From Quasi-Geostrophy to Boussinesq Dynamics and In-Between

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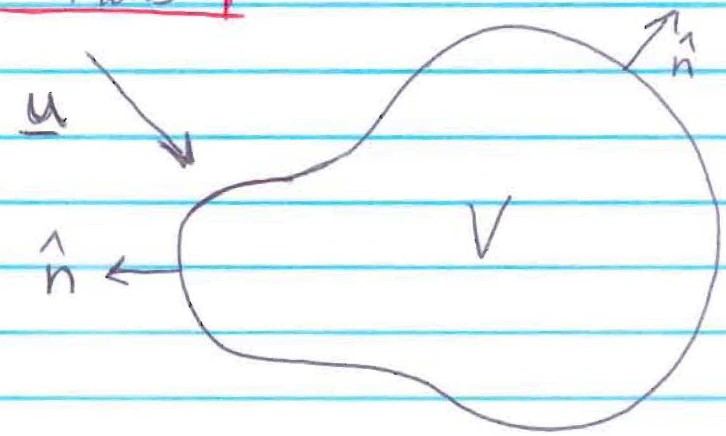
1. Derivation of the Boussinesq Equations
2. Linear and nonlinear structure in terms of waves
3. Traditional and alternative derivations of the Quasi-Geostrophic Equations
4. Reduced models "in between"

NCAR/IMAGe Summer Workshop Lecture 1

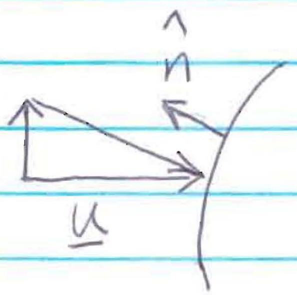
Derivation of the Boussinesq approximation from the conservation laws governing continuum fluid dynamics.

Conservation of Mass

$\rho(\underline{x}, t)$ is density



$$\frac{d}{dt}\int_V \rho \, dV = \frac{\text{mass in} - \text{mass out}}{\text{unit time}} = \int_A \rho(-\underline{u} \cdot \hat{n})\, dA$$



Divergence Thm: $\displaystyle\int_V \underline{\nabla} \cdot \underline{F} \, dV = \int_A \underline{F} \cdot \hat{n} \, dA$

$$\frac{d}{dt}\int_V \rho \, dV = -\int_V \underline{\nabla}\cdot(\rho \underline{u})\, dV$$

Shrink the volume to a point $\Longrightarrow$

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0 \quad \text{or} \quad \frac{D\rho}{Dt} = -\rho\,\underline{\nabla}\cdot\underline{u} \\ &\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla} \end{aligned}}$$

Conservation of Momentum $\qquad \underline{F} = m\underline{a}$

$$\frac{d}{dt}\int_V \rho\underline{u}\, dV = \frac{\text{momentum in} - \text{momentum out}}{\text{unit time}} + \underline{F}$$

$$= \int_A \rho\underline{u}\,(-\underline{u}\cdot\hat{n})\, dA + \underline{F}$$

$$= \int_V -\underline{\nabla}\cdot(\rho\,\underline{u}\,\underline{u})\, dV + \underline{F}$$

Shrink the volume to a point $\Longrightarrow$

$$\frac{\partial}{\partial t}(\rho\underline{u}) + \underline{\nabla}\cdot(\rho\,\underline{u}\,\underline{u}) = \underline{F}' \qquad \left(\text{per unit volume}\right)$$

$$\frac{\partial}{\partial t}(\rho \underline{u}) + \underline{u}\cdot\underline{\nabla}(\rho\underline{u}) + \rho\underline{u}(\underline{\nabla}\cdot\underline{u}) = \underline{F} \quad \text{(drop prime)}$$

$$\frac{D}{Dt}(\rho\underline{u}) + \rho\underline{u}(\underline{\nabla}\cdot\underline{u}) = \underline{F}$$

$$\rho\frac{D\underline{u}}{Dt} + \underline{u}\frac{D\rho}{Dt} + \rho\underline{u}(\underline{\nabla}\cdot\underline{u}) = \underline{F}$$

use Conservation of Mass $\frac{D\rho}{Dt} = -\rho\,\underline{\nabla}\cdot\underline{u}$

$$\rho\frac{D\underline{u}}{Dt} - \rho\underline{u}\,\underline{\nabla}\cdot\underline{u} + \rho\underline{u}\,\underline{\nabla}\cdot\underline{u} = \underline{F}$$

$$\boxed{\rho\frac{D\underline{u}}{Dt} = \underline{F}}$$

What is $\underline{F}$?

