From Quasi-Geostrophy to Boussinesq Dynamics and In-Between

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1. Derivation of the Boussinesq Equations
2. Linear and nonlinear structure in terms of waves
3. Traditional and alternative derivations of the Quasi-Geostrophic Equations
4. Reduced models “in between”
Derivation of the Boussinesq approximation from the conservation laws governing continuum fluid dynamics.

**Conservation of Mass**

\[ \rho(x, t) \text{ is density} \]

\[
\frac{d}{dt} \int_V \rho \, dV = \text{mass in} - \text{mass out} = \int_A \rho (-\mathbf{u} \cdot \hat{n}) \, dA
\]

**Divergence Thm:**

\[
\int_V \mathbf{D} \cdot \mathbf{E} \, dV = \int_A \mathbf{E} \cdot \hat{n} \, dA
\]
\[
\frac{d}{dt} \int_V p \, dV = -\int_V \nabla \cdot (pu) \, dV
\]

Shrink the volume to a point \(\Rightarrow\)

\[
\frac{\partial p}{\partial t} + \nabla \cdot (pu) = 0 \quad \text{or} \quad \frac{Dp}{Dt} = -p \nabla \cdot u
\]

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla
\]

Conservation of Momentum \(F = ma\)

\[
\frac{d}{dt} \int_V pu \, dV = \frac{\text{momentum in}}{\text{unit time}} - \frac{\text{momentum out}}{\text{unit time}} + F
\]

\[
= \int_A \rho u (-u \cdot \hat{n}) \, dA + F
\]

\[
= \int_V -\nabla \cdot (\rho u u) \, dV + F
\]

Shrink the volume to a point \(\Rightarrow\)

\[
\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho uu) = F' \quad \text{(per unit volume)}
\]
\[ \frac{\partial}{\partial t} (\rho u) + \mathbf{u} \cdot \nabla (\rho u) + \rho u \mathbf{v} \cdot \mathbf{u} = F \quad \text{(drop prime)} \]

\[ \frac{D}{Dt} (\rho u) + \rho u \frac{Dp}{Dt} = F \]

\[ \rho \frac{D\mathbf{u}}{Dt} + u \frac{D\rho}{Dt} + \rho u \mathbf{v} \cdot \mathbf{u} = F \]

use Conservation of Mass \( \frac{D\rho}{Dt} = -\rho \mathbf{v} \cdot \mathbf{u} \)

\[ \rho \frac{D\mathbf{u}}{Dt} - \rho u \mathbf{v} \cdot \mathbf{u} + \rho u \mathbf{v} \cdot \mathbf{u} = F \]

\[ \rho \frac{D\mathbf{u}}{Dt} = F \quad \text{What is } F? \]

**Body Forces**

\[ \begin{bmatrix} \text{force} \\ \text{volume} \end{bmatrix} \text{ gravity } \rho g \]

**Surface Forces**

\[ \begin{bmatrix} \text{force} \\ \text{area} \end{bmatrix} \]

normal and tangential stresses
\[ F_s = \nabla \cdot \sigma = -\nabla p + \nabla \cdot T \]

\[ \sigma = -p I + T \]

\( p \) is the compressive-normal stress at any point in a fluid at rest.

\( T \) gives additional tensile-normal and tangential stresses at a point in a fluid in motion.

\( I \) is symmetric follows from conservation of angular momentum.

**For a Newtonian fluid**

\[ T = \lambda \text{Tr}(S) I + \mu S \]

\[ S = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right] \]

symmetric part of the strain rate

\[ \text{Tr}(S) = \nabla \cdot u \]

\( \lambda, \mu \) can be different [think of adding purely compressive additional stresses]
Consider the sum of all the normal stresses

\[ \text{Tr}(\sigma) = -3p + 3\lambda \text{Tr}(\Sigma) + 2\mu \text{Tr}(\Sigma) \]

\[ p - \frac{1}{3} \text{Tr}(\Sigma) = (\frac{2}{3} \mu + \lambda) \nabla \cdot \mathbf{u} \]

The quantity \((\frac{2}{3} \mu + \lambda)\) is measured to be significantly different from zero only when compressibility effects are very strong, e.g., when shock waves are present.

