings – two-dimensional in the plane perpendicular to the closest vortex filament

- As the temporal increment $\tau \rightarrow 0$ the velocity difference probability density function approaches the new vortex nearest neighbor velocity pdf, because changes in the flow field resulting from new vortex creation overwhelm contributions from advection by existing filaments

  NEW vorticity changes do not have to be big (pdf normalized by rms)

- Lagrangian tracers randomly sample a random collection of vortices

- Random stirring mimics effects of vortex stretching

Toward a transport model:

1. Can we model (analytically and experimentally) transport using the statistics of Lagrangian trajectories in a point vortex flow?

2. Can we objectively (interactively) coherent vortical structures simulations of real three-dimensional turbulence?

3. Can we relate these statistics to the large scale flow?

4. Can we use these in place of point vortices in a transport model?