Body Forces $\left[\frac{\text{Force}}{\text{volume}}\right]$ gravity $\rho\underline{g}$

Surface Forces $\left[\frac{\text{force}}{\text{area}}\right]$

normal and tangential stresses

$$\underline{F}_s = \underline{\nabla} \cdot \underline{\underline{\sigma}} = -\underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}}$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$$

p is the compressive-normal stress at any point in a fluid at rest

$\underline{\underline{\tau}}$ gives additional tensile-normal and tangential stresses at a point in a fluid in motion

$\underline{\underline{\tau}}$ symmetric follows from conservation of angular momentum

$\boxed{\text{For a Newtonian fluid}}$

$$\underline{\underline{\tau}} = \lambda \, Tr(\underline{\underline{S}}) \underline{\underline{I}} + 2\mu \underline{\underline{S}}$$

* $$\underline{\underline{S}} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T\right]$$

symmetric part of the strain rate

$$Tr(\underline{\underline{S}}) = \underline{\nabla} \cdot \underline{u}$$

* λ, μ can be different [think of adding purely compressive additional stresses]

Consider the sum of all the normal stresses

$$Tr(\underline{\underline{\sigma}}) = -3p + 3\lambda \, Tr(\underline{\underline{S}}) + 2\mu \, Tr(\underline{\underline{S}})$$

$$p - \frac{1}{3} Tr(\underline{\underline{\sigma}}) = \left(\frac{2}{3}\mu + \lambda\right) \underline{\nabla} \cdot \underline{u}$$

The quantity $\left(\frac{2}{3}\mu + \lambda\right)$ is measured to be significantly different from zero only when compressibility effects are very strong, e.g. when shock waves are present.

Stokes assumption: $\lambda = -\frac{2}{3}\mu \implies$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \underline{\underline{S}} - \frac{2}{3}\mu (\underline{\nabla} \cdot \underline{u}) \underline{\underline{I}}$$

$$\boxed{\underline{F}_S = \underline{\nabla} \cdot \underline{\underline{\sigma}} = -\underline{\nabla}\left[p + \frac{2}{3}\mu \, \underline{\nabla} \cdot \underline{u}\right] + \underline{\nabla} \cdot (2\mu \underline{\underline{S}})}$$

μ = coefficient of viscosity

So far 4 equations, 5 unknowns
u, v, w, ρ, p

Assume an ideal gas with $\boxed{p = \rho R T}$

and the $\boxed{\text{Thermal Energy Equation}}$ $\Longrightarrow$

6 equations, 6 unknowns

$\boxed{\text{1}^{\text{st}} \text{ Law of Thermodynamics}}$

Fluid at rest: $\delta Q = d\hat{u} + \delta W$

heat added to CV

change in internal energy

work done by CV

Fluid in motion: $\delta Q = d\left(\hat{u} + \frac{\underline{u} \cdot \underline{u}}{2}\right) + \delta W$

$\boxed{\text{Energy Equation}}$ (thermal + kinetic)

$$\frac{d}{dt}\int_V \rho\left(\hat{u} + \frac{\underline{u}\cdot\underline{u}}{2}\right) dV = \int_A \rho\left(\hat{u} + \frac{\underline{u}\cdot\underline{u}}{2}\right)(-\underline{u}\cdot\hat{n})\, dA$$