Stokes assumption: \( \lambda = -\frac{2}{3} \mu \implies \)

\[ \sigma = -p \mathbb{I} + 2\mu \Sigma = -\frac{2}{3} \mu (\nabla \cdot \mathbf{u}) \mathbb{I} \]

\[ \mathbf{F}_s = \nabla \cdot \sigma = -\nabla \left[ p + \frac{2}{3} \mu \nabla \cdot \mathbf{u} \right] + \nabla \cdot (2\mu \Sigma) \]

\( \mu \) = coefficient of viscosity

So far 4 equations, 5 unknowns \( u, v, w, p, P \)
Assume an ideal gas with $p = pRT$
and the **Thermal Energy Equation** $⇒$
6 equations, 6 unknowns

**1st Law of Thermodynamics**

Fluid at rest: $\dot{S}Q = d\hat{u} + \dot{S}W$
- heat added to CV
- change in internal energy

Fluid in motion: $\dot{S}Q = d\left(\hat{u} + \frac{u \cdot \dot{u}}{2}\right) + \dot{S}W$

**Energy Equation** (thermal + kinetic)

$$\frac{d}{dt} \int_{V} \rho \left(\hat{u} + \frac{u \cdot \dot{u}}{2}\right) dV = \int_{A} \rho \left(\hat{u} + \frac{u \cdot \dot{u}}{2}\right) (-u \cdot \hat{n}) dA$$

$$+ \frac{\dot{S}Q}{\dot{S}t} - \frac{\dot{S}W}{\dot{S}t}$$
- heat added to CV per unit time
- work done by CV per unit time
work has units of force \cdot velocity

Work done on CV by pressure forces

\[ \int_A p(-\mathbf{n}) \cdot \mathbf{u} \, dA \]

Work done by CV is \[ \int_A p \mathbf{n} \cdot \mathbf{u} \, dA \]

Pressure

\[ -\frac{\Delta W}{\Delta t} = -\int_A p \mathbf{n} \cdot \mathbf{u} \, dA \]

Viscous Forces

\[ -\frac{\Delta W}{\Delta t} = \int_A \mathbf{T}_{ij} \cdot \mathbf{u} \mathbf{n} \, dA \]

\[ \left[ \int_A \mathbf{T}_{ij} \cdot \mathbf{u} \mathbf{n} \, dA \right] \]

Gravity

\[ -\frac{\Delta W}{\Delta t} = \int_V \rho g \cdot \mathbf{u} \, dV \]
Heat added to CV/unit time \[ \frac{\delta Q}{\delta t} = \int_A \mathbf{q} \cdot (-\hat{n}) \, dA \]

\( \mathbf{q} \) is the heat flux vector.

Use Divergence Thm.: shrink CV to a point

\[ \frac{\partial}{\partial t} \left[ \rho \left( \mathbf{u} + \frac{u \cdot \mathbf{u}}{2} \right) \right] + \nabla \cdot \left[ \mu \left( \mathbf{u} + \frac{u \cdot \mathbf{u}}{2} \right) u \right] = -\nabla \cdot (\rho u) + \nabla \cdot [u \cdot \mathbf{T}] - \nabla \cdot \mathbf{q} + \rho g \cdot \mathbf{u} \]

But we know:

\[ \frac{\partial}{\partial t} (\rho \mathbf{F}) + \nabla \cdot (\rho F \mathbf{u}) = \rho \frac{D \mathbf{F}}{Dt} \]

Using vector identities and Conservation of Mass \( \Rightarrow \)

\[ \rho \frac{D}{Dt} \left[ \mathbf{u} + \frac{u \cdot \mathbf{u}}{2} \right] = -\nabla \cdot (\rho u) + \nabla \cdot [u \cdot \mathbf{T}] - \nabla \cdot \mathbf{q} + \rho g \cdot \mathbf{u} \]
Subtract off equation for Mechanical (kinetic) energy which has no new info:

\[
\frac{\rho Du}{Dt} = u \cdot \left[ -\nabla p + \rho g + \nabla \cdot \mathbf{T} \right]
\]

\[
\left( \frac{\partial (u \cdot u)}{\partial t} \right) = -u \cdot \nabla p + \rho g \cdot u + u \cdot \left[ \nabla \cdot \mathbf{T} \right]
\]