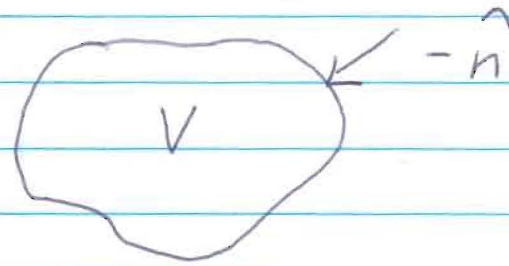
$$+ \frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t}$$

heat added to CV per unit time

work done by CV per unit time

work has units of Force $\cdot$ velocity

Work done on CV by pressure forces



$$\int_A p(-\hat{n}) \cdot \underline{u} \, dA$$

Work done by CV is $\int_A p \, \hat{n} \cdot \underline{u} \, dA$

$\boxed{\text{Pressure}}$ $$-\frac{\delta W}{\delta t} = -\int_A p \, \hat{n} \cdot \underline{u} \, dA$$

$\boxed{\text{Viscous Forces}}$ $$-\frac{\delta W}{\delta t} = \int_A \underline{\underline{\tau}} : \underline{u} \, \hat{n} \, dA$$

$$\left[\int_A \tau_{ij} \, u_i \, n_j \, dA \right]$$

$\boxed{\text{Gravity}}$ $$-\frac{\delta W}{\delta t} = \int_V \rho \underline{g} \cdot \underline{u} \, dV$$

Heat added to CV/unit time $\quad \dfrac{\delta Q}{\delta t} = \int_A \underset{\sim}{q} \cdot (-\hat{n})\, dA$

$\underset{\sim}{q}$ is the heat flux vector

Use Divergence Thm. $\Longrightarrow$ shrink CV to a point

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{u} + \frac{\underline{u}\cdot\underline{u}}{2}\right)\right] + \underline{\nabla}\cdot\left[\rho\left(\hat{u} + \frac{\underline{u}\cdot\underline{u}}{2}\right)\underline{u}\right]$$

$$= -\underline{\nabla}\cdot(p\underline{u}) + \underline{\nabla}\left[\underline{u}\cdot\underline{\underline{\tau}}\right] - \underline{\nabla}\cdot\underset{\sim}{q} + \rho\underset{\sim}{g}\cdot\underline{u}$$

But we know:

$$\frac{\partial}{\partial t}(\rho F) + \underline{\nabla}\cdot(\rho F\underline{u}) = \rho\frac{DF}{Dt}$$

Using vector identities and Conservation of Mass $\Longrightarrow$

$$\rho\frac{D}{Dt}\left[\hat{u} + \frac{\underline{u}\cdot\underline{u}}{2}\right] = -\underline{\nabla}\cdot(p\underline{u}) + \underline{\nabla}\cdot\left[\underline{u}\cdot\underline{\underline{\tau}}\right]$$

$$- \underline{\nabla}\cdot\underset{\sim}{q} + \rho\underset{\sim}{g}\cdot\underline{u}$$

Subtract off equation for Mechanical (kinetic) energy which has no new info:

$$\underline{u} \cdot \rho \frac{D\underline{u}}{Dt} = \underline{u} \cdot \left[-\underline{\nabla} p + \rho \underline{g} + \underline{\nabla} \cdot \underline{\underline{\tau}} \right]$$

$$\rho \frac{D\left(\frac{\underline{u}\cdot\underline{u}}{2}\right)}{Dt} = -\underline{u}\cdot\underline{\nabla} p + \rho \underline{g}\cdot\underline{u} + \underline{u}\cdot\left[\underline{\nabla}\cdot\underline{\underline{\tau}}\right]$$

After the subtraction:

$$\boxed{\rho \frac{D\hat{u}}{Dt} = -p\,\underline{\nabla}\cdot\underline{u} - \underline{\nabla}\cdot\underline{q} + \underline{\underline{\tau}} : \underline{\nabla}\,\underline{u}}$$

is the thermal energy equation

$\boxed{\text{Summary}}$ Newtonian Fluid $\underline{\underline{\tau}} \propto \underline{\underline{S}}$;