After the subtraction:

\[
\frac{\rho D\mathbf{u}}{Dt} = -p \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{g} + \mathbf{T} : \nabla \mathbf{u}
\]

is the thermal energy equation.
Summary

Newtonian Fluid \( \mathbf{I} \propto \dot{\mathbf{v}} \)

Stokes' assumption \( \dot{\mathbf{v}} = -\frac{2}{3} \mu \mathbf{j} \) Ideal Gas

\[
\frac{D \mathbf{E}}{D t} + \mathbf{p} \nabla \cdot \mathbf{u} = 0, \quad \mathbf{p} = \rho RT
\]

\[
\rho \frac{D \mathbf{u}}{D t} = \rho g - \nabla \left[ \rho + \frac{2}{3} \mu \nabla \cdot \mathbf{u} \right] + \nabla \cdot (\rho \mathbf{u} \mathbf{u})
\]

\[
= \rho g - \nabla \rho + \nabla \cdot \mathbf{T}
\]

\[
\rho \frac{D \mathbf{u}}{D t} = -\rho \nabla \cdot \mathbf{u} - \nabla \rho + \mathbf{T} \cdot \nabla \mathbf{u}
\]

\[
\mathbf{I} = \rho \mathbf{u} \mathbf{u} - \frac{2}{3} \mu \nabla \cdot \mathbf{u} \mathbf{I}
\]

\[
\mathbf{S} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]
\]

\( \mathbf{q} = \) heat flux vector, \( \mu = \) coefficient of viscosity

6 equations, 6 unknowns \( \mathbf{u}, \rho, \mathbf{p}, T \)

If \( \mathbf{q} \) is given in terms of \( T \)
The Boussinesq Approximation

Thermal Heat Equation:

\[ \rho \frac{D\hat{u}}{Dt} = -\nabla q - p \nabla \cdot u + \chi : \nabla u \]

In general, \(-p \nabla \cdot u = \rho \frac{\partial \rho}{\partial T} DT\)

Now assume \(\rho\) depends mainly on \(T\)

[neglect dependence on \(p\) and salt]

\[-p \nabla \cdot u = \rho \frac{\partial \rho}{\partial T} DT \approx \frac{\partial \rho}{\partial T} \bigg|_p \frac{DT}{Dt} \]

\[-p \alpha \frac{DT}{Dt} \]

With \(\alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_p\) is the thermal expansion coefficient; dimension \(\text{[1/°K]}\)
By Ideal Gas Law \( p = \rho RT \)
\[
\alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_p = -\frac{1}{\rho} \left( \frac{-\rho}{RT^2} \right) = \frac{1}{T}
\]

So \( -p \nabla \cdot \mathbf{u} \approx -\rho RT \frac{DT}{DT} = -\rho (c_p - c_v) \frac{DT}{DT} \)

[see handout]

with \( \hat{u}(T) = c_v T \) [see handout] and

neglecting viscous heating \( \Rightarrow \)

\[
\rho \frac{D}{Dt} (c_v T) \approx -\nabla \cdot \mathbf{q} - \rho (c_p - c_v) \frac{DT}{DT}
\]

\[
\Rightarrow \rho c_p \frac{DT}{DT} = -\nabla \cdot \mathbf{q}
\]

Finally Fourier's law of Heat Conduction

\[
\Rightarrow \rho c_p \frac{DT}{DT} = -\nabla \cdot \left[ -k \nabla T \right] = k \nabla^2 T
\]
Now assume the background state \((p_0, T_0)\) is in hydrostatic balance

\[
\rho = p_0 + \tilde{\rho}(x, t), \quad |\tilde{\rho}| \ll p_0
\]

\[
T = T_0 + \tilde{T}(x, t)
\]

\[
p = p_0(z) + \tilde{p}(x, t) \quad \text{and} \quad \nabla p_0(z) = \rho_0 g
\]

\[
\frac{\partial p}{\partial x} \propto -\alpha \frac{\partial T}{\partial x} \quad \text{and} \quad |\tilde{\rho}| \ll p_0 \implies
\]

\[
\frac{\partial p}{\partial p_0} \propto -\alpha (T - T_0) \quad \text{or} \quad \frac{\partial \tilde{p}}{\partial \rho_0} \propto (T - T_0)
\]