Stokes assumption $\lambda = -\frac{2}{3}\mu$; Ideal Gas

$$* \quad \frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0 \quad , \quad * \quad p = \rho R T$$

$$* \quad \rho \frac{D\underline{u}}{Dt} = \rho \underline{g} - \underline{\nabla}\left[p + \frac{2}{3}\mu \underline{\nabla} \cdot \underline{u}\right] + \underline{\nabla} \cdot (2\mu \underline{\underline{S}})$$

$$= \rho \underline{g} - \underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}}$$

$$* \quad \rho \frac{D\hat{u}}{Dt} = -p \underline{\nabla} \cdot \underline{u} - \underline{\nabla} \cdot \underline{q} + \underline{\underline{\tau}} : \underline{\nabla} \underline{u}$$

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{S}} - \frac{2}{3}\mu \underline{\nabla} \cdot \underline{u} \, \underline{\underline{I}}$$

$$\underline{\underline{S}} = \frac{1}{2}\left[\underline{\nabla} \underline{u} + (\underline{\nabla} \underline{u})^T\right]$$

$\underline{q}$ = heat flux vector, μ = coefficient of viscosity

6 equations, 6 unknowns $\underline{u}, \rho, p, T$

if $\underline{q}$ is given in terms of T

$\boxed{\text{The Boussinesq Approximation}}$

Thermal Heat Equation:

$$\rho \frac{D\hat{u}}{Dt} = -\underline{\nabla} \cdot q - p \underline{\nabla} \cdot \underline{u} + \underline{\underline{\tau}} : \underline{\nabla} \underline{u}$$

In general $-p \underline{\nabla} \cdot \underline{u} = \frac{p}{\rho} \frac{D\rho}{Dt}$

Now assume ρ depends mainly on T

[neglect dependence on p and salt]

$$-p \underline{\nabla} \cdot \underline{u} = \frac{p}{\rho} \frac{D\rho}{Dt} \approx \frac{p}{\rho} \left.\frac{\partial \rho}{\partial T}\right|_p \frac{DT}{Dt}$$

$$= -p \alpha \frac{DT}{Dt}$$

with $\alpha = -\frac{1}{\rho} \left.\frac{\partial \rho}{\partial T}\right|_p$ is the thermal

expansion coefficient; dimension $\frac{1}{T}$ $\left[\frac{1}{{}^\circ K}\right]$

By Ideal Gas Law $p = \rho R T$

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big|_p = -\frac{1}{\rho}\left(\frac{-p}{RT^2}\right) = \frac{1}{T}$$

So $$-p\,\underline{\nabla}\cdot\underline{u} \approx -\frac{\rho R T}{T}\frac{DT}{Dt} = -\rho(c_p - c_v)\frac{DT}{Dt}$$

[see handout]

with $\hat{u}(T) = c_v T$ [see handout] and

neglecting viscous heating $\Longrightarrow$

$$\rho \frac{D}{Dt}(c_v T) \approx -\underline{\nabla}\cdot\underline{q} - \rho(c_p - c_v)\frac{DT}{Dt}$$

$$\Longrightarrow \quad \rho c_p \frac{DT}{Dt} = -\underline{\nabla}\cdot\underline{q}$$

Finally Fourier's Law of Heat Conduction

$$\Longrightarrow \boxed{\rho c_p \frac{DT}{Dt} = -\underline{\nabla}\cdot\left[-k\underline{\nabla}T\right] = k\nabla^2 T}$$

Now assume the background state ρ_0, T_0 is in hydrostatic balance

$$\rho = \rho_0 + \tilde{\rho}(\underline{x}, t) \quad , \quad |\tilde{\rho}| \ll \rho_0$$

$$T = T_0 + \tilde{T}(\underline{x}, t)$$

$$p = p_0(z) + \tilde{p}(\underline{x}, t) \; ; \; \underline{\nabla} p_0(z) = \rho_0 \underline{g}$$

$$\frac{\partial \rho}{\rho} \approx -\alpha \, \partial T \quad \text{and} \quad |\tilde{\rho}| \ll \rho_0 \quad \Longrightarrow$$

$$\frac{\rho - \rho_0}{\rho_0} \approx -\alpha (T - T_0) \quad \text{or} \quad -\frac{\tilde{\rho}}{\alpha \rho_0} \approx (T - T_0)$$

$\boxed{\text{Thermal Energy}}$ $\rho c_p \frac{DT}{Dt} = k \nabla^2 T \implies$

$$\rho_0 c_p \left(\frac{-1}{\alpha \rho_0} \right) \frac{D\tilde{\rho}}{Dt} \approx k \left(\frac{-1}{\alpha \rho_0} \right) \nabla^2 \tilde{\rho} \qquad \text{or}$$

$$\boxed{\frac{D\tilde{\rho}}{Dt} \cong \kappa \nabla^2 \tilde{\rho} \; ; \; \kappa = \frac{k}{\rho_0 c_p}}$$

$\boxed{\text{Mass}}$ $\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \rho = 0$

$$\frac{\partial \tilde{\rho}}{\partial t} + (\rho_0 + \tilde{\rho}) \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \tilde{\rho} = 0$$

with dominant balance $\boxed{\underline{\nabla} \cdot \underline{u} = 0}$

$\boxed{\text{Momentum}}$ $(\rho_0 + \tilde{\rho}) \frac{D\underline{u}}{Dt} = -\underline{\nabla}\left[p_0(z) + \tilde{p}\right]$

$$+ (\rho_0 + \tilde{\rho}) \underline{g} + \underline{\nabla} \cdot \left[2\mu \underline{\underline{S}} - \frac{2}{3}\mu \underline{\nabla} \cdot \underline{u} \; \underline{\underline{I}}\right]$$

$$\left.\begin{array}{ll} \text{Subtract} & -\underline{\nabla} p_0(z) + \rho_0 \underline{g} = 0 \\ \text{use} & \underline{\nabla} \cdot \underline{u} = 0 \end{array}\right\} \Rightarrow$$

$$\left(1 + \frac{\tilde{\rho}}{\rho_0}\right) \frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0} \underline{\nabla} \tilde{p} + \frac{\tilde{\rho}}{\rho_0} \underline{g} + \frac{\mu}{\rho_0} \nabla^2 \underline{u}$$

with dominant balance

$$\boxed{\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0} \underline{\nabla} \tilde{p} + \frac{\tilde{\rho}}{\rho_0} \underline{g} + \frac{\mu}{\rho_0} \nabla^2 \underline{u}}$$

$|g| \gg \left|\frac{D\underline{u}}{Dt}\right|$, viscous effects allowed to be large

Summary

$\underline{\nabla} \cdot \underline{u} = 0$ (mass)

$$\rho_0 \frac{D\underline{u}}{Dt} = -\underline{\nabla} \tilde{p} + \tilde{\rho} \underline{g} + \nu \nabla^2 \underline{u}$$

(momentum)

$$\frac{D\tilde{\rho}}{Dt} = \kappa \nabla^2 \tilde{\rho}$$ (thermal energy)

with background state

$\rho = \rho_0 + \tilde{\rho}$, $|\tilde{\rho}| \ll \rho_0$

$T = T_0 + \tilde{T}$

$p = p_0(z) + \tilde{p}$

$\underline{\nabla} p_0(z) = \rho_0 \underline{g}$

$\tilde{\rho} \simeq \alpha \rho_0 (T - T_0)$

$\boxed{\text{Homework}}$

* Assume another layer of hydrostatic balance

$$\tilde{\rho} = \rho_1(z) + \rho'(\underline{x},t) = -bz + \rho'(\underline{x},t)$$

$$\tilde{p} = p_1(z) + p'(\underline{x},t)$$

with $-\frac{1}{\rho_0}\underline{\nabla} p_1(z) = \frac{bz}{\rho_0}\underline{g}$, $\underline{g} = -g\hat{z}$

* Move into the rotating frame $\underline{\Omega} = \Omega\hat{z}$
* Define $\rho' = \left(\frac{b\rho_0}{g}\right)^{1/2}\theta$, $N = \left(\frac{gb}{\rho_0}\right)^{1/2}$