**Thermal Energy**

\[
\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T \implies
\]

\[
\rho_0 c_p \left( \frac{1}{\rho_0} \right) \frac{\partial \tilde{\rho}}{\partial t} \propto k \left( \frac{1}{\rho_0} \right) \nabla^2 \tilde{\rho}
\]

\[
\frac{\partial \tilde{\rho}}{\partial t} \propto k \nabla^2 \tilde{\rho} \quad \text{or} \quad K = \frac{k}{\rho_0 c_p}
\]
Mass \[ \frac{\partial \rho}{\partial t} + \rho \mathbf{V} \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0 \]

\[ \frac{\partial \rho}{\partial t} + (\rho + \tilde{\rho}) \mathbf{V} \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \tilde{\rho} = 0 \]

with dominant balance \[ \nabla \cdot \mathbf{u} = 0 \]

Momentum \[ (\rho + \tilde{\rho}) \frac{\partial \mathbf{u}}{\partial t} = -\nabla \left[ \rho \mathbf{o}(z) + \tilde{\rho} \right] \]

\[ + (\rho + \tilde{\rho}) \frac{g}{\rho} + \mathbf{V}_0 \left[ 2 \mu \mathbf{s} - \frac{3}{3} \mu \nabla \cdot \mathbf{u} \right] \]

Subtract \[ -\nabla \rho \mathbf{o}(z) + \rho \mathbf{g} = 0 \]

use \[ \nabla \cdot \mathbf{u} = 0 \]

\[ (1 + \frac{\tilde{\rho}}{\rho_0}) \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla \tilde{\rho} + \frac{\rho_0}{\rho_0} \frac{g}{\rho_0} + \frac{\mu}{\rho_0} \nabla^2 \mathbf{u} \]

with dominant balance \[ \frac{\partial \mathbf{u}}{\partial t} \]

\[ \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla \tilde{\rho} + \frac{\rho_0}{\rho_0} \frac{g}{\rho_0} + \mu \rho_0 \nabla^2 \mathbf{u} \]

\[ |g| \gg \left| \frac{\partial \mathbf{u}}{\partial t} \right| \] viscous effects allowed to be large
Summary

\[ \nabla \cdot \mathbf{u} = 0 \quad \text{(mass)} \]

\[ \frac{\rho_0 \, \partial \mathbf{u}}{\partial t} = -\nabla \tilde{p} + \tilde{\rho} \mathbf{g} + \nu \nabla^2 \mathbf{u} \quad \text{(momentum)} \]

\[ \frac{\nabla \tilde{\rho}}{\partial t} = K \nabla^2 \tilde{\rho} \quad \text{(thermal energy)} \]

with background state

\[ \tilde{\rho} = \rho_0 + \tilde{\rho} , \quad |\tilde{\rho}| < \rho_0 \]

\[ T = T_0 + \tilde{T} \]

\[ \tilde{\rho} = \rho_0(z) + \tilde{\rho} \]

\[ \nabla \rho_0(z) = \rho_0 \mathbf{g} \]

\[ \tilde{\rho} \approx \alpha \rho_0 (T - T_0) \]
Homework

* Assume another layer of hydrostatic balance
  \[ \tilde{\gamma} = \gamma_i(z) + \gamma'(x,t) = -b z + \gamma'(x,t) \]
  \[ \tilde{p} = p_i(z) + \gamma'(x,t) \]
  with \[ -\frac{1}{\rho_0} \nabla p_i(z) = \frac{b z}{\rho_0} \frac{g}{g}, \quad g = -g \hat{z} \]

* Move into the rotating frame \[ \tilde{z} = \Omega \hat{z} \]

* Define \[ \epsilon' = \left( \frac{b \rho_0}{g} \right)^{1/2} \theta, \quad N = \left( \frac{g b \gamma}{\rho_0} \right)^{1/2} \]
  \[ p = \frac{p'}{\rho_0} \]

\[ \nabla \cdot u = 0 \]

\[ \frac{D u + 2 \Omega \hat{z} \times u + N \theta \hat{z}}{D t} = -\nabla p + \nu \nabla^2 u \]

\[ \frac{D \theta - N (u \cdot \hat{z})}{D t} = K \nabla^2 \theta \]
Shortcomings of Bassiniesq and what to do about it:

* In the compressible atmosphere, \( \rho \) depends on \( p \) as well as \( T \)

* Density \( \rho \) decreases with height as pressure \( p \) decreases

* The natural background state is not constant density \( \rho_0 \), but rather a background density that decreases with height \( [p] \)

\( \Rightarrow \) a background of constant

Potential Temperature
General Thermal Energy Equation

\[ \rho c_v \frac{DT}{Dt} = -V \cdot q - p V \cdot u + T \cdot \nabla \cdot \nu u \]

= \[-V \cdot q + p \frac{Dp}{Dt} \quad \text{[mass conservation]} \]

= \[-V \cdot q - \rho c_v \left( \frac{1}{2} \frac{Dp}{Dt} \right) = -V \cdot q - \rho c_v \frac{Dp}{Dt} \quad \frac{1}{2} \]

Ideal Gas

\[ p = \rho R T \]

\[ \rho c_v \frac{DT}{Dt} = -V \cdot q - \rho c_v \frac{Dp}{Dt} \quad \frac{RT}{p} \]

= \[-V \cdot q - \rho c_v \frac{R DT}{Dt} + \frac{p c_v RT}{p^2} \frac{DP}{DT} \]

= \[-V \cdot q - \rho c_v \frac{RT}{p} \frac{DT}{Dt} + \frac{DP}{DT} \]

\[ \rho c_p \frac{DT}{Dt} = -V \cdot q + \frac{DP}{DT} \]

Boussinesq assumes \( \frac{DP}{DT} \) small
Divide by $q, T$:
\[
\frac{C_p D T}{T} = -\frac{\nabla \cdot q}{q T} + \frac{R D p}{P D T}
\]
\[
C_p \frac{D \ln T}{D t} - R \frac{D \ln p}{D T} = -\frac{\nabla \cdot q}{q T}
\]

Introduce Potential Temperature $\Theta = T \left( \frac{P_s}{P} \right)^{\frac{R}{C_p}}$

not to be confused with the notation $\Theta$ in the previous notes !!.

$p_s = \text{surface pressure}$

$\Theta = \text{temperature a parcel would have if it moves adiabatically [no heat leaves the parcel]}$ from its current height down to the surface.

$\ln \Theta = \ln T + \frac{R}{C_p} \left[ \ln p_s - \ln p \right]$
\[ C_p \ln \theta = C_p \ln T - R \ln p_s - R \ln p \]

\[ \frac{C_p D \ln \theta}{D t} = \frac{C_p D \ln T}{D t} - \frac{R D \ln p}{D t} \]

\[ \frac{\nabla \cdot q}{\rho} = - \frac{\nabla \cdot q}{\rho} \]

2nd Law of Thermodynamics

\[ S Q = T d S \]
\[ \frac{S Q}{S t} = - \frac{\nabla \cdot q}{\rho} \]

\[ \frac{1}{T} \frac{S Q}{S t} = d S = - \frac{\nabla \cdot q}{\rho} \]

\[ \frac{1}{T} \frac{S Q}{S t} \]

\[ \frac{d S}{d t} = - \frac{\nabla \cdot q}{\rho} \]

\[ T \]

\[ \Rightarrow \text{surfaces of constant } \theta \text{ are}
\]

\[ \text{surfaces of constant entropy} \]

\[ \ast \text{the natural background state is one of constant } \theta \text{ [not constant } p] \]

Finally

\[ \frac{d \theta}{d t} = \frac{\theta}{T} \left( \frac{k}{c_p} \right) \nabla^2 T \]