$P = p'/\rho_0$ $\Longrightarrow$

$\underline{\nabla}\cdot\underline{u} = 0$,

$$\frac{D\underline{u}}{Dt} + 2\Omega\hat{z}\times\underline{u} + N\theta\hat{z} = -\underline{\nabla}P + \nu\nabla^2\underline{u}$$

$$\frac{D\theta}{Dt} - N(\underline{u}\cdot\hat{z}) = \kappa\nabla^2\theta$$

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Shortcomings of Boussinesq and what to do about it:

* In the compressible atmosphere, ρ depends on p as well as T

* Density ρ decreases with height as pressure p decreases

* The natural background state is <u>not</u> constant density ρ_0, but rather a background density that decreases with height [pressure]

$\Longrightarrow$ a background of constant

$$\boxed{\text{Potential Temperature}}$$

General Thermal Energy Equation

$$\rho c_v \frac{DT}{Dt} = -\underline{\nabla} \cdot \underline{q} - p\, \underline{\nabla} \cdot \underline{u} + \cancel{\underline{\underline{\tau}} : \underline{\nabla}\, \underline{u}}$$

$$= -\underline{\nabla} \cdot \underline{q} + \frac{p}{\rho} \frac{D\rho}{Dt} \quad \left[\text{mass conservation} \right]$$

$$= -\underline{\nabla} \cdot \underline{q} - p\rho \left[-\frac{1}{\rho^2} \frac{D\rho}{Dt} \right] = -\underline{\nabla} \cdot \underline{q} - p\rho \frac{D}{Dt} \left[\frac{1}{\rho} \right]$$

Ideal Gas

$$p = \rho R T$$

$$\rho c_v \frac{DT}{Dt} = -\underline{\nabla} \cdot \underline{q} - p\rho \frac{D}{Dt} \left[\frac{RT}{p} \right]$$

$$= -\underline{\nabla} \cdot \underline{q} - \frac{p\rho R}{p} \frac{DT}{Dt} + \frac{p\rho RT}{p^2} \frac{Dp}{Dt}$$

$$= -\underline{\nabla} \cdot \underline{q} - \rho R \frac{DT}{Dt} + \frac{Dp}{Dt}$$

$$\rho c_p \frac{DT}{Dt} = -\underline{\nabla} \cdot \underline{q} + \frac{Dp}{Dt}$$

$$\boxed{\text{Boussinesq assumes } \frac{Dp}{Dt} \text{ small}}$$

$$\frac{}{Dt}$$

Divide by ρT:

$$\frac{c_p}{T}\frac{DT}{Dt} = -\frac{\nabla\cdot q}{\rho T} + \frac{R}{p}\frac{Dp}{Dt}$$

$$c_p \frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} = -\frac{\nabla\cdot q}{\rho T}$$

Introduce $\boxed{\text{Potential Temperature} \quad \theta = T\left(\frac{p_s}{p}\right)^{R/c_p}}$

not to be confused with the notation θ in the previous notes !!

p_s = surface pressure

θ = temperature a parcel would have if it moves adiabatically [no heat leaves the parcel] from its current height down to the surface.

$$\ln\theta = \ln T + \frac{R}{c_p}\left[\ln p_s - \ln p\right]$$

$$c_p \ln\theta = c_p \ln T - R\ln p_s - R\ln p$$

$$c_p \frac{D}{Dt}\ln\theta = c_p \frac{D}{Dt}\ln T - R\frac{D}{Dt}\ln p$$

$$= -\frac{\nabla\cdot \underline{q}}{\rho T}$$

$\boxed{\text{2nd Law of Thermodynamics}}$

$\boxed{\delta Q = T dS}$, $\dfrac{\delta Q}{\delta t} = -\dfrac{\nabla\cdot \underline{q}}{\rho}$

$$\frac{1}{T}\frac{\delta Q}{\delta t} = \frac{dS}{dt} = \frac{-\nabla\cdot \underline{q}}{\rho T}$$

$\Longrightarrow$ surfaces of constant θ are surfaces of constant entropy

* the natural background state is one of constant θ [not constant ρ]

Finally $\boxed{\dfrac{D\theta}{Dt} = \dfrac{\theta}{T}\left(\dfrac{k}{c_p\rho}\right)\nabla^2 T}$